Holonomic Approximation Theorem and Applications

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Introduction

- Holonomic approximation is a powerful tool for proving homotopy principle, which asserts that given a formal solution of a partial differential relation, it can be homotoped to a genuine solution.
- We present two applications of holonomic approximation theorem which are special cases of Gromov's homotopy principle for open Diff V-invariant differential relations over open manifolds.

The Language of Jets

We introduce the language of jets:

Let $d_r = d(n,r)$ be the number of all partial derivatives D^α of order r of a function $\mathbb{R}^n \to \mathbb{R}$. Let $N_r = N(n,r) = 1 + d_1 + \cdots + d_r$. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^q$ and a point $x \in \mathbb{R}^n$,

$$J_f^r(x) = \left(x, f(x), f'(x), \ldots, f^{(r)}(x)\right)$$

is called the r-jet of f at x. Equivalently, we can regard f as the section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$. The space $x \times \mathbb{R}^{qN_r}$ can be viewed as a space of all a priori possible values of the jets of the maps $f : \mathbb{R}^n \to \mathbb{R}^q$ at the point $x \in \mathbb{R}^n$. In this context the space

$$\mathbb{R}^n \times \mathbb{R}^{qN_r}$$

is called the space of r-jets of sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ and denoted by $J^r(\mathbb{R}^n, \mathbb{R}^q)$

Given a section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, the section

$$J_f^r : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q),\ x \mapsto J_f^r(x),$$

of the trivial bundle

$$p^r : J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \to \mathbb{R}^n$$

is called the r-jet extension of f.

In order to define the r-jet space for an arbitrary fibre bundle, we need the following invariant definition of jets:

Consider a smooth fibre bundle $p : X \to V$. Fix $v \in V$. Two local sections $f : \mathcal{O}p\, v \to X$ and $g : \mathcal{O}p\, v \to X$ are called r-tangent at the point v if $f(v) = g(v)$ and

$$J^r_{\phi_* f}(\phi(v)) = J^r_{\phi_* g}(\phi(v))$$

for a local trivialization $\phi : U \to R^n \times R^q$ of X in a neighborhood U of the point $x = f(v) = g(v)$, where $\phi_* f$ and $\phi_* g$ are images of the sections f and g. The r-tangency class of a section $f : \mathcal{O}p\, v \to X$ at a point $v \in V$ is called the r-jet of f at v and denoted by $J_f^r(v)$.

For smooth fibre bundle $p : X \to V$, we define its r-jet space

$$X^{(r)} = \{(v, f) : f : \mathcal{O}p\, v \to X\}/\sim$$

where $\sim$ is the equivalence relation defined by r-tangency.

Define $p_0^r : X^{(r)} \to X$ by $p_0^r([v,f]) = f(v)$, and $p^r = p \circ p_0^r : X^{(r)} \to V$.

Remark. For local trivializations $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^q$, the extensions

$$\varphi^r : (p_0^r)^{-1}(U) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

sending the r-tangency classes of local sections of X to the r-tangency classes of their images in $J^r(\mathbb{R}^n, \mathbb{R}^q)$ define a natural smooth structure on $X^{(r)}$ such that $p^r : X^{(r)} \to V$ becomes a smooth fibre bundle.

Remark. Since the r-tangency of two sections implies their s-tangency for $0 \le s < r$, we get projections $p_s^r : X^{(r)} \to X^{(s)}$. Thus, we have a chain a projections:

$$X = X^{(0)} \leftarrow X^{(1)} \leftarrow X^{(2)} \leftarrow \cdots \leftarrow X^{(r)} \leftarrow \cdots$$

Holonomic Approximation Theorem

We are interested in the following question: Given an r-jet section and an arbitrary small neighborhood of the image of this section in the jet space, can one find a holonomic section in this neighborhood?

