Closed Geodesics on Kummer K3 Surfaces

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Abstract

K3 surfaces are compact, connected complex manifolds of complex dimension 2 that possess compatible complex and Riemannian structures, i.e. are Kähler, and along with complex tori completely classify all 4-dimensional Calabi-Yau manifolds, structures central to supersymmetric theory. We examine the geometry of a particular construction of a K3 surface by Kummer as a broader exploration of the geometry of four-dimensional Einstein manifolds (i.e. manifolds whose Ricci curvature is a scalar multiple of the Riemannian metric), noting results on the existence and stability of geodesics on Kummer K3 Surfaces. Finding closed geodesics on K3 surfaces, along with their stability, helps answer multiple open conjectures in physics and furthers geometric understanding of 4-D Einstein manifolds, whose classification remains an open problem.

1 Review of Tensor Algebra on Manifolds

We review some tensor algebra necessary for defining central concepts in Riemannian geometry, including a Riemannian metric and curvature.

Definition 1.1. Let V be a finite-dimensional vector field over the field k, and let V^* be its dual space. A (r,s) tensor τ is a map, k-linear in each of its arguments,

$$\tau : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to k.$$

We call a $(r,0)$-tensor covariant and a $(0,s)$-tensor contravariant. Note that a $(0,1)$ tensor is an element of V^*, and a $(1,0)$ tensor is an element of V (through the canonical isomorphism $V^{**} \cong V$).

The space of (r,s) tensors on V, denoted $V^{(r,s)}$, is a vector space. Choose a basis $e_1,\ldots,e_n$ of V and a basis $f_1,\ldots,f_n$ of V^*. Then we can write an (r,s) tensor τ in the basis $\{f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s} : 1 \leqslant i_k, j_k \leqslant n\}$.

We also have a isomorphism between $(1,1)$-tensors and linear endomorphisms of V, given by

$$\mathrm{End}(V) \to V^{(1,1)},\ A \mapsto \big((v,\ell) \mapsto \ell(Av)\big)$$
$$V^{(1,1)} \to \mathrm{End}(V),\ \sum_{i,j} a_{ij} f_i \otimes e_j \mapsto (v \mapsto \sum_{i,j} a_{ij} f_i(v) e_j)$$

.

We can consequently define the space of (smooth) tensors on a manifold M via the language of vector bundles.

Definition 1.2. We denote by $T^{(r,s)}M$ the (r,s)-tensor bundle on a smooth manifold M, which is a vector bundle on M with a fiber at a point p, denoted $T_p^{(r,s)}M$, given by $(T_pM)^{(r,s)}$ the (r,s) tensor space on T_pM.

Definition 1.3. A (C^k) (r,s)-tensor on the manifold M is a (C^k) section $\tau : M \to T^{(r,s)}M$.

That is, every point $p \in M$ $\tau(p) = \tau_p$ is a (r,s)-tensor on M. We denote $\mathcal{T}^{(r,s)}M$ the space of C^∞ tensors on M. $\mathcal{T}^{(1,0)}M$ is therefore the space of C^∞ vector fields on M.

Unless stated otherwise, we assume that all (r,s)-tensors are C^∞. We can naturally represent an (r,s)-tensor τ as a function:

$$\underbrace{\mathcal{T}^{(0,1)}M \times \cdots \times \mathcal{T}^{(0,1)}M}_{r} \times \underbrace{\mathcal{T}^{(1,0)}M \times \cdots \times \mathcal{T}^{(1,0)}M}_{s} \to C^\infty(M)$$
$$(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s) \mapsto \Big(p \mapsto \tau_p\big((\alpha_1)_p, \ldots, (\alpha_r)_p, (X_1)_p, \ldots, (X_s)_p\big)\Big)$$

We denote a C^∞ function in the image of this map as $\tau(\alpha_1, \ldots, \alpha_r, X_1, \ldots, X_s)$.

2 Fundamental Definitions of Riemannian Geometry

By equipping a smooth manifold with a Riemannian metric g, we introduce the notion of lengths, as well as a precise means to take derivatives and curvature. We define these briefly here.

