The h-Principle and singularities

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Abstract

In this expository paper, we introduce the foundations of h-principle, singularity theory of smooth functions, and several complex variables. The purpose of this article is to introduce the Igusa's theorem, which roughly says that "higher singularities" are "unnecessary", and a plurisubharmonic analogue of Igusa's theorem proven by Paul Falcone.

Contents

1 Overview

In this expository paper we will introduce the fundamentals of h-principle, singularity theory of smooth functions, and several complex variables.

A partial differential relation is said to satisfy the homotopy principle (h-principle) if every formal solution, (roughly) a solution of the relation treating partial derivatives as independent variables, is homotopic to a genuine solution. We will introduce the notion of jet space where a partial differential relation can be thought of a subset of it. Then we will introduce holonomic approximation theorem which is a powerful tool for proving the h-principle and give some examples of the h-principle.

Next, we will introduce the basics of singularity theory for smooth functions and the Igusa's theorem, which roughly says that "higher singularities" are unnecessary.

Finally, we will introduce some backgrounds in complex analysis and then state a plurisubharmonic analogue of Igusa's theorem proven by Paul Falcone.

These notes were made while reading [1], [2], and [3] and was a preparation for my senior honor thesis which will be devoted to the analogues of Igusa's theorem in the context of plurisubharmonic functions. Most proofs will be omitted and we refer the readers to the references [1], [2], and [3] for complete proofs.

2 Preliminary definitions and notations

We will briefly review the definitions of fibre bundles and vector bundles.

Definition 2.1. *A map $p : E \to B$ is called a fibre bundle with fibre F if for every $x \in B$ there exists a neighborhood U of x and a diffeomorphism $\psi : p^{-1}(U) \to U \times F$ such that*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\psi} & U \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr_U} \\ U & = & U \end{array} \tag{1}$$

commutes. Here, pr_U is the projection. Such ψ is called a local trivialization.

A map $s : B \to E$ such that $p \circ s = id_B$ is called a section of the fibre bundle, i.e. $s(x) \in p^{-1}(x)$.

Here are some examples of fibre bundles:

1. Projection map $X \times Y \to X$ is a fibre bundle with a global trivialization.

2. Möbius strip is a fibre bundle over S^1 with fibre $\mathbb{R}$.

3. Klein bottle is a fibre bundle over S^1 with fibre S^1.

Definition 2.2. *An n-dimensional vector bundle is a fibre bundle $\pi : E \to B$ whose fibre is an n-dimensional vector space V and such that the local trivializations $\psi : \pi^{-1}(U) \to U \times V$ are fibrewise linear, i.e. we require the change of trivializations to be fibrewise linear.*

For convenience, we will use the notation $\mathcal{O}p\, A$ for "an open neighborhood" of $A \subset V$, i.e. an arbitrarily small but non-specified open neighborhood of A.

3 Holonomic Approximation

In this section we introduce the holonomic approximation theorem which is a powerful tool for proving h-principle.

3.1 Language of jet spaces

In this section we introduce the language of jets. We will start with the coordinate definition of jets.

Definition 3.1.1. *Let $d_r = d(n,r)$ be the number of all partial derivatives D^α of order r of a function $\mathbb{R}^n \to \mathbb{R}$. Let $N_r = N(n,r) = 1 + d_1 + \cdots + d_r$. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^q$ and a point $x \in \mathbb{R}^n$,*

$$J^r_f(x) = \left(x, f(x), f'(x), \ldots, f^{(r)}(x)\right)$$

is called the r-jet of f at x. Equivalently, we can regard f as the section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$. The space $x \times \mathbb{R}^{qN_r}$ can be viewed as a space of all a priori possible values of the jets of the maps $f : \mathbb{R}^n \to \mathbb{R}^q$ at the point $x \in \mathbb{R}^n$. In this context the space

$$\mathbb{R}^n \times \mathbb{R}^{qN_r}$$

is called the space of r-jets of sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ and denoted by $J^r(\mathbb{R}^n, \mathbb{R}^q)$

Given a section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, the section

$$J^r_f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q), \; x \mapsto J^r_f(x),$$

of the trivial bundle

$$p^r : J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \to \mathbb{R}^n$$

is called the r-jet extension of f.

In order to define the r-jet space for an arbitrary fibre bundle, we need the following invariant definition of jets.

Consider a smooth fibre bundle $p : X \to V$. Fix $v \in V$. Two local sections $f : \mathcal{O}p\, v \to X$ and $g : \mathcal{O}p\, v \to X$ are called r-tangent at the point v if $f(v) = g(v)$ and

$$J^r_{\phi_* f}(\phi(v)) = J^r_{\phi_* g}(\phi(v))$$

for a local trivialization $\phi : U \to R^n \times R^q$ of X in a neighborhood U of the point $x = f(v) = g(v)$, where $\phi_* f$ and $\phi_* g$ are images of the sections f and g. The r-tangency class of a section $f : \mathcal{O}p\, v \to X$ at a point $v \in V$ is called the r-jet of f at v and denoted by $J^r_f(v)$.

Definition 3.1.2. *For smooth fibre bundle $p : X \to V$, we define its r-jet space*

$$X^{(r)} = \{(v, f) | f : \mathcal{O}p\, v \to X\} / \sim$$

where $\sim$ is the equivalence relation defined by r-tangency.

Define $p^r_0 : X^{(r)} \to X$ by $p^r_0([v, f]) = f(v)$, and $p^r = p \circ p^r_0 : X^{(r)} \to V$.