The problem of finding a holonomic approximation of a section of the r-jet space near a submanifold A is usually unsolvable. However, we can always find a holonomic approximation of a section $F : V \to X^{(r)}$ near a slightly deformed submanifold $\tilde{A}$ if the original submanifold $A \subset V$ is of positive codimension.

Theorem 1. (Holonomic Approximation) Let $A \subset V$ be a polyhedron of positive codimension and

$$F : \mathcal{O}p\, A \to X^{(r)}$$

a section. Then for arbitrarily small $\delta, \epsilon > 0$ there exists a δ-small (in C^0 sense) diffeotopy

$$h^\tau : V \to V,\ \tau \in [0,1],$$

and a holonomic section

$$\tilde{F} : \mathcal{O}p\, h^1(A) \to X^{(r)}$$

such that

$$\text{dist}\left(\tilde{F}(v), F|_{\mathcal{O}p\, h^1(A)}(v)\right) < \epsilon$$

for all $v \in \mathcal{O}p\, h^1(A)$

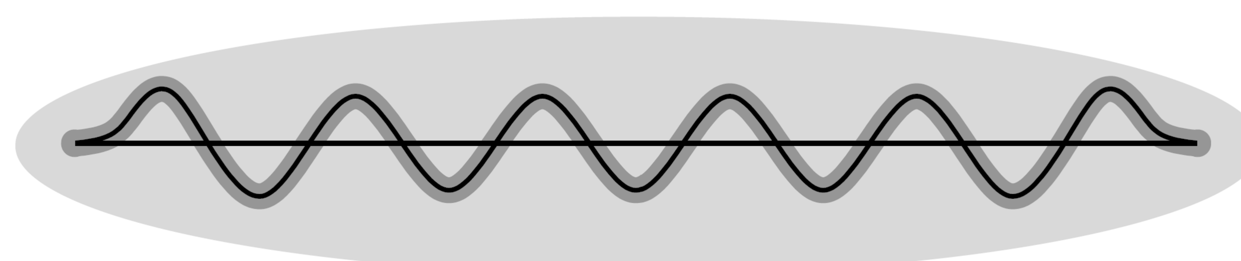

Figure 1: The sets $A, h^1(A), \mathcal{O}p\, A$ (gray) and $\mathcal{O}p\, h^1(A)$ (deep gray).

Remarks. By the term polyhedron we mean that A is a subcomplex of a certain smooth triangulation of the manifold V. We assume that V is endowed with a Riemannian metric and the bundle $X^{(r)}$ is endowed with Euclidean structure in a neighborhood of the section $F(V) \subset X^{(r)}$. A diffeotopy $h^\tau : V \to V, \tau \in [0,1]$, is called δ-small if $h^0 = id_V$ and $\text{dist}(h^\tau(v), v) < \delta$ for all $v \in V$ and $\tau \in [0,1]$.

There is also a parametric version of holonomic approximation theorem:

Theorem 2. (Parametric holonomic approximation) Let $A \subset V$ be a polyhedron of positive codimension, $B \subset A$ be a subpolyhedron and

$$F_z : \mathcal{O}p\, A \to X^{(r)}$$

a family of sections parametrized by a cube $I^m = [0,1]^m$. Suppose that the sections F_z are holonomic for all $z \in \partial I^m$ and holonomic over $\mathcal{O}p\, B \subset V$ for all $z \in I^m$. Then for arbitrarily small $\delta, \epsilon > 0$ there exists a family of δ-small (in C^0 sense) diffeotopies

$$h_z^\tau : V \to V,\ \tau \in [0,1],\ z \in I^m,$$

and a family of holonomic sections

$$\tilde{F}_z : \mathcal{O}p\, h_z^1(A) \to X^{(r)}, z \in I^m,$$

such that

1. $h_z^\tau(v) = v$ and $\tilde{F}_z(v) = F_z(v)$ for $(z,v) \in (I^m \times \mathcal{O}p\, B) \cup (\partial I^m \times A)$;
2. $\text{dist}\left(\tilde{F}_z(v), F_z|_{\mathcal{O}p\, h_z^1(A)}(v)\right) < \epsilon$ for all (z,v) such that $v \in \mathcal{O}p\, h_z^1(A)$.