Definition 2.1. A Riemannian manifold (M, g) is a pair consisting of a compact C^∞ manifold M and C^∞ $(0,2)$-tensor g, called the metric, such that

1. g is symmetric, i.e. $g(X,Y) = g(Y,X)$ for all vector fields X and Y.
2. g is positive definite, i.e. $g(X,X) > 0$ for every nonzero vector field X.

In other words, Riemannian metric g is a smooth choice of an inner product at each $p \in M$. Therefore, specifying g specifies an isomorphism between

$T^{(1,0)}M$ and $T^{(0,1)}M$ (through the usual isomorphism of a vector space and its dual via an inner product).

This extends, however, to an isomorphism between $T^{(r+1,s)}M$ and $T^{(r,s+1)}M$ for any (r,s); we call the operation $T^{(r+1,s)}M \to T^{(r,s+1)}M$ lowering an index (denoted by $\flat$) and $T^{(r,s+1)}M \to T^{(r+1,s)}M$ by raising an index (denoted by $\sharp$). In particular, for a (r,s)-tensor τ with i-th argument a vector, we define $\tau^\flat$ by

$$\tau^\flat(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{r+s}) = \tau(\alpha_1, \ldots, {\alpha_i}^\flat, \ldots, \alpha_{r+s})$$

where the flat isomorphism for a vector is already known. Similarly for a (r,s)-tensor τ with i-th argument a covector, we define $\tau^\sharp$ by

$$\tau^\sharp(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{r+s}) = \tau(\alpha_1, \ldots, {\alpha_i}^\sharp, \ldots, \alpha_{r+s}).$$

Note that if $r \geqslant 2$ (resp. $s \geqslant 2$), there exist multiple ways to lower an index (resp. raise an index) for a (r,s)-tensor τ. In particular, raising or lowering an index is particular to a specific index, though we do not usually denote this in our notation. A Riemannian metric g naturally produces a volume form μ_g given in local coordinates by

$$\mu_g = \sqrt{|\det(g)|}\, dx^1 \wedge \cdots \wedge dx^n$$

where we take the matrix representation of the metric g in a coordinate system $x_1, \ldots, x_n$.

With a metric, we can also define a compatible connection that determines the notion of a covariant derivative with respect to a metric; namely

Definition 2.2. A connection is a bilinear map $\nabla : \mathcal{T}^{(1,0)}M \times \mathcal{T}^{(1,0)}M \to \mathcal{T}^{(1,0)}M$, written in its arguments as $\nabla_X Y$ satisfying, for all $f \in C^\infty(M)$ and $X, Y \in \mathcal{T}^{(1,0)}M$,

1. $\nabla_{fX} Y = f \nabla_X Y$
2. $\nabla_X fY = X(f)Y + f\nabla_X Y$

Defining a connection on the tangent bundle likewise induces a map $\nabla : \mathcal{T}^{(1,0)}M \times \mathcal{T}^{(0,1)}M \to \mathcal{T}^{(0,1)}M$ such that

$$\nabla_X \alpha(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y).$$

From there, we can all extend ∇ to all tensors by requiring that

$$\begin{aligned} \nabla_X(\alpha \otimes \beta) &= \nabla_X \alpha \otimes \beta + \alpha \otimes \nabla_X \beta \\ \nabla_X(\alpha + \beta) &= \nabla_X \alpha + \nabla_X \beta. \end{aligned}$$

Thus, for any (r,s) $\nabla : \mathcal{T}^{(1,0)}M \times \mathcal{T}^{(r,s)}M \to \mathcal{T}^{(r,s)}M$. This defines a covariant derivative on any tensor (r,s)-tensor τ, denoted $\nabla\tau$, where $\nabla\tau$ is a $(r+1,s)$-tensor such that

$$\nabla\tau(X, \alpha_1, \ldots, \alpha_{r+s}) = (\nabla_X \tau)(\alpha_1, \ldots, \alpha_{r+s}).$$

Definition 2.3. The Levi-Civita connection on a Riemannian manifold (M, g) is the connection ∇ such that

1. ∇ is metric-compatible, i.e. $\nabla g = 0$
2. ∇ is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[X, Y]$ is the Lie bracket, for all vector fields X and Y.