Remark. *For local trivializations* $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^q$*, the extensions*

$$\varphi^r : (p_0^r)^{-1}(U) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

sending the r-tangency classes of local sections of X *to the r-tangency classes of their images in* $J^r(\mathbb{R}^n, \mathbb{R}^q)$ *define a natural smooth structure on* $X^{(r)}$ *such that* $p^r : X^{(r)} \to V$ *becomes a smooth fibre bundle.*

Remark. *Since the r-tangency of two sections implies their s-tangency for* $0 \le s < r$*, we get projections* $p_s^r : X^{(r)} \to X^{(s)}$*. Thus, we have a chain a projections:*

$$X = X^{(0)} \leftarrow X^{(1)} \leftarrow X^{(2)} \leftarrow \cdots \leftarrow X^{(r)} \leftarrow \cdots$$

Given a section $F : V \to X^{(r)}$, we denote by $bs\, F$ the underlying section $p_0^r \circ F : V \to X$.
A section $F : V \to X^{(r)}$ is called holonomic if $F = J^r_{bs\,F}$. In particular, holonomic sections $\mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ have the form

$$x \to \left(x, f(x), f'(x), \dots, f^{(r)}(x)\right).$$

Note that the space $J^r(\mathbb{R}^n, \mathbb{R}^q)$ has a canonical parametrization

$$J^r : \mathbb{R}^n \times P_r(n,q) \to J^r(\mathbb{R}^n, \mathbb{R}^q), \quad (u, f) \mapsto J^r_f(u)$$

where $P_r(n,q)$ is the space of all polynomial maps $\mathbb{R}^n \to \mathbb{R}^q$ of degree less than or equal to r. Such parametrization has the property that the images of the horizontal fibres are holonomic sections. In particular, we have the following holonomic splitting theorem.

Theorem (Holonomic Splitting) 3.1.3. *Any holonomic section* $F : V \to X^{(r)}$ *has a holonomically trivialized tubular neighborhood over any open ball* $U \subset V$*, i.e. there is an embedding*

$$P_F : U \times \mathbb{R}^K \to X^{(r)}|_U$$

onto a neighborhood of $F|_U$*, where* $K = dim P_r(n,q) = \frac{q(n+1)!}{n!r!}$*, such that* $P_F(u,0) = F(u)$*,* $u \in U$*, and for each* $z \in \mathbb{R}^K$ *the map*

$$u \mapsto P_F(u,z),\ u \in U,$$

is a holonomic section $U \to X^{(r)}|_U$

3.2 Thom transversality Theorem

In this subsection, we introduce Thom transversality theorem, which will be used later when discussing singularities.

Recall that a map $f : V \to W$ is called transverse to a submanifold $\Sigma \subset W$ if for each $x \in V$, either $f(x) \notin \Sigma$ or $f(x) \in \Sigma$ and $T_{f(x)}W = T_{f(x)}\Sigma + df(T_x V)$. By the implicit function theorem, if a map $f : V \to W$ is transverse to Σ then $f^{-1}(\Sigma)$ is a submanifold of V of the same codimension in V as that of Σ in W.

To state Thom transversality theorem, we introduce the following notion of stratified sets.

Definition 3.2.1. *A closed subset S of a manifold V is called stratified if it is presented as a union $\bigcup_0^N S_j$ of locally closed submanifolds S_j, called strata, such that for each $k \in \{0,\dots,N\}$ we have*

$$\bar{S_k} = \bigcup_{j=k}^N S_j.$$

The dimension of a stratified set is the maximal dimension of its strata and its codimension is the minimal codimension of its strata.

Here are some examples of stratified sets:

1. A manifold V with boundary has a stratification with two strata $S_0 = \operatorname{Int} V$ and $S_1 = \partial V$.

2. Given a smooth triangulation of a manifold V, any closed subset which is a union of simplices of the triangulation is stratified by the strata wich are interiors of the simplices, and this kind of stratified sets are called polyhedra.

Here is another example that will be useful to us later:

Proposition 3.2.2. *Denote Σ^i the algebraic subset of the space $M_{q\times n}$ of $q \times n$ matrices which consists of matrices of rank $\leq m - i$ where $m = min(n,q)$, and by S_i the space of matrices of rank $m - i$. Then S_i are locally closed submanifolds of $M_{q\times n}$ of codimension $i(|q-n|+i)$, and the union $\bigcup_{j=i}^m S_j$ is a natural stratification of Σ^i.*

As a corollary, we have the following:

Corollary 3.2.3. *Let $\Sigma^i \subset J^1(V,W)$ be the space of 1-jets of maps of rank $\leq min(n,q) - i$. Then Σ^i is a stratified subset of codimension $i(|q-n|+i)$*

A map $f : V \to W$ is called transverse to a stratified set $\Sigma = \bigcup_0^N S_j \subset W$ if it is transverse to each stratum S_j. For a transverse map f, the preimage $f^{-1}(\Sigma)$ of a stratified subset $\Sigma \subset W$ is a stratified subset of V of the same codimension.

Now we are ready to state the Thom transversality theorem.

Theorem 3.2.4. *(Thom Transversality Theorem) Let $X \to V$ be a smooth fibre bundle and Σ a stratified subset of the r-jet space $X^{(r)}$. Then for a generic section $f : V \to X$ its r-jet extension $J^r_f : V \to X^{(r)}$ is transverse to Σ.*

3.3 Holonomic approximation and its applications

We are interested in the following question: Given a r-jet section and an arbitrary small neighborhood of the image of this section in the jet space, can one find a holonomic section in this neighborhood?

The problem of finding a holomonic approximation of a section of the r-jet space near a submanifold A is usually unsolvable. However, we can always find a holonomic approximation of a section $F : V \to X^{(r)}$ near a slightly deformed submanifold $\tilde{A}$ if the original submanifold $A \subset V$ is of positive codimension.

Theorem 3.3.1. *(Holonomic Approximation) Let $A \subset V$ be a polyhedron of positive codimension and*

$$F : \mathcal{O}p\, A \to X^{(r)}$$

a section. Then for arbitrarily small $\delta, \epsilon > 0$ there exists a δ-small (in C^0 sense) diffeotopy

$$h^\tau : V \to V,\ \tau \in [0,1],$$

and a holonomic section

$$\tilde{F} : \mathcal{O}p\, h^1(A) \to X^{(r)}$$

such that

$$dist\big(\tilde{F}(v), F|_{\mathcal{O}p\, h^1(A)}(v)\big) < \epsilon$$

for all $v \in \mathcal{O}p\, h^1(A)$

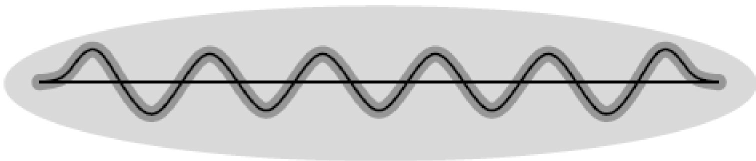

Figure 1: The sets $A, h^1(A), \mathcal{O}p\, A$ (gray) and $\mathcal{O}p\, h^1(A)$ (deep gray).

