Notes on Elliptic Sine Gordan Equation (Liu Wei)

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July 2023

Abstract

The elliptic sine-Gordon equation is a semilinear elliptic PDE with a special double well potential. It has multi-end solutions. We show that these solutions have explicit expressions given by the Hirota form. Moreover, we use their asymptotic behavior at infinity to classify all multi-end solutions.

1 Introduction

The elliptic sine-Gordon equation is a semilinear elliptic PDE with a special double well potential. It originated from the classical(hyperbolic) sine-Gordon equation, which was first used to study surfaces with constant negative curvature in the nineteenth century. It also appears in various physical contexts such as Josephson junction. The hyperbolic sine-Gordon equation is given by

$$\partial_z^2 u - \partial_x^2 u + \sin u = 0.$$

In our paper, the elliptic version of this equation will be investigated:

$$-\Delta u = \sin u.$$

We are particularly interested in this equation for the fact that it is a special form of the Allen-Cahn equation:

$$\Delta u = W'(u),$$

where W is the double well potential. If we choose $W(u) = \frac{1}{4}(1-u^2)^2$, then it reduces to the classical Allen-Cahn equation:

$$-\Delta u = u - u^3.$$

If we choose $W(u) = 1 + \cos u$, then we get the elliptic sine-Gordon equation. It has a number of magical properties due to the fact that it is an integrable PDE so we can use the method of inverse scattering transform [4] to analyze its solutions. Surprisingly, the solutions to the elliptic sine-Gordon equation have a nice explicit formula. In particular, the nodal set of these solutions behaves like multiple straight-half lines at infinity away from the origin. We call such solutions multi-end solutions, and it turns out that the explicit solutions we find are the unique multi-end solutions in the plane. The explicit formula and the uniqueness theorem to the elliptic sine-Gordon equation are very surprising because we can rarely find such rich results in general elliptic PDEs. It is worth mentioning that $W(u) = 1 + \cos u$ is essentially the only double well potential such that the corresponding Allen-Cahn equation is integrable[5].

The paper is organized as follows. In section 2, we will give an explicit formula for a family of solutions U_n by discussing their Hirota forms[2]. We prove that they are indeed $2n$-end solutions by using the exact formula to analyze their asymptotic behaviors at infinity. In section 5, we classify all the $2n$-end solutions by their asymptotic behavior at infinity. This guarantees the uniqueness of these multi-end solutions.

Acknowledgement

This paper is written in SURIM 2023. I would like to extend my utmost appreciation to my mentor Professor Otis Chodosh for his effort and input which made this project happen. I also want to give huge thanks to Juncheng Wei for his useful answers to my questions regarding his paper. We would additionally like to thank Lernik Asserian and everyone who made SURIM 2023 possible.

2 A Family of Multiple-End Solutions of The Elliptic Sine Gordon Equation

In this section, for each $n \in \mathbf{N}$, we aim to find a family of explicit, real-valued, nonsingular solutions of the elliptic sine-Gordon equation in $\mathbf{R}^2$:

$$-\partial_x^2 u - \partial_y^2 u = \sin u. \tag{2.1}$$

For convenience, we replace u by $u+\pi$ and transform the equation into

$$\partial_x^2 u + \partial_y^2 u = \sin u. \tag{2.2}$$

It turns out that these solutions are $2n$-ended, and this family of solutions has $2n$ free parameters. We follow the proof by Liu and Wei[1] to obtain solutions of closed form, and we shall first rewrite the sine-Gordon equation in the bilinear form. Recall D is the bilinear derivative operator [3].

Using the bi-logarithmic transformation

$$u = 2i \ln \frac{\bar{F}}{F}.$$

Notice that we take the principal branch to make the function single-valued. We compute

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i} = \frac{1}{2i}\left(\frac{\bar{F}^2}{F^2} - \frac{F^2}{\bar{F}^2}\right),$$

$$\partial_x u = 2i\left(\frac{D_x \bar{F}}{\bar{F}} + \frac{D_x F}{F}\right),$$

$$\partial_x^2 u = 2i\left(\frac{(D_x \bar{F})\bar{F} - (D_x \bar{F})^2)}{\bar{F}^2} - \frac{(D_x F)F - (D_x F)^2}{F^2}\right).$$

Finally, recall that

$$D_x(a \cdot b) = a_{xx} b - 2a_x b_x + a b_{xx},$$

we get

$$\partial_x^2 u = i\left(\frac{D_x^2(\bar{F}\cdot\bar{F})}{\bar{F}} - \frac{D_x^2(F\cdot F)}{F^2}\right).$$

By the above computations, equation (2.2) can be rewritten into

$$\left[(D_x^2 + D_y^2)(F\cdot F) + \frac{1}{2}(\bar{F}^2 - F^2)\right]\bar{F}^2 - \left[(D_x^2 + D_y^2)(\bar{F}\cdot\bar{F}) + \frac{1}{2}(F^2 - \bar{F}^2)\right]F^2 = 0. \tag{2.3}$$

We then get the following bilinear form of equation (2.2) for some real parameter λ:

$$(D_x^2 + D_y^2)(F\cdot F) + \frac{1}{2}(\bar{F}^2 - F^2) = \lambda F^2. \tag{2.4}$$

On one hand, if F satisfies equation (2.4), then u is a solution to the sine-Gordon equation.