Such a connection is unique to any g.

We only consider the Levi-Civita connection on a Riemannian manifold and hence denote it ∇. From the covariant derivative we develop a notion of parallel transport of any tensor α_p at a point p by solving the differential equation $\nabla \tau = 0$ with initial condition $\tau_p = \alpha_p$. In particular, parallel transport of a vector field integrates to a geodesic on (M, g):

Definition 2.4. A smooth map $\gamma : [a, b] \to M$ is a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$, where $\dot\gamma$ is (the extension to all of M of) the tangent vector field to γ.

Definition 2.5. A *geodesic* $\gamma : \mathbb{R} \to M$ is closed if γ is periodic, i.e. induces a smooth map $\gamma : S^1 \to M$.

Lastly, we define curvature on a manifold.

Definition 2.6. The Riemann curvature tensor R is a map $R : \mathcal{T}^{(1,0)}M \times \mathcal{T}^{(1,0)}M \to \mathrm{End}(\mathcal{T}^{(1,0)}M)$ defined by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z.$$

Definition 2.7. The Ricci curvature r is given by

$$r(X, Y) = \mathrm{tr}\ R(X, Y)$$

Definition 2.8. The scalar curvature s is defined as

$$s = \mathrm{tr}_g\ r = \mathrm{tr}\ r^\sharp$$

Here tr_g indicates that our trace is dependent on the $\sharp$ isomorphism induced by g; since $r^\sharp$ is a $(1, 1)$-tensor it is isomorphic to a linear endomorphism, whose trace we take.

3 Einstein Manifolds and Calabi-Yau Manifolds

We can now define Einstein manifolds as a special subclass of Riemannian manifolds.

Definition 3.1. An Riemannian manifold (M, g) is Einstein if $r = \lambda g$ for a constant $\lambda \in \mathbb{R}$.

The original motivation for Einstein manifolds was in the Lorentzian case (i.e a metric with signature $(1, p)$ or $(p, 1)$), where the solution to Einstein field equations in a vacuum is Einstein (with respect to the Lorentzian metric, for which we can define Ricci curvature in the same manner).

Einstein manifolds in a Riemannian context, however, hold their own importance, being themselves the critical point of many functionals we can put on the space of metric on a manifold, as well as of physical importance (for instance, modeling gravitational instantons in a quantum theory of gravity). For instance, we have

Proposition 3.1. *Let $S : \mathcal{M} = \{g : g \text{ is a Riemannian metric on } M\} \to \mathbb{R}$ be given by*

$$S(g) = \int_M s_g \mu_g$$

Define $\mathcal{M}_1 = \{g \in \mathcal{M} : \int_M \mu_g = 1\}$. Then, if dim $M > 2$, g is a critical point of $S|_{\mathcal{M}_1}$ if and only if g is Einstein.

$\mathcal{M}$ is a smooth manifold with $g \mapsto \int_M \mu_g$ a smooth map, so we can make sense of $T\mathcal{M}_1$ and critical points of S on it.

Before we prove this proposition, we must define a few differential operators on a Riemannian manifold.

We note that the inner product induced by g on vectors on T_pM extends to $T_p^{(r,s)}M$ by letting $\{e_{i_1} \otimes \dots e_{i_r} \otimes e_{j_1}^\flat \otimes \dots e_{j_s}^\flat\}$ be an orthonormal basis for $\{e_i\}$ an orthonormal basis of T_pM. Thus for any two (r, s)-tensors h and k, we have an inner product of h_p and k_p, which we write $g(h, k)$, which is smooth with respect to p. This becomes an inner product on $\mathcal{T}^{(r,s)}M$ by

$$\langle h, k \rangle = \int_M g(h, k) \; \mu_g.$$

With this we can define

Definition 3.2. Let $\alpha \in \Omega^k(M)$ be an exterior differential k-form. We define the operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ as the adjoint of the exterior differential d with respect to the inner product of $(k, 0)$-tensors, i.e.

$$\langle \delta\alpha, \beta \rangle = \langle \alpha, d\beta \rangle$$

for all $\beta \in \Omega^{k-1}(M)$.