Remarks. *By the term polyhedron we mean that A is a subcomplex of a certain smooth triangulation of the manifold V. We assume that V is endowed with a Riemannian metric and the bundle $X^{(r)}$ is endowed with Euclidean structure in a neighborhood of the section $F(V) \subset X^{(r)}$. A diffeotopy $h^\tau : V \to V, \tau \in [0,1]$, is called δ-small if $h^0 = id_V$ and $dist(h^\tau(v), v) < \delta$ for all $v \in V$ and $\tau \in [0,1]$.*

We also have a parametric version of holonomic approximation theorem.

Theorem 3.3.2. *(Parametric holonomic approximation) Let $A \subset V$ be a polyhedron of positive codimension, $B \subset A$ be a subpolyhedron and*

$$F_z : \mathcal{O}p\, A \to X^{(r)}$$

a family of sections parametrized by a cube $I^m = [0,1]^m$. Suppose that the sections F_z are holonomic for all $z \in \partial I^m$ and holonomic over $\mathcal{O}p\, B \subset V$ for all $z \in I^m$. Then for arbitrarily small $\delta, \epsilon > 0$ there exists a family of δ-small (in C^0 sense) diffeotopies

$$h_z^\tau : V \to V,\ \tau \in [0,1],\ z \in I^m,$$

and a family of holonomic sections

$$\tilde{F}_z : \mathcal{O}p\, h_z^1(A) \to X^{(r)}, z \in I^m,$$

such that

1. *$h_z^\tau(v) = v$ and $\tilde{F}_z(v) = F_z(v)$ for $(z, v) \in (I^m \times \mathcal{O}p\, B) \cup (\partial I^m \times A)$;*

2. *$dist\big(\tilde{F}_z(v), F_z|_{\mathcal{O}p\, h_z^1(A)}(v)\big) < \epsilon$ for all (z, v) such that $v \in \mathcal{O}p\, h_z^1(A)$.*

Next we will look at two interesting applications of holomonic approximation theorem.

Theorem 3.3.3. *Let V be the annulus $\delta^2 \leq x_1^2 + x_2^2 \leq 4$ in $\mathbb{R}^2$. There exists a family of functions $f_t : V \to \mathbb{R}$, $t \in [0,1]$, such that $\mathrm{grad} f_t \neq 0$, $f_0 = -x_1^2 - x_2^2$ and $f_1 = x_1^2 + x_2^2$*

Proof. Note that $J^1(V, \mathbb{R}) = V \times \mathbb{R} \times \mathbb{R}^2$. We shall identify the last component with $\mathbb{C}$. Note that $\mathrm{grad} f_0 = -\mathrm{grad} f_1$. Let

$$F_t = \left((1-t)f_0 + tf_1, e^{i\pi t}\mathrm{grad} f_0\right).$$

F_t joins $F_0 = J^1_{f_0}$ with $F_1 = J^1_{f_1}$. For $t \neq 0, 1$, the section F_t is not holonomic. We can make F_t independent of t and thus holonomic for $t \in \mathcal{O}p\,\partial I$ by reparametrizing. Applying the parametric holonomic approximation theorem with $A = S^1 \subset V$, we can have a family of holonomic ϵ-approximation $\tilde{F}_t = J^1_{\tilde{f}_t} : U_t \to J^1(V, \mathbb{R})$ where U_t is a neighborhood of the perturbed circle $h^1_t(S^1)$.

Indeed we can choose $\tilde{F}_t$ and U_t such that $U_t = V$ and $\tilde{F}_t = F_t$ for $t \in \mathcal{O}p\,\partial I$. For sufficiently small ϵ the functions $\tilde{f}_t$ do not have critical points on U_t because $\mathrm{grad}\tilde{f}_t \approx e^{i\pi t}\mathrm{grad} f_0 \neq 0$ near S^1. Let $\{\phi^\tau_t : V \to V, \tau \in [0,1]\}_{t\in[0,1]}$ be a family of isotopies such that for each $t \in [0,1]$ the isotopy $\phi^\tau_t, \tau \in [0,1]$, shrinks V into the neighborhood U_t and $\phi^\tau_0 = \phi^\tau_1 = id_V$. Then the family $g_t = \tilde{f}_t \circ \phi^1_t$ consists of functions without critical points on V and interpolates between f_0 and f_1. □

By a similar application of the parametric holomomic approximation theorem, we can get the Smale's sphere eversion theorem, which we state below.

Recall that two immersions are called regularly homotopic if they can be connected by a family of immersions. Denote by V the thickened sphere

$$(1-\delta)^2 \leq x_1^2 + x_2^2 + x_3^2 \leq (1+\delta)^2$$

in $\mathbb{R}^3$. Let

$$\mathrm{inv} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\},\ \mathrm{inv}(x) = \frac{x}{||x||^2},$$

be the inversion,

$$r : \mathbb{R}^3 \to \mathbb{R}^3,\ r(x_1, x_2, x_3) = (x_1, x_2, -x_3),$$

the reflection and $i_V : V \to \mathbb{R}^3$ the inclusion.

Theorem (Smale's sphere eversion) 3.3.4. *The map*

$$r \circ inv \circ i_V : V \to \mathbb{R}^3,$$

which inverts V inside out, is regularly homotopic to the inclusion $i_V : V \to \mathbb{R}^3$.

For more applications of holonomic approximation theorem, we refer readers to chapter 4 of [1].

4 Differential relations and h-Principle

4.1 Differential relations

In this subsection, we will give the formal definition of differential relation and give some examples.