On the other hand, let u be a solution to equation (2.2), and define

$$\rho(x,y) := \frac{(D_x^2+D_y^2)(F\cdot F)+\frac{1}{2}(\bar{F}^2-F^2)}{F^2}.$$

Dividing ρ into real and imaginary parts, i.e. $\rho(x,y)=\rho_1(x,y)+i\rho_2(x,y)$, we know by equation (2.3) that

$$(\rho_1(x,y)+i\rho_2(x,y))-(\overline{\rho_1(x,y+i\rho_2(x,y))})=2i\rho_2(x,y)=0.$$

This implies that for $F\neq 0$, we get

$$(D_x^2+D_y^2)(F\cdot F)+\frac{1}{2}(\bar{F}^2-F^2)=\rho_1 F^2.$$

Next, we introduce notations necessary for giving the explicit forms of solutions to the sin-Gordon equation. Fix $n\in\mathbf{N}$, and let $p_j, q_j, j=1,2,\cdots n$ be real parameters s.t. $p_j^2+q_j^2=1$. Define

$$\alpha(j,k) := \frac{(p_j-p_k)^2+(q_j-q_k)^2}{(p_j+p_k)^2+(q_j+q_k)^2}. \tag{2.5}$$

Note that

$$p_j - iq_j = \frac{1}{p_j+iq_j},$$

we get

$$\begin{aligned}\alpha(j,k) &= \frac{(p_j-p_k+iq_j-iq_k)(\frac{1}{p_j+iq_j}-\frac{1}{p_k+iq_k})}{(p_j+p_k+iq_j+iq_k)(\frac{1}{p_j+iq_j}+\frac{1}{p_k+iq_k})}\\ &= -\frac{(p_j-p_k+iq_j-iq_k)^2}{(p_j+p_k+iq_j+iq_k)^2}.\end{aligned}$$

We then define α for multi-indices by the following:

$$\alpha(j_1,\cdots,j_m) := 1, \textbf{ if } m=0,1,$$

$$\alpha(j_1,\cdots,j_m) := \prod_{k<l\leq m}\alpha(j_k,j_l), \textbf{ if } m\geq 2.$$

Let $\eta_j = p_j x + q_j y + \eta_j^0$ for real parameters η_j^0. Then, we define

$$f_n := \sum_{k=0}^{\lfloor n/2\rfloor}\left(\sum_{\{n,2k\}}[\alpha(j_1,\cdots,j_{2k})\exp(\eta_{j_1}+\cdots+\eta_{j_{2k}})]\right), \tag{2.6}$$

$$g_n := \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\left(\sum_{\{n,2k\}}[\alpha(j_1,\cdots,j_{2k})\exp(\eta_{j_1}+\cdots+\eta_{j_{2k}})]\right). \tag{2.7}$$

Here, $\sum_{\{n,k\}}$ means summing over all possible k different integers $j_1,\cdots,j_k$ from $\{1,2,\cdots,n\}$.

Now we can introduce a family of solutions to equation (2.2).

Theorem 2.1. (L Wei) For fixed $n\in\mathbf{N}$, let f_n, g_n be defined by equation (2.6),(2.7), then $4\arctan(g_n/f_n)$ is a solution to the elliptic sine-Gordon equation (2.2).

Proof. For fixed $n \in \mathbf{N}$, we want to find explicit n-soliton solutions of the bilinear equation (2.4) with parameter $\lambda = 0$. The equation is given by

$$(D_x^2 + D_y^2)(F \cdot F) + \frac{1}{2}(\bar{F}^2 - F^2) = 0. \tag{2.8}$$

We seek solutions with formal expansion in powers of ε:

$$F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots$$

Inserting back to equation (2.8),we compare the $O(\varepsilon)$ terms:

$$(D_x^2 + D_y^2)(F_1 \cdot 1) + \frac{1}{2}(\bar{F}_1 - F_1) = 0. \tag{2.9}$$

Comparing the $O(\varepsilon^2)$ terms:

$$2(D_x^2 + D_y^2)(F_2 \cdot 1) + (D_x^2 + D_y^2)(F_1 \cdot F_1) + \frac{1}{2}(\bar{F_1}^2 - F_1^2 + 2\bar{F}_2 - 2F_2) = 0. \tag{2.10}$$

Comparing the $O(\varepsilon^3)$ terms:

$$(D_x^2 + D_y^2)(F_3 \cdot 1) + (D_x^2 + D_y^2)(F_2 \cdot F_1) - \frac{1}{2}(\bar{F}_2\bar{F}_1 - F_2F_1 + \bar{F}_3 - F_3) = 0. \tag{2.11}$$

We can further investigate the equations for $O(\varepsilon^m)$ terms for higher orders.

Now, we define F_i in order to satisfy these equations above. Define

$$F_1 := i\sum_{j=1}^{n} \exp(\eta_j).$$

We compute

$$(D_x^2 + D_y^2)(F_1 \cdot 1) = i\sum_{j=1}^{n}(p_j^2 + q_j^2)\exp(\eta_j) = F_1,$$

$$\frac{1}{2}(\bar{F}_1 - F_1) = -F_1.$$

We can see that equation (2.9) is satisfied by this choice of F_1. We also compute that

$$(D_x^2 + D_y^2)(F_1 \cdot F_1) = -2\sum_{j_1 < j_2 \le n}\left[\left((p_{j_1} - p_{j_2})^2 + (q_{j_1} - q_{j_2})^2\right)\exp(\eta_{j_1} + \eta_{j_2})\right].$$

Next, we define

$$F_2 := \sum_{j_1 < j_2 \le n}\left[\alpha(j_1, j_2)\exp(\eta_{j_1} + \eta_{j_2})\right]$$

We compute

$$(D_x^2 + D_y^2)(F_2 \cdot 1) = \sum_{j_1 < j_2 \le n}\left[\alpha(j_1, j_2)\left((p_{j_1} + p_{j_2})^2 + (q_{j_1} + q_{j_2})^2\right)\exp(\eta_{j_1} + \eta_{j_2})\right].$$

Using definition (2.5) of $\alpha(j_1, j_2)$, we find that

$$2(D_x^2 + D_y^2)(F_2 \cdot 1) + (D_x^2 + D_y^2)(F_1 \cdot F_1) = 0.$$

Notice that $\bar{F}_1 = -F_1$ and $\bar{F}_2 = F_2$, we know that equation (2.10) holds by this choice of F_2.