We can also consider symmetric tensors instead of forms. Let $\mathcal{S}^p(M)$ be the space of symmetric $(p, 0)$ tensors on M.

Definition 3.3. We define the divergence $\delta : \mathcal{S}^{p+1}(M) \to \mathcal{S}^p(M)$ as follows. The covariant derivative maps $\nabla : \mathcal{S}^p(M) \to \mathcal{S}^1(M) \otimes \mathcal{S}^p(M)$, which we can compose with symmetrization to get $\delta^* : \mathcal{S}^p(M) \to \mathcal{S}^{p+1}(M)$, whose adjoint is the divergence map.

Lastly, we can define the Laplacian with respect to a metric g

Definition 3.4. The Laplacian with respect to a metric g is given by $\Delta = d\delta + \delta d$.

Proof. It can be shown that

$$\frac{d}{dt} s_{g+th}\Big|_{t=0} = \Delta(tr(h)) + \delta(\delta h) - g(h,r)$$

Therefore,

$$\begin{aligned}
\frac{d}{dt} S(g+th)\Big|_{t=0} &= \frac{d}{dt}\int_M s_{g+th}\mu_{g+th}\Big|_{t=0} \\
&= \int_M \frac{d}{dt}(s_{g+th})\mu_{g+th} + \int_M s_{g+th}\frac{d}{dt}(\mu_{g+th})\Big|_{t=0} \\
&= \int_M \big(\Delta(tr(h)) + \delta(\delta h) - g(h,r)\big)\mu_g + \int_M \frac{s_g}{2}\mathrm{tr}(h)\mu_g \\
&= \int_M (\frac{s_g}{2}\mathrm{tr}(h) - g(h,r))\mu_g \\
&= \langle \frac{s_g}{2} g - r, h\rangle
\end{aligned}$$

since, by Stokes's Theorem,

$$\begin{aligned}
\int_M \big(\Delta(tr(h)) + \delta(\delta h)\big)\mu_g &= \int_M \big(d\delta(tr(h)) + \delta d(tr(h)) + \delta(\delta h)\big)\mu_g \\
&= \langle \delta d(tr(h)) + \delta(\delta h), 1\rangle + \int_M d\delta(tr(h))\mu_g \\
&= \langle d(tr(h)) + \delta h, d(1)\rangle + 0 \\
&= 0.
\end{aligned}$$

Thus, g is a critical point of $S|_{\mathcal{M}_1} \iff \frac{d}{dt}S(g+th)\Big|_{t=0} = 0$ for all $h \in T_g\mathcal{M}_1 = \{h \in \mathcal{S}^2M : \langle h, g\rangle = 0\} \iff \langle r, h\rangle = 0$ for all $h \in \{h \in \mathcal{S}^2M : \langle h, g\rangle = 0\} \iff r = fg$ for $f \in C^\infty(M)$. With the following lemma, this is equivalent to $r = \lambda g$ for $\lambda \in R$. □

Lemma 3.1. *If dim $M > 2$, $r = fg$ for $f \in C^\infty(M)$ if and only if $r = \lambda g$ for $\lambda \in R$.*

Proof. In the previous proof we found

$$S'_g \cdot h = \frac{d}{dt}S(g+th) = \langle \frac{s_g}{2} g - r, h\rangle.$$

The functional S_g is invariant under diffeomorphism of M, so for any $h \in T_g\mathcal{D}$ where $\mathcal{D}$ is the orbit of g in $\mathcal{M}$ under the diffeomorphism group, $S'_g \cdot h = \langle \frac{s_g}{2} g - r_g, h\rangle = 0$. However we have that $T_g\mathcal{D}$ is the image of δ^*_g on $\mathcal{S}^1(M)$.