Definition 4.1.1. *A differential relation of order r imposed on sections $f : V \to X$ of a fibre bundle $X \to V$ is a subset $\mathcal{R}$ of the r-jet space $X^{(r)}$. A differential relation $\mathcal{R}$ is said to be open or closed if it is open or closed in $X^{(r)}$.*

Here are two examples of differential relations:

The immersion relation $\mathcal{R}_{\text{imm}} \subset J^1(V, W)$ over each point $(x, w) \in V \times W$ consists of monomorphisms $T_vV \to T_wW$. Locally, with respect to a chart

$$\phi : U_1 \times U_2 \to \mathbb{R}^n \times \mathbb{R}^q$$

the relation $\mathcal{R}_{\text{imm}}$ over each point $x = (v, w)$ consists of matrices $A \in M_{q\times n}$ of rank n.

The submersion relation $\mathcal{R}_{\text{sub}} \subset J^1(V, W)$ over each point $(x, w) \in V \times W$ consists of epimorphisms $T_vV \to T_wW$. Locally, with respect to a chart

$$\phi : U_1 \times U_2 \to \mathbb{R}^n \times \mathbb{R}^q$$

the relation $\mathcal{R}_{\text{sub}}$ over each point $x = (v, w)$ consists of matrices $A \in M_{q\times n}$ of rank q.

A section $F : V \to \mathcal{R}$ is called a formal solution to $\mathcal{R}$. We denote the space of such sections as $\operatorname{Sec}\mathcal{R}$.

A genuine solution of $\mathcal{R}$ is a holonomic section $F : V \to \mathcal{R}$. In other words, a genuine solution is a section $f : V \to X$ such that $J^r_f(V) \subset \mathcal{R}$.

Here are some examples:

1. A formal solution to a system of differential equations is a solution of the underlying system of "algebraic" equations obtained by substituting derivatives with new independent functions.

2. A formal solution of the immersion relation is a fibrewise injective bundle homomorphism $TV \to TW$.

3. A formal solution of the submersion relation is a fibrewise surjective bundle homomorphism $TV \to TW$.

4.2 Homotopy Principle

It seems that the existence of a formal solution cannot be sufficient for the genuine solvability of a differential relation $\mathcal{R}$. However, it was discovered in the second half of the twentieth century that there exist large and geometrically interesting classes of differential relations for which the solvability of the formal problem is sufficient for genuine solvability. This property was formalized as the following

Homotopy principle (h-principle). *We say that a differential relation $\mathcal{R}$ satisfies the h-principle if every formal solution of $\mathcal{R}$ is homotopic in $Sec\mathcal{R}$ to a genuine solution.*

There are different flavors of the h-principle:

Parametric h-principle. *We say that a differential relation $\mathcal{R}$ satisfies the parametric h-principle if every*

$$\varphi_0 : (D^k, S^{k-1}) \to (Sec\mathcal{R}, Hol\mathcal{R}), \quad k = 0, 1, \dots,$$

there exists a homotopy

$$\varphi_t : (D^k, S^{k-1}) \to (Sec\mathcal{R}, Hol\mathcal{R}), \quad t \in [0,1],$$

fixed on S^{k-1} *such that* $\varphi_1(D^k) \subset Hol\mathcal{R}$.

In other words, the inclusion $Hol\mathcal{R} \to Sec\mathcal{R}$ *is a weak homotopy equivalence.*

Local h-principle. *We say that a differential relation* $\mathcal{R}$ *satisfies the local h-principle near* $A \subset V$ *if for every formal solution* $F_0 : \mathcal{O}p\,A \to \mathcal{R}$ *there exists a homotopy* $F_t : \mathcal{O}p\,A \to \mathcal{R}$, $t \in [0,1]$ *such that* F_1 *is a genuine solution.*

We say that a differential relation $\mathcal{R}$ *satisfies the parametric local h-principle if every*

$$\varphi_0 : (D^k, S^{k-1}) \to (Sec_{\mathcal{O}p\,A}\mathcal{R}, Hol_{\mathcal{O}p\,A}\mathcal{R}), \quad k = 0, 1, \ldots,$$

there exists a homotopy

$$\varphi_t : (D^k, S^{k-1}) \to (Sec_{\mathcal{O}p\,A}\mathcal{R}, Hol_{\mathcal{O}p\,A}\mathcal{R}), \quad t \in [0,1],$$

fixed on S^{k-1} *such that* $\varphi_1(D^k) \subset Hol_{\mathcal{O}p\,A}\mathcal{R}$.

For $B \subset A \subset V$ we denote by $\mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R}$ the space of formal solutions $F : \mathcal{O}p\,A \to \mathcal{R}$ which are holonomic near B.

Relative h-principle. *We say that a differential relation* $\mathcal{R}$ *satisfies the relative h-principle near* $(A,B) \subset V$ *if for every formal solution* $F_0 \in Sec_{\mathcal{O}p\,(A,B)}\mathcal{R}$ *there exists a homotopy through formal solutions* $F_t \in Sec_{\mathcal{O}p\,(A,B)}\mathcal{R}$, $t \in [0,1]$ *such that* $F_t|_{\mathcal{O}p\,B} = F_0|_{\mathcal{O}p\,B}$ *for all* $t \in [0,1]$ *and* F_1 *is a genuine solution.*

We say that a differential relation $\mathcal{R}$ *satisfies the parametric relative h-principle near* (A,B) *if every*

$$\varphi_0 : (D^k, S^{k-1}) \to (Sec_{\mathcal{O}p\,(A,B)}\mathcal{R}, Hol_{\mathcal{O}p\,A}\mathcal{R}), \quad k = 0, 1, \ldots,$$

there exists a homotopy

$$\varphi_t : (D^k, S^{k-1}) \to (Sec_{\mathcal{O}p\,(A,B)}\mathcal{R}, Hol_{\mathcal{O}p\,A}\mathcal{R}), \quad t \in [0,1],$$

fixed on S^{k-1} *such that* $\varphi_1(D^k) \subset Hol_{\mathcal{O}p\,A}\mathcal{R}$ *and for every* $p \in D^k$ *the homotopy* $\varphi_t(p) : \mathcal{O}p(A) \to \mathcal{R}$ *is fixed near* B.

Next, we give an example of the h-principle. We will start with the following two definitions.