To proceed, we define

$$F_3 := i \sum_{j_1<j_2<j_3\le n} \left[\alpha(j_1,j_2,j_3)\exp(\eta_{j_1}+\eta_{j_2}+\eta_{j_3})\right].$$

We compute the coefficient J before $i\exp(\eta_{j_1}+\eta_{j_2}+\eta_{j_3})$ for the $O(\varepsilon^3)$ terms:

$$\begin{aligned}
(D_x^2+D_y^2)(F_2\cdot F_1) =& a(j_1,j_2)\left((p_{j1}+p_{j2}-p_{j3})^2+(q_{j1}+q_{j2}-q_{j3})^2\right)\\
&= a(j_2,j_3)\left((p_{j2}+p_{j3}-p_{j1})^2+(q_{j2}+q_{j3}-q_{j1})^2\right)\\
&= a(j_1,j_3)\left((p_{j1}+p_{j3}-p_{j2})^2+(q_{j1}+q_{j3}-q_{j2})^2\right).\\
(D_x^2+D_y^2)(F_3\cdot F_1) =& a(j_1,j_2,j_3)\left((p_{j1}+p_{j2}+p_{j3})^2+(q_{j1}+q_{j2}+q_{j3})^2\right).\\
-\frac{1}{2}(\bar{F}_2\bar{F}_1-F_2F_1) =& -\left(a(j_1,j_2)+a(j_2,j_3)+a(j_1,j_3)\right).\\
-\frac{1}{2}(\bar{F}_3-F_3) =& -a(j_1,j_2,j_3).
\end{aligned}$$

Note that we have omitted the factor$i\exp(\eta_{j_1}+\eta_{j_2}+\eta_{j_3})$ in the above computation for simplicity.

Setting $v_j := p_j + iq_j$, we get $a(j,k) = -\frac{(v_j-v_k)^2}{(v_j+v_k)^2}$, and thus

$$\begin{aligned}
J =& \frac{(v_{j_1}-v_{j_2})^2}{(v_{j_1}+v_{j_2})^2}\left(1-(v_{j_1}+v_{j_2}-v_{j_3})\left(\frac{1}{v_{j_1}}+\frac{1}{v_{j_2}}-\frac{1}{v_{j_3}}\right)\right)\\
=& \frac{(v_{j_2}-v_{j_3})^2}{(v_{j_2}+v_{j_3})^2}\left(1-(v_{j_2}+v_{j_3}-v_{j_1})\left(\frac{1}{v_{j_2}}+\frac{1}{v_{j_3}}-\frac{1}{v_{j_1}}\right)\right)\\
=& \frac{(v_{j_1}-v_{j_3})^2}{(v_{j_1}+v_{j_3})^2}\left(1-(v_{j_1}+v_{j_3}-v_{j_2})\left(\frac{1}{v_{j_1}}+\frac{1}{v_{j_3}}-\frac{1}{v_{j_2}}\right)\right)\\
=& \frac{(v_{j_1}-v_{j_2})^2(v_{j_2}-v_{j_3})^2(v_{j_1}-v_{j_3})^2}{(v_{j_1}+v_{j_2})^2(v_{j_2}+v_{j_3})^2(v_{j_1}+v_{j_3})^2}\left(1-(v_{j_1}+v_{j_2}+v_{j_3})\left(\frac{1}{v_{j_1}}+\frac{1}{v_{j_2}}+\frac{1}{v_{j_3}}\right)\right).
\end{aligned}$$

Note that we have flipped the sign in the $()$, and there's additional contribution of signs by the $a(j,k)$ and $a(j,k,l)$.

Multiplying J by $(v_{j_1}+v_{j_2})^2(v_{j_2}+v_{j_3})^2(v_{j_1}+v_{j_3})^2 v_{j_1}v_{j_2}v_{j_3}$, we get a homogenous polynomial $L(v_{j_1},v_{j_2},v_{j_3})$ of degree 9. We prove that L is the zero polynomial by the following claim.

Proposition 2.1. $(v_{j_1}^2-v_{j_2}^2)^2$ is a factor of L.

Proof. We first denote v_{j_k} by v_k, and the i-th line of J by (i).

It suffices to show $(1)+(2)+(3)+(4)$ has divisor $(v_1^2-v_2^2)^2$.

Notice that $(1),(4)$ has divisor $(v_1-v_2)^2$. We factorize $(2),(3)$ by

$$\begin{aligned}
(2) =& (v_2-v_3)^2(v_1+v_2)^2(v_1+v_3)^2(v_1v_2v_3-(v_2+v_3+v_1)(v_1v_3+v_1v_2-v_2v_3))\\
=& (v_2-v_3)^2(v_1+v_2)^2(v_1+v_3)^2(v_1-v_2)(v_1-v_3)(v_2+v_3).
\end{aligned}$$

$$\begin{aligned}
(3) =& (v_1-v_3)^2(v_2+v_3)^2(v_1+v_2)^2(v_1v_2v_3-(v_1+v_3+v_2)(v_2v_3+v_1v_2-v_1v_3))\\
=& (v_2-v_3)^2(v_1+v_2)^2(v_1+v_3)^2(v_2-v_3)(v_2-v_1)(v_1+v_3).
\end{aligned}$$

Then, we find the additional factor of $v_1 - v_2$ by computing

$$\frac{(2)+(3)}{v_1 - v_2}\Big|_{v_1=v_2} = (v_1-v_3)^2(2v_1^2)(v_1+v_3)^2(v_1^2-v_3^2) + (v_1-v_3)^2(v_1+v_3)^2(2v_1^2)(-1)(v_1^2-v_3^2) = 0.$$

Similarly, (2), (3) has divisor $(v_1+v_2)^2$, and we can also show that $(1)+(4)$ also have factor $(v_1+v_2)^2$. □

Due to symmetry, the above proposition implies that L is a polynomial of degree at least 12, which is larger than 9. So, L must be identically zero as desired.

For $4 \le j \le n$, define

$$F_j := \exp\left(\left(1-(-1)^j\frac{\pi i}{4}\right)\right)\sum_{l_1<\cdots<l_j\le n}[\alpha(l_1.\cdots,l_j)\exp(\eta_{l_1}+\cdots+\eta_{l_j})].$$

Note that F_j is pure imaginary for odd j and real-valued for even j. Also, we set $F_j = 0$ for $j > n$.

We claim that the $O(\varepsilon^k)$ terms sum up to zero as before. We only consider odd k, and the proof for even k is very similar.