Thus, $\langle \frac{s_g}{2} g - r, \delta_g^* \alpha \rangle = 0$ for all $\alpha \in \mathcal{S}^1(M)$, i.e. $\langle \delta_g(\frac{s_g}{2} g - r_g), \alpha \rangle = 0$ for all $\alpha \in \mathcal{S}^1(M)$, so $0 = \delta_g(\frac{s_g}{2} g - r_g) = \frac{ds_g}{2} - \delta_g . r_g$. Now let $r_g = fg$. then $\delta_g r_g = df$, so we get $\frac{ds_g}{2} - df = 0$, so $ds_g = 2df$. But also, taking the trace of $r_g = fg$, we also get $s_g = nf$ and thus $ds_g = ndf_g$ for $n > 2$, so $ds_g = df = 0$ and f is constant. The other direction is trivial. □

In this sense, a metric on manifold M minimizing or maximizing total scalar curvature while preserving total volume is Einstein. More so, by this last lemma, in manifolds of dimension greater than 2, the Ricci curvature being a multiple of the metric by a smooth function necessarily means that that function is constant.

A special subclass of Einstein manifolds are Calabi-Yau manifolds, which we define as Kähler manifolds with vanishing Ricci curvature (there exists another, more general definition, but they happen to be the same in the simply connected case).

We define this formally below.

Definition 3.5. A Calabi-Yau manifold is a manifold M with a Riemannian metric g, complex structure J, and a symplectic form (i.e. a closed, nondegenerate differential 2-form) ω such that M is Kähler (i.e. $g(X, Y) = \omega(X, JY)$ and $r_g = 0$ (equivalently, the first Chern class of M vanishes, by the Calabi-Yau theorem).

Calabi-Yau manifolds are important structures in supersymmetric string theory in physics, where the universe is theorized to assume the form of four spacial dimensions and a six-dimensional Calabi-Yau manifold. Numerous conjectures about the nature of the universe manifest as questions about the geometry of such a manifold; to begin answering them, we consider them a (complex) dimension downwards. In complex dimension 2, the only Calabi-Yau manifolds are complex tori or K3 Surfaces. We look at the latter (the geometry of complex tori are simple), focusing on a construction by Kummer. We examine the geometry of Kummer K3 Surfaces by looking at their closed geodesics, relating to a conjecture in physics about all $K3$ surfaces possessing a closed geodesic.

4 Kummer K3 Surfaces

We construct a class of K3 surface via a construction by Kummer and explore certain results on stable closed geodesics therein, following Lye [2023] closely.

Take a lattice Γ in $\mathbb{C}^2$. The quotient $T = \mathbb{C}^2/\Gamma$ is a 4-torus with a complex structure induced by the particular lattice. The group $\mu_2 = \{\pm 1\}$ acts on $\mathbb{C}^2$ via the map $(z, w) \mapsto (-z, -w)$, which descends to an action on T with 16 fixed points corresponding to $(z, w) \in \mathbb{C}^2$ such that $(2z, 2w) \in \Gamma$.

If we let $Y = T/\mu_2$, this is a complex space with singularities at each fixed point. In particular, Y looks like the cone $\mathbb{C}^2/\mu_2$ in the neighborhood of each singularity. We define a (complex) blowup on $\mathbb{C}^2/\mu_2$, which we use to resolve each of these singularities; in particular, we identify $\mathbb{C}^2/\mu_2$ with

$$\mathcal{O}_{\mathbb{C}P^1}(-2) = T^*\mathbb{C}P^1 = \{\big((z,w),[\xi:\zeta]\big) : z\zeta^2 = w\xi^2\} \subset \mathbb{C}^2 \times \mathbb{C}P^1,$$

which has a copy of $\mathbb{C}P^1$ when $z = w = 0$, which we call the exceptional divisor, using the map

$$\begin{aligned}(\mathbb{C}^2\backslash\{0\})/\mu_2 &\to \mathcal{O}_{\mathbb{C}P^1}(-2)\backslash\mathbb{C}P^1,\\ [(z,w)] &\mapsto \big((z^2,w^2),[z:w]\big).\end{aligned}$$

where we have the blow-down map π given by inverting the map away from $z = w = 0$, which is a diffeomorphism.