Applications of Holonomic Approximation Theorem

We look at two interesting applications of holonomic approximation theorem:

Theorem 3. Let V be the annulus $\delta^2 \le x_1^2 + x_2^2 \le 4$ in $\mathbb{R}^2$. There exists a family of functions $f_t : V \to \mathbb{R}$, $t \in [0,1]$, such that grad$f_t \neq 0$, $f_0 = -x_1^2 - x_2^2$ and $f_1 = x_1^2 + x_2^2$

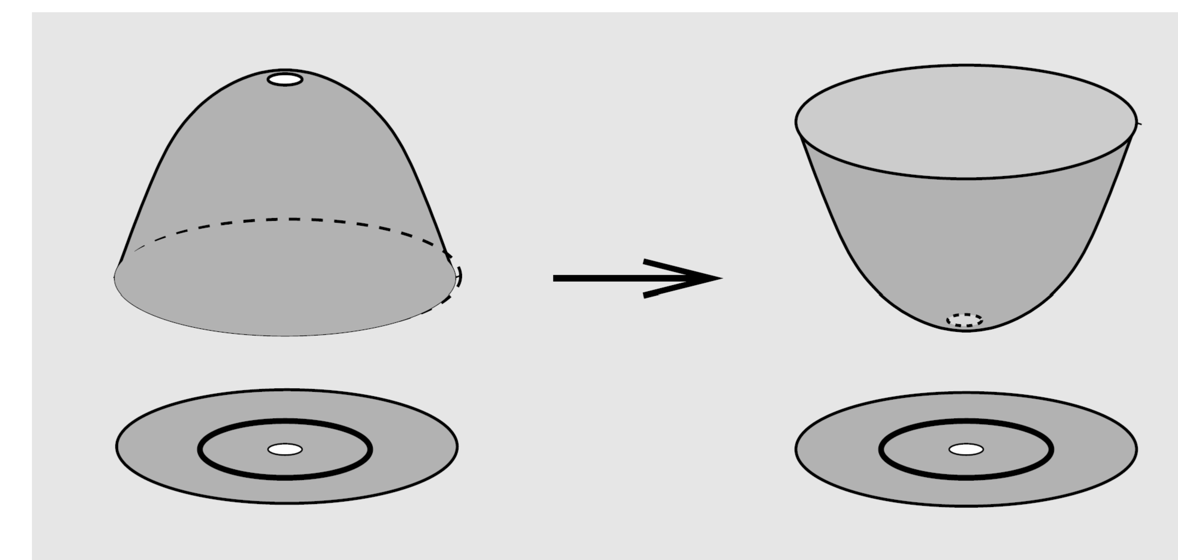

Figure 2: The functions f_0 and f_1

Recall that two immersions are called regularly homotopic if they can be connected by a family of immersions. Denote by V the thickened sphere

$$(1-\delta)^2 \le x_1^2 + x_2^2 + x_3^2 \le (1+\delta)^2$$

in $\mathbb{R}^3$. Let

$$\text{inv} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3 - \{0\},\ \text{inv}(x) = \frac{x}{||x||^2},$$

be the inversion,

$$r : \mathbb{R}^3 \to \mathbb{R}^3,\ r(x_1, x_2, x_3) = (x_1, x_2, -x_3),$$

the reflection and $i_V : V \to \mathbb{R}^3$ the inclusion.

Theorem (Smale's sphere eversion) 4. The map

$$r \circ \text{inv} \circ i_V : V \to \mathbb{R}^3,$$

which inverts V inside out, is regularly homotopic to the inclusion $i_V : V \to \mathbb{R}^3$.

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