For fixed indices $j_1 \le \cdots \le j_k$, define

$$G_l := \sum m(l)\left[\alpha(j_{m_1},\cdots,j_{m_l})\alpha(j_{m_{l+1}}j_{m_k})(h-1)\right].$$

Here,

$$h := \left(v_{j_{m_1}}+\cdots+v_{j_{m_l}}-v_{j_{m_{l+1}}}-\cdots-v_{j_{m_k}}\right)\left(v_{j_{m_1}}^{-1}+\cdots+v_{j_{m_l}}^{-1}-v_{j_{m_{l+1}}}^{-1}-\cdots-v_{j_{m_k}}^{-1}\right),$$

$\sum_{m(l)}$ means summation over indices $m_1,\cdots,m_k \le k$ such that

$$m_1<\cdots<m_l; m_{l+1}<\cdots<m_k.$$

Similarly, we compute the coefficient before $i\exp(\eta_{j_1}+\cdots+\eta_{j_k})$, which is equal to $\sum_l G_l$ by the definition above.

Multiplying G_l by $\left(\prod_{l=1}^k v_{j_l}\right)\left(\prod_{a<b\le k}(v_{j_a}+v_{j_b})^2\right)$, we get a polynomial L_l of degree k^2. We prove that $L_l = 0$ by the following claim.

Proposition 2.2. $(v_{j1}^2 - v_{j2}^2)^2$ is a factor of L_l.

Proof. Denote the polynomial by $L(v_1,\cdots,v_k)$. Notice that $\alpha(v_j,v_j) = 0$, we find inductive relation between the polynomial of k and $k-2$ variables:

$$\begin{aligned}L(v_1,v_2,\cdots,v_k)\big|_{v_1=\pm v_2} =& v_1^2(v_1+v_1)^2\prod_{i=3}^k(v_1-v_i)^2(v_1+v_i)^2L(v_3,\cdots,v_k)\\ =& 4v_1^4\prod_{i=3}^k(v_1^2-v_i^2)^2L(v_3,\cdots,v_k).\end{aligned}$$

Note that the factors $(v_1-v_i)^2$ come from $\alpha(v_1,v_i) = -\frac{(v_1-v_i)^2}{(v_1+v_i)^2}$.

Now, assume the polynomial vanishes for $k-2$ variables, which is true for $k=5$ by proposition 2.1.

Then, by the above inductive relation, $L(v_1,v_2,\cdots,v_k)\big|_{v_1=\pm v_2} = 0$ for k variables, which implies that L has factor $(v_1-v_2)(v_1+v_2) = (v_1^2-v_2^2)$. By symmetry, it has factor $(v_1^2-v_2^2)^2$ after exchanging the first two entries, which concludes the proof.

□

By the above proposition, L_l has degree at least $2k(k-1)$, which is greater than k^2 for $k \geq 4$. So, $L_l = 0$, and the terms of $O(\varepsilon^k)$ must sum up to zero.

Finally, take $\varepsilon = 1$, and let $f_n = Re\, F$, $g_n = Im\, F$. Recall that $\arctan x = \frac{i}{2}\log(\frac{1-ix}{1+ix})$, we have

$$u = 2i \ln \frac{\bar{F}}{F} = 4 \arctan \frac{g_n}{f_n}.$$

Note that we use formal power series to neglect the issue of convergence when we choose $\varepsilon = 1$. □

By theorem 2.1, we get a family of smooth solutions to the elliptic sine-Gordon equation (2.2):

$$U_n := 4 \arctan \frac{g_n}{f_n} - \pi. \tag{2.12}$$

Here, U_n is dependent on p_j, q_j, η_j^0. Also, note that $-\pi < U_n < \pi$ since f_n, g_n are positive functions.

Next, we want to analyze the asymptotic behavior of U_n at infinity. Our goal is to show that U_n behaves like $2n$ half-straight lines at infinity.

Lemma 2.1. For fixed $c \in \mathbb{R}$ and k be a fixed index. Suppose (x_j, y_j) is a sequence of points such that $\eta_k(x_j, y_j) = c$ as $j \to +\infty$, $x_j^2 + y_j^2 \to +\infty$. Moreover, relabeling (p_m, q_m), $m = 1, \cdots, n$, such that as $j \to +\infty$,

$$\eta_m(x_j, y_j) \to \infty, m = 1, \cdots, k-1,$$
$$\eta_m(x_j, y_j) \to -\infty, m = k+1, \cdots, n.$$

Then, we have

$$\lim_{j \to +\infty} U_n(x_j, y_j) = \begin{cases} 4\arctan(\exp(\eta_k + \beta_k)) - \pi, & k \text{ odd}, \\ 4\arctan(\exp(-\eta_k - \beta_k)) - \pi, & k \text{ even}, \end{cases}$$

where $\beta_k = \sum_{j=1}^{k-1} \ln(\alpha(j,k))$.

Proof. If k is odd, then as $j \to +\infty$, the main order term of f_n is

$$\alpha(1, \cdots, k-1) \exp(\eta_1 + \cdots + \eta_{jk-1}).$$

The main order term of g_n is

$$\alpha(1, \cdots, k) \exp(\eta_1 + \cdots + \eta_k).$$

Since $\eta_k(x_j, y_j) = c$, we have U_n converges to

$$4\arctan\left(\frac{\alpha(1,\cdots,k)}{\alpha(1,\cdots,k-1)} e^c\right) - \pi = 4\arctan(\eta_k + \beta_k) - \pi.$$

If k is even, then as $j \to -\infty$, the main order term of f_n is

$$\alpha(1, \cdots, k) \exp(\eta_1 + \cdots + \eta_k).$$

The main order term of g_n is

$$\alpha(1, \cdots, k-1) \exp(\eta_1 + \cdots + \eta_{jk-1}).$$

Hence,U_n converges to

$$4\arctan\left(\frac{\alpha(1,\cdots,k-1)}{\alpha(1,\cdots,k)} e^{-c}\right) - \pi = 4\arctan(\exp(-\eta_k - \beta_k)) - \pi.$$

By lemma 2.1, the nodal set of U_n is asymptotic to $2n$ straight-half lines, and these lines have equations $\eta_j = p_j x + q_j y + \eta_j^0 = 1 - \beta_k$ for $j = 1, \cdots, n$.