Choose $a > 0$ and $0 < \delta \ll 1$. Define the potential maps $f_{Euc}, f_a : (\mathbb{C}^2\backslash\{0\})\mu_2 \to \mathbb{R}$, the Euclidean-Kähler and Eguchi-Hansen potentials respectfully, by

$$\begin{aligned}f_{Euc}(z,w) &= |z|^2 + |w|^2\\ f_a(z,w) &= \sqrt{a^2 + |z|^2 + |w|^2} - a \cdot \operatorname{arcsinh}(\frac{a}{|z|^2 + |w|^2})\end{aligned}$$

We let $\theta : [0,\infty) \to \mathbb{R}$ be a smooth bump function such that $\theta|_{[0,1]} \equiv 1$ and $\theta|_{[1+\delta,\infty)} \equiv 0$, and define a new potential

$$\Phi_a(z,w) = f_{Euc}(z,w) + \theta(|z|^2 + |w|^2)(f_a(z,w) - f_{Euc}(z,w)).$$

which is spherically symmetric for small enough a. For the metric $g = \partial\bar{\partial}\Phi_a$ given by this potential, $\pi_* g$ is a metric on $\mathcal{O}_{\mathbb{C}P^1}(-2)\backslash\mathbb{C}P^1$ which extends to all of $\mathcal{O}_{\mathbb{C}P^1}(-2)$.

Let X be blowup of Y at each of its 16 singularities p_i with exceptional divisors $E_i = \pi^{-1}(0)$ for $i = 1,\ldots,16$. X is the underlying smooth manifold for our Kummer K3 Surface. We equip it with a metric g in a neighborhood of each E_i, choosing parameters a_i and δ by the procedure above, and let g be the standard flat metric outside of these neighborhoods. We also let U_i be the blowup of $B_{1+2\delta}(0)/\mu_2$ in the local model of each p_i; we scale X accordingly so that these U_i are disjoint.

We call g the patchwork metric on X. We have $g|_{E_i} = a_i g_{FS}$ for g_{FS} the Fubini-Study metric on $\mathbb{C}P^1$. On the (corresponding images of) the set $B_1(0)$ and outside of $B_{1+\delta}(0)$, g is Ricci-flat, but on set $N_i = B_{1+\delta}(0)\backslash B_1(0)$ (again we are taking the images in X), g is not Ricci-flat. Lastly, g is flat in the complement of all U_i.

We use the Calabi-Yau theorem to perturb g into a Ricci flat-metric. We state the theorem here:

Theorem 4.1 (Calabi-Yau). *Let (X, g) be a Kähler manifold of complex dimension n with vanishing first Chern class and Kähler form ω. Let $\psi : X \to \mathbb{R}$ be a function such that $r_g = \partial\bar{\partial}\psi$. For A given by*

$$A := \frac{\int_X \omega^n}{\int_X e^{\psi}\omega^n}$$

there is a function $\phi : X \to \mathbb{R}$ *such that* $\int_X \phi\, \omega^n = 0$ *and* $\tilde{\omega} := \omega + i\partial\bar{\partial}\phi$ *is a Kähler form satisfying the Monge-Ampère equation:*

$$\tilde{\omega}^n = Ae^{\psi}\omega^n$$

the metric $\tilde{g}$ *associated with* $\tilde{\omega}$ *is Ricci-flat.*

We look at stable closed geodesics on $(X, \tilde{g})$, which we define as

Definition 4.1. A closed geodesic $\gamma : S^1 \to M$ on a (M, g) is stable if it locally minimizes the length functional,

$$\ell(\gamma) = \int_{S^1} |\dot{\gamma}|(t)\; dt$$

We also assume the following theorem

Theorem 4.2 (Lye). *Let* $(X, \tilde{g})$ *be a Kummer K3 Surface as constructed before. Then for* $|a|^2 = \sum_{i=1}^{16} |a_i|^2$ *small enough, there is an open set* $V_a \subset X$ *containing every exceptional divisor such that no stable, closed geodesic with respect to both* g *and* $\tilde{g}$ *ever enters* V_a*.*

