Closed Geodesics on Kummer K3 Surfaces

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Introduction

1. Calabi–Yau manifolds are important structures in supersymmetric theories in physics.
2. In dimension 4, all Calabi–Yau manifolds are either complex tori or K3 surfaces.
3. The existence of closed geodesics on all K3 surfaces is an open conjecture.
4. K3 surfaces are also Einstein manifolds, whose classification remains open in dimension 4.
5. We look specifically at a construction of K3 surfaces by Kummer.

Definitions

A Riemannian manifold is a smooth manifold M equipped with smooth $(0,2)$-tensor g, called the metric, such that

$$g(X,Y) = g(Y,X)$$
$$g(X,X) > 0 \text{ for } X \neq 0.$$

The **Levi-Civita connection** is the unique connection ∇ such that

$$\nabla g = 0$$
$$\nabla_X Y - \nabla_Y X = [X,Y].$$

A smooth map $\gamma : [a,b] \to M$ is a **geodesic** if

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

A geodesic γ is **closed** if γ is periodic, i.e.,

$$\gamma : S^1 \to M$$

and **stable** if it is a local minimum of the length functional

$$\ell(\gamma) = \int_{S_1} |\dot\gamma(t)|_g \, \mu_g.$$

The **Riemann curvature tensor** R and **Ricci curvature** r are defined by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$$
$$r(X,Y) = \text{tr } R(X,Y).$$

M is **Einstein** if there exits $\lambda \in \mathbb{R}$ such that

$$r = \lambda g.$$

M is **Kähler** if it has a complex structure J such that

$$\nabla J = 0.$$

M is **Calabi–Yau** if it is Kähler with vanishing first Chern class $c_1(M) = 0$.

Kummer Construction

Take a lattice $\Gamma \subset \mathbb{C}^2$. We let $T = \mathbb{C}^2/\Gamma$ be a 4-torus with a complex structure determined by Γ. $\mu_2 = \{\pm 1\}$ acts on $\mathbb{C}^2$ via the map $(z,w) \mapsto (-z,-w)$, which descends to an action on T with 16 fixed points corresponding to $(z,w) \in \mathbb{C}^2$ such that $(2z, 2w) \in \Gamma$.

Let $Y = T/\mu_2$, this is a complex space with singularities at each fixed point. In a neighborhood of each singularity, Y looks like the cone $\mathbb{C}^2/\mu_2$. We define a (complex) blowup on $\mathbb{C}^2/\mu_2$, which we use to resolve each of these singularities, ifentifying $\mathbb{C}^2/\mu_2$ with

$$\mathcal{O}_{\mathbb{C}P^1}(-2) = T^*\mathbb{C}P^1 = \{((z,w),[\xi:\zeta]) : z\zeta^2 = w\xi^2\} \subset \mathbb{C}^2 \times \mathbb{C}P^1,$$

which has a copy of $\mathbb{C}P^1$ when $z = w = 0$, using the map

$$\pi^{-1} : (\mathbb{C}^2\backslash\{0\})/\mu_2 \to \mathcal{O}_{\mathbb{C}P^1}(-2)\backslash\mathbb{C}P^1,$$
$$[(z,w)] \mapsto ((z^2,w^2),[z:w]).$$

where π^{-1} is a diffeomorphism on this set. Choose $a > 0$ and $0 < \delta \ll 1$. We define $f_{Euc}, f_a : (\mathbb{C}^2\backslash\{0\})\mu_2 \to \mathbb{R}$:

$$f_{\text{Euc}}(z,w) = |z|^2 + |w|^2 \qquad \text{(Euclidean-Kähler Potential)}$$
$$f_a(z,w) = \sqrt{a^2 + |z|^2 + |w|^2} - a \cdot \text{arcsinh}\left(\frac{a}{|z|^2 + |w|^2}\right) \qquad \text{(Eguchi-Hansen Potential)}$$

Let $\theta : [0,\infty) \to \mathbb{R}$ be a smooth bump function such that $\theta|_{[0,1]} \equiv 1$ and $\theta|_{[1+\delta,\infty)} \equiv 0$, and let

$$\Phi_a(z,w) = f_{\text{Euc}}(z,w) + \theta(|z|^2 + |w|^2)(f_a(z,w) - f_{\text{Euc}}(z,w))$$
$$g = \partial\bar\partial\Phi_a.$$

g is Kähler such that $\pi_* g$ extends to all of $\mathcal{O}_{\mathbb{C}P^1}(-2)$. We blowup Y as so at every singularity to get X, letting g be $\partial\bar\partial\Phi_{a_i}$ near each exceptional divisor, and the flat metric otherwise. Let

$$U_i = \pi^{-1}\left(B_{1+2\delta}(0)/\mu_2\right)$$

We use the following theorem to get our Ricci-flat K3 metric $\tilde g$ on X.

> **Theorem (Calabi–Yau)** For (M,g) Kähler and $c_1(M) = 0$, there is a metric $\tilde g$ in the same Kähler class as g such that $r_{\tilde g} = 0$.

Results on Closed Geodesics

Let $(X,\tilde g)$ be a Kummer K3 Surface. The existence of closed geodesics on $(X,\tilde g)$ is known due to Oliveira. The following theorems can be shown

> **Theorem (Bourguignon)** There exist no strictly stable geodesics on any K3 Surface, including $(X,\tilde g)$.

> **Theorem (Lye)** For $|a|^2 = \sum_{i=1}^{16} |a_i|^2$ small enough:
> 1. There is an open set $V_a \subset X$ containing every exceptional divisor such that no stable, closed geodesic with respect to both g and $\tilde g$ ever enters V_a
> 2. No stable, closed geodesic with respect to $\tilde g$ stays completely in an Eguchi-Hansen patch U_i.

Geometry of Kummer K3 Surfaces

Though the geometry of a Kummer K3 $(X,\tilde g)$ is not constructively known, it can be related to (X,g):

> **Theorem (Lye)** Let $F : X \to X$ be an (anti-)holomorphic isometry of g. F is an isometry of $\tilde g$ as well.

For $|a|$ small, there exists a $C > 0$ independent of $|a|$ such that for a $\tilde g$-geodesic γ:

$$\left|D_t^g \dot\gamma(t)\right|_g \leq C|a|^{\frac{1}{2}}|\dot\gamma|_g^2$$

i.e. γ is 'close' to a g-geodesic.

Open Questions

These results constrain the form of stable, closed geodesics on Kummer K3 surfaces. However,

1. Do there exist stable geodesics entering and later leaving the Eguchi-Hansen patches?
2. Can we find closed geodesics of index n on $K3$ surfaces, near each exceptional divisor but also beyond?
3. Do these methods extend to other K3 surfaces?
4. Can these geodesics, or $\tilde g$, be found constructively?

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