Definition 4.2.1. *A fibre bundle* $X \to V$ *is natural if for any diffeomorphism* $h : U \to U'$ *between two open subsets* $U, U' \subset V$ *there is given a fibre-preserving diffeomorphism* $h_* : p^{-1}(U) \to p^{-1}(U')$ *such that for any two diffeomorphisms* $h : U \to U'$, $h' : U' \to U''$ *we have* $(h' \circ h)_* = (h')_* \circ h_*$, *and for any point* $v \in V$ *the germ of* h_* *along* $p^{-1}(v)$ *depends only on the germ of* h *at* v.

Here are some examples:

1. The trivial bundle $V \times W \to V$ is natural.

2. The tangent bundle $TV \to V$ is natural. The cotangent bundle $T^*V \to V$ is natural.

3. If $X \to V$ is natural then $X^{(r)} \to V$ is natural.

Definition 4.2.2. *For natural fibre bundle $X \to V$, a differential relation $\mathcal{R} \subset X^{(r)}$ is called Diff V-invariant if the action*

$$s \mapsto h_*^{(r)}(s)$$

for any local diffeomorphism $h : U \to U'$ for $U, U' \subset V$ leaves $\mathcal{R}$ invariant.

Here are some examples:

1. The immersion relation $\mathcal{R}_{\text{imm}}$ is Diff V-invariant.

2. The submersion relation $\mathcal{R}_{\text{sub}}$ is Diff V-invariant.

With the help of holomonic approximation theorem, we can show the local h-principle for open Diff V-invariant relations:

Theorem 4.2.3. *(Local h-principle for open Diff V-invariant relations) Let $X \to V$ be a natural fibre bundle. Then any open Diff V-invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies all forms of the local h-principle near any polyhedron $A \subset V$.*

Moreover, we have the following result:

Theorem 4.2.4. *(Local h-principle implies global for open manifolds) Let V be an open manifold and $X \to V$ a natural fibre bundle. Let $\mathcal{R} \subset X^{(r)}$ be a Diff V-invariant differential relation. Then the parametric local h-principle implies the parametric global h-principle for $\mathcal{R}$.*

Combining the above two theorems:

Theorem 4.2.5. *(Gromov's h-principle for open Diff V-invariant relations over open manifolds) Let V be an open manifold and $X \to V$ a natural fibre bundle. Then any open Diff V-invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies the parametric h-principle. In particular, immersions, submersions, k-mersions satisfies the parametric h-principle as long as the underlying manifold V is open.*

5 Singularities of smooth maps

5.1 Thom-Boardman singularities

Consider manifolds V and W of dimension n and q respectively. Let $f : V \to W$ be a smooth map. For $k \geq 0$, a point $p \in V$ is said to be of type Σ^k, if $\operatorname{corank} d_p f := \min(n, q) - \operatorname{rank} d_p f = k$. We say that f is Σ^k-nonsingular if it has no points of type Σ^j for $j > k$.

Recall that we defined above $\Sigma^i \subset J^1(V, W)$ the space of 1-jets of maps of rank $\leq \min(n, q) - i$, and Σ^i has codimension $i(|q-n|+i)$. Since $J^1(V, W)$ fibres over $J^0(V, W) = V \times W$ with fiber $\operatorname{Hom}(T_x V, T_y W)$ over (x, y) and Σ^k is the associated bundle whose fibre over (x, y) is the set of homomorphisms $T_xV \to T_yW$ of corank k. Thus, $\Sigma^k(f) := (J^1_f)^{-1}(\Sigma^k)$ is the set of Σ^k-points of f. By Thom transversality theorem, for a generic map f, $\overline{\Sigma^k(f)}$ is stratified by smooth strata $\Sigma^j(f)$, $j > k$, of codimension $j(|n-q|+j)$.

If J^1_f is transverse to Σ^k then we can further consider the locus

$$\Sigma^{k,j} := \Sigma^j(f|_{\Sigma^k(f)})$$

. The condition that $p \in \Sigma^{k,j}$ can be expressed in terms of the 2-jets of f at p. The corresponding subset of $J^2(V, W)$ is denoted by $\Sigma^{k,j}$.

In general, for any multi-index $I = (i_1, \ldots, i_l)$ with

$$i_1 + \max(0, n-q) \geq i_2 \geq \cdots \geq i_l \geq 0$$

we can similarly define inductively

$$\Sigma^{i_1,\ldots,i_l}(f) := \Sigma^{i_l}(f|_{\Sigma^{i_1,\ldots i_{l-1}}})$$

and the corresponding

$$\Sigma^I = \Sigma^{i_1,\ldots,i_l} \subset J^l(V, W)$$

such that if $J^{l-1}(f)$ is transverse to $\Sigma^{i_1,\ldots i_{l-1}} \subset J^{l-1}(V, W)$ then

$$(J^l_f)^{-1}(\Sigma^{i_1,\ldots i_{l-1}}) = \Sigma^{i_l}(f|_{\Sigma^{i_1,\ldots i_{l-1}}})$$

Next we introduce Boardman's formula which counts the codimension of Σ^I. Given a multi-index $(k_1 \geq \cdots \geq k_l \geq 0)$ we denote by $\mu(k_1, \ldots, k_l)$ the number of sequences $j_1 \geq \cdots j_l \geq 0$ such that $j_s \leq k_s$ for all s and $j_1 > 0$.

Theorem 5.1.1. *(Boardman's formula) Let $I = (i_1, \ldots, i_l)$. Denote $\bar{i_1} := i_1 + max(0, n-q)$ and suppose that $\bar{i_1} \geq i_2 \geq \cdots \geq i_l \geq 0$. Then*

$$codim\ \Sigma^I = (\bar{i_1} + q - n)\mu(\bar{i_1}, \ldots, i_l) - (\bar{i_1} - i_2)\mu(i_2, \ldots, i_l) - \cdots - (i_{l-1} - i_l)\mu(i_l)$$

In the case of $n \geq q$, generic singularities of type $\Sigma^{1,0}$ are called folds, and generic singularities of type $\Sigma^{1,1,0}$ are called cusps. By "generic" we mean that the transversality of the corresponding jet section to the singularities Σ^1 and $\Sigma^{1,1}$ in the jet spaces.

See below for examples of folds and cusps in the case $n = q = 2$.

In the case $n \geq q$, a generic singular point of type $\Sigma^{1,1,1,0}$ is somtimes called a swallowtail, and the singularity $\Sigma^{1,0}$ for $n = 2, q = 3$ is called the Whitney's umbrella.