We use this to prove the following

Theorem 4.3 (Lye). *Assume the same setup as before. Then for* $|a|^2$ *small enough, no stable, closed geodesic with respect to* $\tilde{g}$ *stays completely in an Eguchi-Hansen patch* U_i*.*

Proof. Suppose such a stable closed geodesic γ exists. By the previous theorem, we can assume γ stays away from E_i. Let $d(t)$ be the distance squared between $\gamma(t)$ and E_i. We find this by considering the family of geodesics $\rho_t(s)$ where for every t ρ_t connects the point $\gamma(t)$ to the point on E_i closest to it (such a point exists because $E_i \cong \mathbb{C}P^1$ is compact). Since these are geodesics,

$$D_s\partial_s\rho_t(s) = 0$$

Also $D_t\partial_s = D_s\partial t$. Since $d(t)$ is smooth and defined on a compact set S^1, it must have a maximum, i.e. $T \in S^1$ such that $d'(T) = 0$ and $d''(T) \leqslant 0$. We show that this produces a contradiction.

We have

$$\begin{aligned} d'(t) &= \frac{d}{dt}\int_0^1 |\partial_s\rho_t(s)|^2 \; ds \\ &= \int_0^1 \frac{d}{dt}\langle \partial_s\rho_t(s), \partial_s\rho_t(s)\rangle \, ds \\ &= 2\mathrm{Re}\int_0^1 \langle D_t\partial_s\rho_t(s), \partial_s\rho_t(s)\rangle \, ds \\ &= 2\mathrm{Re}\int_0^1 \langle D_s\partial_t\rho_t(s), \partial_s\rho_t(s)\rangle \, ds \\ &= 2\mathrm{Re}\int_0^1 \partial_s\langle \partial_t\rho_t(s), \partial_s\rho_t(s)\rangle \, ds \\ &= 2\mathrm{Re}\,\langle \partial_t\rho_t(s), \partial_s\rho_t(s)\rangle|_{s=0}^{s=1}. \end{aligned}$$

The lower limit here vanishes since $\partial_s\rho_t(s)|_{s=0}$ is normal to E_i. Thus

$$\begin{aligned} d'(t) &= 2\mathrm{Re}\,\langle \partial_t\rho_t(s), \partial_s\rho_t(s)\rangle|_{s=1} \\ d''(t) &= 2\mathrm{Re}\,\big(\langle D_t\partial_t\rho_t(s), \partial_s\rho_t(s)\rangle + \langle \partial_t\rho_t(s), D_t\partial_s\rho_t(s)\rangle\big)|_{s=1}. \end{aligned}$$

Now we work in our original coordinates for the Eguchi-Hansen metrics g, on $(B_1(0)\backslash\{0\})/\mu_2$. Take $z = (z_1, z_2)$ coordinates on this set. Let $u = z_1^2 + z_2^2$. We have

$$\langle U, V\rangle_g = \sqrt{1 + \frac{a_i^2}{z^2}}\Big(\langle U, V\rangle_{\mathbb{C}^2} - \frac{a_i^2}{a_i^2 + u^2}\frac{\langle U, z\rangle_{\mathbb{C}^2}\langle z, V\rangle_{\mathbb{C}^2}}{u}\Big).$$

We can locally write $\rho_t(s) = \varphi(s,t)z(t)$ for $\varphi(1,t) = 1$ and $\partial_s\varphi > 0$ for all t. We insert this to get

$$d'(t) = 2(\partial_s\varphi(1,t)\;\mathrm{Re}\langle z(t), z'(t)\rangle_g)$$

Thus $d'(T) = 0 \iff \mathrm{Re}\langle z(T), z'(T)\rangle_g = 0 \iff \mathrm{Re}\langle z(T), z'(T)\rangle_{\mathbb{C}^2} = 0$.