In the special case of $n = 2$, we can choose $p_1 = p_2 = p$, $q_1 = -q_2 = 1$, and $\eta_1^0 = \eta_2^0 = \ln \frac{p}{q}$. Then, $\eta_{1,2} = px \pm qy + \ln \frac{p}{q}$ and $\alpha(1,2) = \frac{q^2}{p^2}$. We compute that

$$\begin{aligned}
\varphi_{p,q} :=& 4\arctan\left(\frac{\eta_1+\eta_2}{1+\alpha(1,2)e^{\eta_1+\eta_2}}\right) - \pi \\
=& 4\arctan\left(\frac{\frac{p}{q}(e^{px+qy}+e^{px-qy})}{1+\frac{q^2}{p^2}\cdot e^{2px}\frac{p^2}{q^2}}\right) - \pi \\
=& 4\arctan\left(\frac{p\cosh(qy)}{q\cosh(px)}\right) - \pi.
\end{aligned}$$

This corresponds to a 4-ended solution to the elliptic sine-Gordon equation (2.2).

Note that in the special case $p = q = \frac{1}{\sqrt{2}}$, the solution is the classical saddle solution:

$$4\arctan\left(\frac{\cosh(\frac{y}{\sqrt{2}})}{\cosh(\frac{x}{\sqrt{2}})}\right)$$

□

3 Inverse Scattering Transform And The Classification of Multiple-End Solutions

Recall the correspondence $\varphi + \pi \longleftrightarrow u$, the multi-end solutions to $\Delta u = \sin u$ corresponds to solutions to $-\Delta\varphi = \sin\varphi$ whose π level sets are asymptotic to $2n$ half-straight lines. In this section, we will prove a classification theorem of the solutions to the elliptic sine-Gordon equation (2.2) by using the inverse scattering transform developed in [4]. We will follow the proof by Liu and Wei [1] in this section. The main result is given by the following theorem:

Theorem 3.1. Suppose φ is an $2n$-ended solution of the equation $-\Delta\varphi = \sin\varphi$. Then there exist parameters p_j, q_j, η_j^0, $j = 1, \cdots, n$, such that $\varphi = U_n$, where U_n is defined in equation (2.12).

Proof. First, recall the Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\sigma_j^2 = I$, and

$$\sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3;$$

$$\sigma_3\sigma_2 = i\sigma_1 = -\sigma_2\sigma_3;$$

$$\sigma_2\sigma_1 = i\sigma_3 = -\sigma_1\sigma_2.$$

Equation $\Delta u = \sin u$ has a Lax pair:

$$\Phi_x = A\Phi, \tag{3.1}$$

$$\Phi_y = B\Phi. \tag{3.2}$$

Here Φ is vector-valued or 2 by 2 matrix-valued, depending on contexts.

The matrices A, B are defined by

$$A := \frac{i}{4}\left[\left(\lambda - \frac{\cos u}{\lambda}\sigma_3\right) - (u_x - iu_y)\sigma_2 - \frac{\sin u}{\lambda}\sigma_1\right],$$
$$B := \frac{1}{4}\left[-\left(\lambda + \frac{\cos u}{\lambda}\right)\sigma_3 + (u_x - iu_y)\sigma_2 - \frac{\sin u}{\lambda}\sigma_1\right].$$

The compatibility of equations (3.1) and (3.2) gives

$$A_y + AB = B_x + BA.$$

This is equivalent to the elliptic sine-Gordon equation (2.2) by the following computation.

$$\begin{aligned} A_y + AB = &\frac{i}{4}\left[\frac{\sin u}{\lambda}u_y\sigma_3 - (u_{xy} - iu_{yy})\sigma_2 - \frac{\cos u}{\lambda}u_y\sigma_1\right] \\ &+ \frac{i}{16}\left[-(\lambda^2 - \frac{\cos^2 u}{\lambda^2}) - (u_x - iu_y)^2 + \frac{\sin^2 u}{\lambda^2}\right] \\ &- \frac{i}{16}\left[\frac{2\cos u}{\lambda}(u_x - iu_y)(-i\sigma_1) - 2\sin u(i\sigma_2) + \frac{2\sin u}{\lambda}(u_x - iu_y)(-i\sigma_3)\right]. \end{aligned}$$

$$\begin{aligned} B_x + BA = &\frac{1}{4}\left[\frac{\sin u}{\lambda}u_x\sigma_3 + (u_{xx} - iu_{xy})\sigma_2 - \frac{\cos u}{\lambda}u_x\sigma_1\right] \\ &+ \frac{i}{16}\left[-(\lambda^2 - \frac{\cos^2 u}{\lambda^2}) - (u_x - iu_y)^2 + \frac{\sin^2 u}{\lambda^2}\right] \\ &+ \frac{i}{16}\left[\frac{2\cos u}{\lambda}(u_x - iu_y)(-i\sigma_1) + 2\sin u(i\sigma_2) - \frac{2\sin u}{\lambda}(u_x - iu_y)(-i\sigma_3)\right]. \end{aligned}$$

Then, $A_y + AB = B_x + BA$ is equivalent to

$$\begin{aligned} &i\left[\frac{\sin u}{\lambda}u_y\sigma_3 - (u_{xy} - iu_{yy})\sigma_2 - \frac{\cos u}{\lambda}u_y\sigma_1\right] \\ &= \left[\frac{\sin u}{\lambda}u_x\sigma_3 + (u_{xx} - iu_{xy})\sigma_2 - \frac{\cos u}{\lambda}u_x\sigma_1\right] + \left[\frac{\cos u}{\lambda}(u_x - iu_y)\sigma_1 - \sin u\sigma_2 - \frac{\sin u}{\lambda}(u_x - iu_y)\sigma_3\right]. \end{aligned}$$

Note that only terms with σ_2 remain, which gives $u_{xx} + u_{yy} = \sin u$.