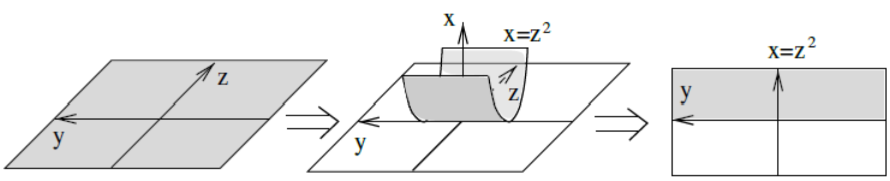

Figure 2: The fold $(y, z) \mapsto (y, z^2)$

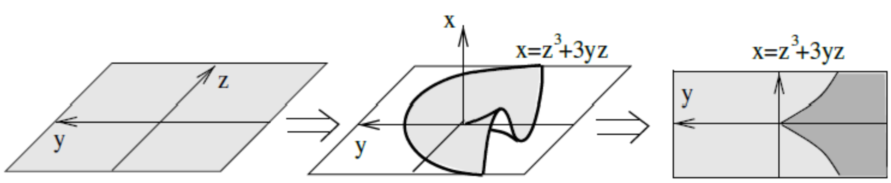

Figure 3: The fold $(y, z) \mapsto (y, z^3 + 3yz)$

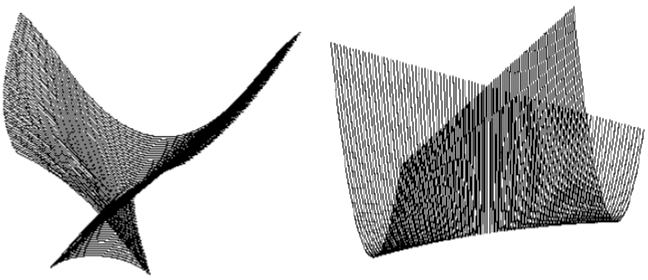

Figure 4: Swallowtail and Whitney umbrella

5.2 The Morse Lemma

In this section, we introduce Morse lemma, which classifies nondegenerate critical points of functions $f : v \to \mathbb{R}$. For a function $f : V \to \mathbb{R}$, $\Sigma^1(f)$ is the set of critical points of f and $\Sigma^2(f) = \phi$. Note that all critical points are nondegenerate if and only if $J^1_f : V \to J^1(V, \mathbb{R})$ is transverse to Σ^1.

Theorem 5.2.1. *(Morse Lemma) Suppose that a function $\varphi : \mathcal{O}p_{\mathbb{R}^n} 0 \to \mathbb{R}$ has a nondegenerate critical point at the origin with $\varphi(0) = 0$. Then there exists a diffeomrophism $h : \mathcal{O}p_{\mathbb{R}^n} 0 \to \mathcal{O}p_{\mathbb{R}^n} 0$ such that $h(0) = 0$ and*

$$\varphi \circ h(x) = Q_{n,s}(x) := -\sum_{i=1}^{s} x_i^2 + \sum_{j=s+1}^{n} x_j^2.$$

The number s is called the Morse index of the critical point of φ.

5.3 Fibred form for Σ^1-singularities

In this section, we give a further study the Σ^1 singularities. Let V, W be manifolds of dimension n and q respectively. We have the following fibred form for Σ^1 singularities:

Theorem 5.3.1. *(Fibred form for Σ^1) Consider a smooth map $f : V \to W$ and denote $d := min(n, q) - 1$. Then the germ of f at any point $p \in \Sigma^1(f)$ is equivalent to the germ at the origin of a fibred map*

$$h : \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}^d \times \mathbb{R}^{q-d}, \quad (y, x) \mapsto (y, h_y(x)),$$

where $h_y : \mathcal{O}p_{\mathbb{R}^n} 0 \to \mathbb{R}^{q-d}$ is a family of maps parametrized by $y \in \mathcal{O}p_{\mathbb{R}^d} 0$ and h_0 has a critical point at the origin.

The following is an useful application of the fibred form which gives a description of the submanifold $\Sigma^I(h)$ for $I = (\underbrace{1, \dots, 1}_{j}, 0)$.

Theorem 5.3.2. *(Equations for $\Sigma^{1,\dots,1}(h)$ in the case $n = q$) Let $n = q$ and function h has the above fibred form. Then the submanifold $\Sigma^I(h)$ for $I = (\underbrace{1, \dots, 1}_{j}, 0)$ is given by the equations*

$$\frac{\partial h_y(x)}{\partial x} = 0, \dots, \frac{\partial^j h_y(x)}{\partial x^j} = 0, \frac{\partial^{j+1} h_y(x)}{\partial x^{j+1}} \neq 0$$

Here are two examples:

1. Fix integers $r \geq 1$ and $s \in \{0, \ldots, n-1\}$, and for any $d \geq r-1$ consider the d-parametric family of functions $h_t^{n,r,s} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$, $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, given by the formula

$$h_t^{n,r,s}(x,z) = z^{r+1} + \sum_{j=1}^{r-1} t_j z^j + Q_{n-1,s}(x).$$

Consider the map

$$H^{n,r,s}(t,x,z) = (t, h_t^{n,r,s}(x,z)).$$

It is Σ^2-nonsingular and $\Sigma^{1_{r+1}}$-nonsingular, and its Σ^{1_j}-singularities for $j \leq r$ are given by the equations

$$\frac{\partial^i h_t^{n,r,s}}{\partial z^i} = 0,\ i = 1, \ldots, j, \quad \frac{h_t^{n,r,s}}{\partial x_m} = 0,\ m = 1, \ldots, n-1.$$

2. Suppose that $d \geq mr - 1$. Consider the d-parametric family of paths $\gamma_t^{m,r} : \mathbb{R} \to \mathbb{R}^m, t \in \mathbb{R}^d$, and the map $\Gamma^{d,m,r} : \mathbb{R}^{d+1} \to \mathbb{R}^d \times \mathbb{R}^m$ given by

$$\gamma^{m,r}(z) = \left(\sum_1^r t_j z^j, \sum_1^r t_{r+j} z^j, \ldots, \sum_1^r t_{(m-2)r+j} z^j, \sum_1^{r-1} t_{(m-1)r+j} z^j + z^{r+1} \right),$$

$$\Gamma^{d,n,r}(t,z) = (t, \gamma_t^{m,r}(z)).$$

$\Gamma^{d,m,r}$ is Σ^2-nonsingular, and Σ^{1_k}-nonsingular for $k > r$. Its Σ^{1_j}-singularities for $j \leq r$ are given by the equations

$$\frac{\partial^i \gamma_t^{m,r}}{\partial z^i} = 0, \quad i = 1, \ldots, j.$$

Note that for $d = 1$, $m = 2$ and $r = 1$, the map $\Gamma^{1,2,1} : \mathbb{R}^2 \to \mathbb{R}^3$ has the form $(t,z) \mapsto (t, tz, z^2)$. This is the Whitney umbrella. See Figure 4.

