Holonomic Approximation Theorem

We are interested in the following question: Given an r-jet section and an arbitrary small neighborhood of the image of this section in the jet space, can one find a holonomic section in this neighborhood?

The problem of finding a holonomic approximation of a section of the r-jet space near a submanifold A is usually unsolvable. However, we can always find a holonomic approximation of a section $F : V \to X^{(r)}$ near a slightly deformed submanifold $A$ if the submanifold $A \subset V$ is of positive codimension.

**Theorem 1. (Holonomic Approximation)** Let $A \subset V$ be a polyhedron of positive codimension and

$F : Op_{B} A \to X^{(r)}$

a section. Then for arbitrarily small $\delta, \varepsilon > 0$ there exists a $\delta$-small (in $C^{1}$ sense) diffeotopy

$h^{B} : V \to V, \tau \in [0, 1]$

and a holonomic section

$\tilde{F} : Op_{h^{B}}(A) \to X^{(r)}$

such that

$$\text{dist} (\tilde{F}(v), F_{|Op_{h^{B}}(A)}(v)) < \varepsilon$$

for all $v \in Op_{h^{B}}(A)$.

**Remarks.** By the term polyhedron we mean that $A$ is a subcomplex of a certain smooth triangulation of the manifold $V$. We assume that $V$ is endowed with a Riemannian metric and the bundle $X^{(r)}$ is endowed with Euclidean structure in a neighborhood of the section $F(V) \subset X^{(r)}$. A diffeotopy $h^{B} : V \to V, \tau \in [0, 1]$, is called $\delta$-small if $h^{B} = id_{V}$ and $\text{dist}(h^{B}(v), v) < \delta$ for all $v \in V$ and $\tau \in [0, 1]$.

There is also a parametric version of holonomic approximation theorem:

**Theorem 2. (Parametric holonomic approximation)** Let $A \subset V$ be a polyhedron of positive codimension, $B \subset A$ be a subpolyhedron and

$F : Op_{B} A \to X^{(r)}$

a family of sections parametrized by a cube $I^{m} \times [0, 1]$. Suppose that the sections $F_{t}$ are holonomic for all $t \in \partial I^{m}$ and holonomic over $Op_{B} C \subset V$ for all $z \in I^{m}$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exists a family of $\delta$-small (in $C^{1}$ sense) diffeotopies

$h_{t}^{B} : V \to V, \tau \in [0, 1], z \in I^{m}$

and a family of holonomic sections

$\tilde{F}_{t} : Op_{h_{t}^{B}}(A) \to X^{(r)}, z \in I^{m}$,

such that:

1. $h_{t}^{B}(v) = v$ and $\tilde{F}_{t}(v) = F_{t}(v)$ for $(z, v) \in (I^{m} \times Op_{B}) \cup (\partial I^{m} \times A)$,
2. $\text{dist} (\tilde{F}_{t}(v), F_{|Op_{h_{t}^{B}}(A)}(v)) < \varepsilon$ for all $(z, v)$ such that $v \in Op_{h_{t}^{B}}(A)$.

References


Acknowledgements

I would like to thank my faculty mentor, Professor Eliashberg, who introduced me to this intriguing subject. I am grateful of his guidance and help throughout the summer. I would also like to thank SURIM program for funding and support for the research and Dr. Lenn Auerian, the director of SURIM program, for organizing this amazing undergraduate research program.