Define $K(\lambda) := \lambda - \frac{1}{\lambda}$. For fixed $y \in \mathbb{R}$, as $x \to \pm\infty$, lemma 2.1 implies that u approaches 0 or 2π, and we have

$$A \to \frac{Ki}{4}\sigma_3.$$

We prove the existence of matrix-valued solutions $\Phi_\pm$ of equation (3.1) such that $\Phi_\pm(x, y) \to \exp(\frac{Ki}{4}\sigma_3 x)$ as $x \to \pm\infty$ using Picard iteration and certain assumptions on λ. The existence of the Jost solutions $\Phi_\pm$ and their analytic properties are given by the following lemma.

Lemma 3.1. Assume $Im\lambda \geq 0$ and $\lambda \neq 0$. Then, there exist a solution $\Phi_{+,1}$ to the equation $\partial_x\Phi_{+,1} = A\Phi_{+,1}$ such that $\Phi_{+,1}\exp(-Kix/4) - (1,0)^T \to 0$ as $x \to \infty$, and a solution $\Phi_{-,2}$ to the equation $\partial_x\Phi_{-,2} = A\Phi_{-,2}$ such that $\Phi_{-,2}\exp(Kix/4) - (0,1)^T \to 0$ as $x \to -\infty$. Moreover, $\Phi_{+,1}$ and $\Phi_{-,2}$ are analytic with respect to λ in the region $\{\lambda : Im\lambda > 0, \lambda \neq 0\}$.

Proof. Define

$$A^*(u,\lambda) := A(u,\lambda) - \frac{Ki\sigma_3}{4}.$$

Write

$$A^* = \begin{bmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{bmatrix}.$$

Then, $A^*_{ij} \to 0$ as $x \to \pm\infty$ since $A \to \frac{Ki\sigma_3}{4}$. Now, we introduce the column vector

$$\varphi_{+,1} = \Phi_{+,1} \exp\left(-\frac{Kix}{4}\right) = (\varphi_{+,11}, \varphi_{+,21})^T.$$

For fixed (y,λ), consider the following integarl equations:

$$\begin{cases} \varphi_{+,11}(x,y,\lambda) = 1 + \int_{+\infty}^x [A^*_{11}\varphi_{+,11} + A^*_{12}\varphi_{+,21}](s,y,\lambda)ds, \\ \varphi_{+,21}(x,y,\lambda) = \int_{+\infty}^x \exp\left(\frac{Ki}{2}(s-x)\right)[A^*_{21}\varphi_{+,11} + A^*_{22}\varphi_{+.21}](s,y,\lambda)ds. \end{cases} \tag{3.3}$$

We claim that equation (3.3) implies $\partial_x \Phi_{+,1} = A\Phi_{+,1}$.

In fact,

$$\Phi_{+,1} = \left[e^{\frac{Kix}{4}}\varphi_{+,11} \middle| e^{\frac{Kix}{4}}\varphi_{+,21} \right].$$

$$\partial_x \Phi_{+,1} = \left[(\frac{Ki}{4} + A^*_{11})e^{\frac{Kix}{4}}\varphi_{+,11} + A^*_{12}e^{\frac{Kix}{4}}\varphi_{+,21} \middle| (\frac{Ki}{4} + A^*_{22})e^{\frac{Kix}{4}}\varphi_{+,21} + A^*_{21}e^{\frac{Kix}{4}}\varphi_{+,11} - \frac{Ki}{2}e^{\frac{Kix}{4}}\varphi_{+,21} \right].$$

Here, we use the following formula when computing the second entry:

$$\frac{d}{dx}\int_{+\infty}^x e^{\frac{Ki}{2}(s-x)} f(s)ds = f(x) - \int_{+\infty}^x e^{\frac{Ki}{2}(s-x)} f(s)ds.$$

Recall that

$$A = \begin{bmatrix} A^*_{11} + \frac{Ki}{4} & A^*_{12} \\ A^*_{21} & A^*_{22} - \frac{Ki}{4} \end{bmatrix},$$

we get

$$A\Phi_{+,1} = \left[(\frac{Ki}{4} + A^*_{11})e^{\frac{Kix}{4}}\varphi_{+,11} + A^*_{12}e^{\frac{Kix}{4}}\varphi_{+,21} \middle| (A^*_{22} - \frac{Ki}{4})e^{\frac{Kix}{4}}\varphi_{+,21} + A^*_{21}e^{\frac{Kix}{4}}\varphi_{+,11} \right] = \partial_x \Phi_{+,1}.$$

Next, for $Im\lambda \geq 0$, we impose the boundary condition $\varphi_{+,1} = \Phi_{+,1}\exp\left(-\frac{Kix}{4}\right) \to (1,0)^T$ as $x \to +\infty$. Using Picard iteration, we can prove that system (3.3) has a unique solution,

We start from the constant solution $(1,0)^T$. Then, we define $(\varphi^{(n)}_{+,11}, \varphi^{(n)}_{+,21})$ by the following. Let $(\varphi^{(0)}_{+,11}, \varphi^{(0)}_{+,21}) := (1,0)$, and

$$\begin{cases} \varphi^{(n)}_{+,11}(x,y,\lambda) = 1 + \int_{=\infty}^x \left[A^*_{11}\varphi^{(n-1)}_{+,11} + A^*_{12}\varphi^{(n-1)}_{+,21} \right](s,y,\lambda)ds, \\ \varphi^{(n)}_{+,21}(x,y,\lambda) = \int_{+\infty}^x \exp\left(\frac{Ki}{2}(s-x)\right)\left[A^*_{21}\varphi^{(N-1)}_{+,11} + A^*_{22}\varphi^{(n-1)}_{+,21} \right](s,y,\lambda)ds. \end{cases} \tag{3.4}$$

Note that when $Im\lambda \geq 0$ and $\lambda \neq 0$, we get

$$Re\left(\frac{Ki}{2}\right) = -\frac{1}{2}\left(1 + \frac{1}{|\lambda|^2}\right) Im\lambda \leq 0.$$

This guarantees the convergence of the second integral:

$$\int_{+\infty}^x \exp\left(\frac{Ki}{2}(s-x)\right)\left[A^*_{21}\varphi^{(N-1)}_{+,11} + A^*_{22}\varphi^{(n-1)}_{+,21} \right](s,y,\lambda)ds.$$

Note that this implies the solution is analytic in the region $\{\lambda : Im\lambda > 0, \lambda \neq 0\}$.