5.4 A_n-singularities of functions

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ of smooth function at the origin and suppose that 0 is a critical point of f. The corank of the critical point 0 of f is defined to be the corank of $d^2 f$ at 0.

Definition 5.4.1. *We say a function has corank* 1 *if it only has corank* 0 *(regular) points and corank* 1 *critical points.*

Here is an example of corank 1 function. Let $f_m(x_1, \ldots, x_n) = x_1^m + Q(x_2, \ldots, x_n)$ where Q is a non-degenerate quadratic form.

Definition 5.4.2. *A singularity of a function is said to be an A_{m+1}-singularity if it is equivalent to the singularity at* 0 *of some f_m of the above form.*

5.5 Morin's normal forms

The following theorem of Morin classified the generic family of B-nonsingular (with corank of the second differential at critical points ≤ 1) functions locally: they are equivalent to one of the families $h_t^{n,r,s}$ defined above.

Theorem 5.5.1. *(B. Morin) Let $f_t : \mathcal{O}p_{\mathbb{R}^n}0$, $t \in \mathcal{O}p_{\mathbb{R}^d}0$, be a family of functions such that f_0 has an A_r-singularity ($r \le d+1$) at the origin, and the family $J^{r+1}_{f_t}$ is transverse $A_r \subset J^{r+1}(\mathbb{R}^n, \mathbb{R})$. Then the family f_t is equivalent to the family $h^{n,r,s}_t$ defined above for some integer $s \in \{0, 1, \dots, n\}$.*

With the help of the following lemma

Lemma 5.5.2. *suppose that a fibred map $h : \mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1} \to \mathbb{R}^{q-1} \times \mathbb{R}$, $(y, x) \mapsto (y, h_y(x))$, has a transverse Σ^{1_r}-singularity at the origin ($1_r = (\underbrace{1, \dots, 1}_{j})$). Then the function $h_0 : \mathbb{R}^{n-q+1} \to \mathbb{R}$ has an A_r-singularity at the origin.*

we get that following normal form:

Theorem 5.5.3. *(B. Morin) Let $f : \mathcal{O}p_{\mathbb{R}^n}0 \to \mathbb{R}^1$ be the germ of a map with transverse Σ^{1_r}-singularity at the origin, $r \ge 1$*

1. *If $n \ge q$, then f is equivalent to the germ at 0 of the map $H^{n-q+1,r,s}$ defined above for some integer $s \in \{0, \dots, n-q\}$.*

2. *If $n \le q$, then f is equivalent to the germ at 0 of the map $\Gamma^{n-1,q-n+1,r}$ defined above.*

In the case $n = q = 2$, we get the theorem of H. Whitney:

Theorem 5.5.4. *(H. Whitney) A generic map $f : V \to W$ between surfaces has only fold and cusp singularities. Its germ is equivalent to $(t, z) \mapsto (t, z^2)$ at a fold point, and to $(t, z) \mapsto (t, z^3 + tz)$ at a cusp point. See Figure 2 and Figure 3.*

5.6 Igusa's theorem

Let N be an n-dimensional, compact, smooth manifold, and $g : N \to \mathbb{R}$ a function without critical points near ∂N. Let $\mathcal{H}(N, g)$ be the space of generalized Morse functions, i.e. functions with corank 1 with at worst A_2 singularities, on N that coincide with g near the boundary of N.

Note that the condition of a function not having higher singularities is a differential relation in $J^3(N, \mathbb{R})$. We denote this differential relation by $\mathcal{R}_{\mathcal{H}}$.

We denote by $\mathrm{Sec}\,\mathcal{R}_{\mathcal{H}}$ the space of smooth sections $N \to \mathcal{R}_{\mathcal{H}}$. Now, we are ready to state Igusa's theorem.

Theorem 5.6.1. *(Igusa) Let N be a compact, n-dimensional smooth manifold as before, and let $g : N \to \mathbb{R}$ be a smooth function without critical points near ∂N. Then the map $J^3 : \mathcal{H}(N, g) \to Sec\,\mathcal{R}_{\mathcal{H}}$ is n-connected.*

A stronger result was given by Eliashberg and Misachev:

Theorem 5.6.2. *Let N be a compact, n-dimensional smooth manifold as before, and let $g : N \to \mathbb{R}$ be a smooth function without critical points near ∂N. Then the map $J^3 : \mathcal{H}(N, g) \to Sec\,\mathcal{R}_{\mathcal{H}}$ a weak homotopy equivalence. In particular, the differntial relation $\mathcal{R}_{\mathcal{H}}$ satisfies the parametric h-principle.*

6 Functions of Several Complex Variables

In this section, we introduce some background in complex analysis we need.

6.1 J-convexity and plurisubharmonicity

In this subsection we give the definition of almost complex manifold and complex manifold and the notion of J-convexity. We start by recalling that a complex vector space is a real vector spcae V of dimension $2n$ with an endomorphism J satisfying $J^2 = -Id$. For $a+bi \in \mathbb{C}$ and $v \in V$, $(a+bi)v$ is defined to be $av + bJv$.