Also

$$\begin{aligned} \langle D_t\partial_t\rho_t(s), \partial_s\rho_t(s)\rangle|_{s=1} &= \partial_s\varphi(1,t)\langle D_t z'(t), z(t)\rangle_g \\ \langle \partial_t\rho_t(s), D_t\partial_s\rho_t(s)\rangle|_{s=1} &= (\partial_t\partial_s\varphi)\langle z'(t), z(t)\rangle_g \\ &\quad + (\partial_s\varphi)\Big(\frac{2a_i^2}{\sqrt{a_i^2 + u(t)^2}}\left|\langle z'(t), z(t)\rangle_g\right|^2 + \frac{u(t)^2}{a_i^2 + u(t)^2}\left|z'(t)\right|_g^2\Big) \end{aligned}$$

where the real part of the first term in the second equality thus vanishes at T.

Putting it together, we have

$$\begin{aligned} d''(T) &= 2\mathrm{Re}\,\big(\langle D_t\partial_t\rho_t(s), \partial_s\rho_t(s)\rangle + \langle \partial_t\rho_t(s), D_t\partial_s\rho_t(s)\rangle\big)|_{s=1,t=T} \\ &= 2(\partial_s\varphi(1,T))\Big(\mathrm{Re}\langle D_t z', z\rangle_g + \frac{2a_i^2}{\sqrt{a_i^2 + u^2}}\left|\langle z', z\rangle_g\right|^2 + \frac{u^2}{a_i^2 + u^2}\left|z'\right|_g^2\Big) \\ &\geqslant 2(\partial_s\varphi(1,T))\Big(\mathrm{Re}\langle D_t z', z\rangle_g + \frac{2a_i^2}{\sqrt{a_i^2 + u^2}}\left|\langle z', z\rangle_g\right|^2\Big) \end{aligned}$$

We have

$$\mathrm{Re}\langle D_t z', z\rangle_g \geqslant -\left|z\right|_g \left|D_t z'\right|_g \geqslant -\left|z\right|_g C\left|a\right|^{\frac{1}{2}} \left|z'\right|_g^2 = -C\frac{u^2}{\sqrt{a_i^2+u^2}}\left|a\right|^{\frac{1}{2}}\left|z'\right|_g^2$$

Therefore:

$$\begin{aligned} d''(T) &\geqslant \frac{2u^2(\partial_s\varphi)}{\sqrt{a_i^2+u^2}}\left|z'\right|_g^2\Big(-C\left|a\right|^{\frac{1}{2}}+\frac{1}{\sqrt{a_i^2+u^2}}\Big) \\ &\geqslant \frac{u^2(\partial_s\varphi)}{\sqrt{a_i^2+u^2}}\left|z'\right|_g^2\Big(-C\left|a\right|^{\frac{1}{2}}+\frac{1}{\sqrt{a_i^2+1}}\Big) \\ &> 0 \end{aligned}$$

for $|a|$ small enough, a contradiction. □

By this theorem, we know that no stable, closed geodesic is ever in the Eguchi-Hansen patches in X a Kummer K3 surface. It has also been found that there are no strictly stable (i.e. strict local minima of the length functional) closed geodesics in $(X, \tilde{g})$ as a whole [Bourguignon, 1976]. This provides information on the geometry of $(X, \tilde{g})$ near each exceptional divisor E_i, suggesting the presence of no "bottlenecks" in its shape.

However, these nonexistence results do not extend to all of $(X, \tilde{g})$, so we still seek to find stable, closed geodesics on X. On this front, Oliveira [2023] finds 48 closed geodesics on $K3$ surfaces, though the stability of these is not known. In future work, we seek to find stable close geodesics on X, even as we lighten the stability constraint (to perhaps a geodesic of index 1).

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