For simplicity, we omit the y, λ dependence in the following estimates:

$$\left|\varphi^{(1)}_{+,11}(x)\right| \leq 1 + \int_x^{+\infty} |A^*_{11}(s)| ds, \left|\varphi^{(1)}_{+,21}(x)\right| \leq 1 + \int_x^{+\infty} |A^*_{21}(s)| ds.$$

Define

$$Q(x) := \int_x^{+\infty} (|A^*_{11}(s)| + |A^*_{12}(s)| + |A^*_{21}(s)| + |A^*_{22}(s)|) ds.$$

Then,

$$\left|\varphi^{(1)}_{+,11}(x)\right| \leq 1 + Q(x), \left|\varphi^{(1)}_{+,21}\right| \leq Q(x).$$

Next, we use the above estimates and integration by part to estimate $\varphi^{(2)}_{+,j1}$:

$$\begin{aligned}
\left|\varphi^{(2)}_{+,11}(x)\right| \leq & 1 + \int_x^{+\infty} |A^*_{11}(s)| \left(1 + \int_x^{+\infty} |A^*_{11}(t)| dt\right) + |A^*_{22}(s)| \int_x^{+\infty} |A^*_{21}(t)| dt ds \\
\leq & 1 + \int \left(1 + \int_x^{+\infty} |A^*_{11}(t)| + |A^*_{21}(t)| dt\right) dQ \\
\leq & 1 + \int (1 + Q) dQ \\
= & 1 + Q(x) + \frac{1}{2} Q^2(x).
\end{aligned}$$

Similarly,

$$\left|\varphi^{(2)}_{+,21}(x)\right| \leq \int (1 + Q) dQ = Q(x) + \frac{1}{2} Q^2(x).$$

Inductively, we obtain

$$\left|\varphi^{(n)}_{+,11}(x)\right| \leq \sum_{j=0}^{n} \frac{Q^j(x)}{j!}, \left|\varphi^{(n)}_{+,21}(x)\right| \leq \sum_{j=1}^{n} \frac{Q^j(x)}{j!}.$$

This gives

$$\left|\varphi^{(n)}_{+,11}(x)\right| \leq \exp(Q(x)), \left|\varphi^{(n)}_{+,21}(x)\right| \leq \exp(Q(x)) - 1. \tag{3.5}$$

After passing through subsequence, $\left(\varphi^{(n)}_{+,11}, \varphi^{(n)}_{+,21}\right)$ converges to $(\varphi_{+,11}, \varphi_{+,21})$, which is analytic with respect to λ in the region $\{\lambda : Im\lambda > 0, \lambda \neq 0\}$.

Note that we integrate from ∞ to x, so $(\varphi_{+,11}, \varphi_{+,21}) \to (1, 0)$ as $x \to +\infty$.

We also have $\partial_x \Phi_{+.1} = A\Phi_{+,1}$ after passing to the limit. By Picard–Lindelöf theorem, this is the unique solution to the system (3.3).

Next, we consider the integral equations:

$$\begin{cases} \varphi_{-,12}(x, y, \lambda) = \int_{-\infty}^{x} \exp\left(-\frac{Ki}{2}(s - x)\right)[A^*_{11}\varphi_{-,12} + A^*_{12}\varphi_{-,22}](s, y, \lambda) ds, \\ \varphi_{-,22}(x, y, \lambda) = 1 + \int_{-\infty}^{x} [A^*_{21}\varphi_{-,12} + A^*_{22}\varphi_{-,22}]. \end{cases} \tag{3.6}$$

Using the same argument, we yield a solution $(\varphi_{-,12}, \varphi_{-,22})$ such that $\varphi_{-,2}(x, y, \lambda) \to (0, 1)^T$ as $x \to -\infty$. This solution is also analytic with respect to λ in the region $\{\lambda : Im\lambda > 0, \lambda \neq 0\}$.

□

For each fixed $y \in \mathbb{R}$, Φ_+, Φ_- are solutions to the same ODE system. Hence, they are related by

$$\Phi_+(x,y,\lambda) = \Phi_-(x,y,\lambda) \begin{bmatrix} a(\lambda,y) & b(\lambda,y) \\ b^*(\lambda,y) & a^*(\lambda,y) \end{bmatrix} \tag{3.7}$$

, for some functions a, a^*, b, b^* independent of x.

Lemma 3.2. For each $\lambda \in \mathbb{C}\backslash\{0\}$ with $Im\lambda \geq 0$, we have

$$\Phi_{+,1}(x,y,\lambda) = i\sigma_2\Phi_{+,2}(x,y,-\lambda).$$

For each $\lambda \in \mathbb{C}\backslash\{0\}$ with $Im\lambda \leq 0$, we have

$$\Phi_{-,1}(x,y,\lambda) = i\sigma_2\Phi_{-,2}(x,y,-\lambda).$$

Proof. Let us write $\Phi_\pm$ as

$$\Phi_\pm = \begin{bmatrix} \Phi_{\pm,11} & \Phi_{\pm,12} \\ \Phi_{\pm,21} & \Phi_{\pm,22} \end{bmatrix}.$$

For $j = 1,2$, define

$$\Theta_{\pm,j} := \begin{bmatrix} \Phi_{\pm,2j} \\ -\Phi_{\pm,1j} \end{bmatrix} = i\sigma_2\Phi_{\pm,j}.$$

Notice that $\sigma_2 A(u,\lambda) = A(u,-\lambda)\sigma_2$, we have

$$\partial_x\Theta_{+,1}(x,y,\lambda) = A(u,-\lambda)\Theta_{+,1}(x,y,\lambda).$$

By the asymptotic behavior of $\Phi_{\pm,j}$, we have

$$\Theta_{+,1}\exp(-\frac{Kix}{4}) \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

as $x \to +\infty$.