Definition 6.1.1. *An almost complex structure on a smooth maifold V of real dimension $2n$ is an endomorphism $J : TV \to TV$ satisfying $J^2 = -Id$. The pair (V, J) is called an almost complex manifold. Such pair is called a complex manifold if the almost complex structure is integrable, i.e., J is induced by complex coordinates on V.*

Let (V, J) be an almost complex manifold. For a smooth function $\phi : V \to \mathbb{R}$, we associate the 2-form

$$\omega_\phi := -dd^{\mathbb{C}}\phi,$$

where the differential operator $d^{\mathbb{C}}$ is defined by

$$d^{\mathbb{C}}\phi(X) = d\phi(JX)$$

for vector field X.

Definition 6.1.2. *A function $\phi : V \to \mathbb{R}$ on an almost complex manifold is called (strictly) J-convex if $\omega_\phi(X, JX) > 0$ for all nonzero tangent vectors X.*

Next, we introduce the notion of plurisubharmonicity. It turns out that a C^2 function being J-convex is equivalent to being plurisubharmonic.

Recall that a C^2 function $U \to \mathbb{R}$ on an open domain $U \subset \mathbb{C}$ is subharmonic if

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} > 0$$

.

We can extend this definition to continuous functions.

Definition 6.1.3. *A continuous function $\phi : U \to \mathbb{R}$ is called subharmonic if it satisfies $\Delta\phi \geq m$ where $m : U \to \mathbb{R}$ is a positive continuous function and the inequality is understood in the following distributional sense:*

$$\int_U \phi \, \Delta\delta \, dx \, dy \geq \int_U m \, \delta \, dx \, dy$$

for any nonnegative smooth function $\delta : U \to \mathbb{R}$ with compact support. The function

$$m_\phi := sup\{m | \Delta\phi \geq m\}$$

is called the modulus of subharmonicity of ϕ.

Remark. *Note that subharmonicity is invariant under biholomorphic change of coordinates, so it can be defined for continuous functions on Riemann surfaces. However, modulus of subharmonicity is not invatiant under biholomorphic change of coordinates.*

Next, we give a useful criterion for subharmonicity of continuous functions.

Lemma 6.1.4. *A continuous function $\phi : U \to \mathbb{R}$ on a domain $U \subset \mathbb{C}$ satisfies $\Delta\phi \geq m$ for a positive continuous function $m : U \to \mathbb{R}$ if and only if*

$$\phi(z) + \frac{m(z)r^2}{4} \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + e^{i\theta})d\theta$$

for all $z \in U$ and sufficiently small $r > 0$.

Proof. We follow the proof in [2]. Fix $z \in U$. Define

$$\psi(w) := \phi(w) = \frac{1}{4}m(z)|w - z|^2.$$

For $r > 0$ sufficiently small, the inequality in the lemma is equivalent to

$$\psi(z) = \phi(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + re^{i\theta})d\theta - \frac{m(z)r^2}{4} = \frac{1}{2\pi}\int_0^{2\pi} \psi(z + re^{i\theta})d\theta,$$

and thus to

$$\psi(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi(z + re^{i\theta})d\theta.$$

Hence, the inequality is equivalent to $\Delta\psi(z) \geq 0$ in the distributional sense, and thus to $\phi(z) \geq \frac{1}{4}m(z)\Delta_w|w - z|^2 = m(z)$. □

For an almost complex manifold (V, J), a complex curve in V is a 1-dimensional complex submanifold of (V, J). Note that the restriction of the almost complex structure J to a complex curve is always integrable.

Definition 6.1.5. *A continuous function $f : V \to \mathbb{R}$ is called (strictly) plurisubharmonic if its restriction to every complex curve is subharmonic.*

The next lemma says that for C^2 functions, J-convexity is equivalent to plurisubharmonicity.

Lemma 6.1.6. *A C^2 function ϕ on an almost complex manifold (V,J) is J-convex if and only if its restriction to every complex curve is subharmonic.*

Proof. By definition, ϕ is J-convex if and only if $-dd^{\mathbb{C}}\phi(X, JX) > 0$ for all $0 \neq X \in T_xV$, $x \in V$. Now for every such $X \neq 0$ there exists a complex curve $C \subset V$ passing through x with $T_xC = \operatorname{span}_{\mathbb{R}}\{X, JX\}$. Note that $-dd^{\mathbb{C}}\phi(X, JX) > 0$ precisely if $\phi|_C$ is subharmonic in x. This is because

$$-dd^{\mathbb{C}}\delta = 2i\frac{\partial^2\delta}{\partial z\partial\bar{z}}dz \wedge d\bar{z} = \Delta_z\delta dx \wedge dy.$$

□

7 A Plurisubharmonic Analogue to Igusa's Theorem

7.1 Falcone's Theorem on a Plurisubharmonic Analogue to Igusa's Theorem

Finally, we are ready to state Paul Falcone's result on plurisubharmonic analogue to Igusa's theorem.

Theorem 7.1.1. *Let $f_t : M \to \mathbb{R}$, $t \in B$ be a family of plurisubharmonic functions with corank 1 singularities on a complex manifold M. Let λ be the kernel line field which is defined over the locus of $A_{\geq 2}$-singularities. Suppose for $t \in \partial B$ the functions f_t have only generalized Morse (i.e. A_1, A_2) singularities, and hence λ can be canonically trivialized. Supoose that this trivialization extends to a trivialization of λ over the whole $A_{\geq 2}$-locus. Then the family f_t can be C^1 perturbed relative to ∂B in the class of plurisubharmonic function to a generalized Morse family. If f_t is already generalized Morse on a neighborhood of a closed subset $A \subset M$ then the functions can be left unchaged on a neighborhood of A.*

In particular, as a corollary:

Corollary 7.1.2. *Igusa's theorem hold for 2-parametric families of plurisubharmonic functions assuming the above hypotheses.*

7.2 Future Work

These notes were made while reading [1], [2], and [3] during SURIM under the guidance of Prof. Eliashberg and was a preparation for my senior honor thesis which will be devoted to the analogues of Igusa's theorem in the context of plurisubharmonic functions. The goal of my senior thesis is to further generalize Falcone's result and attempt to prove Igusa's theorem in the class of plurisubharmonic functions in full generality. Falcone's result was obtained by surgeries of singularities. I will attempt to apply a method called removal of singularities introduced by Eliashberg and Gromov (in [4]).

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