It follows from the uniqueness of solutions to the ODE that

$$\Theta_{+.1}(x,y,\lambda) = -\Phi_{+,2}(x,y,-\lambda).$$

Similarly,

$$\Theta_{-,1}(x,y,\lambda) = -\Phi_{-,2}(x,y,-\lambda).$$

□

Lemma 3.3. Suppose $\lambda \in \mathbb{R}\backslash\{0\}$. WE have $a^*(\lambda,y) = a(-\lambda,y)$, $b^*(\lambda,y) = -b(-\lambda,y)$. Therefore,

$$\Phi_+(x,y,\lambda) = \Phi_-(x,y,\lambda) \begin{bmatrix} a(\lambda,y) & b(\lambda,y) \\ -b(-\lambda,y) & a(-\lambda,y) \end{bmatrix}$$

.

Proof. By definition, Φ_+ and Φ_- are related by

$$\begin{cases} \Phi_{+,1} = a\Phi_{-,1} + b^*\Phi_{-,2}, \\ \Phi_{+,2} = b\Phi_{-,1} + a^*\Phi_{-,2}. \end{cases} \tag{3.8}$$

The second equation of (3.8), we get

$$\Theta_{\pm,2} = b\Theta_{-,1} + a^*\Theta_{-,2}.$$

Using lemma 3.2, we have

$$\Phi_{+,1}(x,y,-\lambda) = -b(\lambda,y)\Phi_{-,2}(x,y,-\lambda) + a^*(\lambda,y)\Phi_{-,1}(x,y,-\lambda). \tag{3.9}$$

By the first equation of (3.8),

$$\Phi_{+,1}(x,y,-\lambda) = a(-\lambda,y)\Phi_{-,1}(x,y,-\lambda) + b^*(-\lambda,y)\Phi_{-,2}(x,y,-\lambda). \tag{3.10}$$

Note that in equation (3.9) and (3.10), we use $\lambda \in \mathbb{R}$ in a crucial way, i.e. $Im\lambda \le 0$ and $Im\lambda \ge 0$.

Comparing equation (3.9) and (3.10), we get

$$a^*(\lambda,y) = a(-\lambda,y), b^*(\lambda,y) = -b(-\lambda,y).$$

□

Next, we investigate the following property of the functions $a(\lambda,y)$, $b(\lambda,y)$.

Lemma 3.4. Assume $a(\lambda,y) = a(\lambda,0)$, and

$$b(\lambda,y) = b(\lambda,0)\exp\left(-\frac{1}{2}(\lambda+\lambda^{-1})y\right). \tag{3.11}$$

Proof. Recall that Φ_+ satisfies equation (3.1) but not equation (3.2). In order to get the dependence on y, we need the following claim.

Proposition 3.1. $\Phi^* := \Phi_+ \exp -\frac{1}{4}(\lambda+\frac{1}{\lambda})\sigma_3 y$ satisfies equation (3.2), i.e. $\partial_y\Phi^* = B\Phi^*$.

Proof. We have

$$\partial_x(B\Phi^*) = B_x\Phi^* + BA\Phi^* = A_y\Phi^* + AB\Phi^*.$$

$$\partial_x(\partial_y\Phi^*) = \partial_y(\partial_x\Phi^*) = \partial_y(A\Phi^*) = A_y\Phi^* + A(\partial_y\Phi^*).$$

Denote $B\Phi^*$ and $\partial_y\Phi^*$ by α,β. Then, we have

$$\partial_x(\alpha-\beta) = A(\alpha-\beta).$$

This gives

$$\alpha-\beta = e^{Ax}\cdot v_0 \tag{3.12}$$

for some column vector $v_0 \in \mathbb{R}^2$.

Notice that as $x \to +\infty$, $B \to -\frac{1}{4}(\lambda+\frac{1}{\lambda})\sigma_3$ so that

$$\partial_y\Phi^* = B\Phi^* = -\frac{1}{4}(\lambda+\frac{1}{\lambda})\sigma_3\Phi_+\exp -\frac{1}{4}(\lambda+\frac{1}{\lambda})\sigma_3 y.$$

This boundary condition and equation (3.12) imply that $v_0 = 0$, i.e. $\alpha = \beta$. □

Now, the proposition and equation (3.7) imply that

$$\begin{aligned}
&\partial_y\Phi_-\begin{bmatrix} a(\lambda,y) & b(\lambda,y) \\ -b(-\lambda,y) & a(-\lambda,y)\end{bmatrix}\exp\left(-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sigma_3 y\right) \\
&+\Phi_-\begin{bmatrix} \partial_y a(\lambda,y) & \partial_y b(\lambda,y) \\ -\partial_y b(-\lambda,y) & \partial_y a(-\lambda,y)\end{bmatrix}\exp\left(-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sigma_3 y\right) \\
&+\Phi_-\begin{bmatrix} a(\lambda,y) & b(\lambda,y) \\ -b(-\lambda,y) & a(-\lambda,y)\end{bmatrix}\exp\left(-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sigma_3 y\right) \\
&= B\Phi_-\begin{bmatrix} a(\lambda,y) & b(\lambda,y) \\ -b(-\lambda,y) & a(-\lambda,y)\end{bmatrix}\exp\left(-\frac{1}{4}\left(\lambda+\frac{1}{\lambda}\right)\sigma_3 y\right).
\end{aligned}$$

Let $x \to -\infty$ and notice that $\Phi_- \to \exp(\frac{Ki}{4}\sigma_3 x)$, we have

$$\begin{bmatrix} \partial_y a(\lambda, y) & \partial_y b(\lambda, y) \\ -\partial_y b(-\lambda, y) & \partial_y a(-\lambda, y) \end{bmatrix} + \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix} \left(-\frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)\sigma_3 y\right)$$
$$= \left(-\frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)\sigma_3 y\right) \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix}.$$

Consider the upper two entries, we obtain

$$\partial_y a = 0, \partial_y b = -\frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) b.$$

The lemma immediately follows by basic ODE computation.

□

Remark. Note that the computation only requires the top two entries of the matrices, so we don't necessarily need lemma 3.3 for the information about a^*, b^*. In other words, the proof should also work for general λ in the upper half λ-plane.

Without loss of generality, we can assume φ has no end parallel to the x-axis after rotation. Choose parameters p_j, q_j, η_j^0 such that the nodal set of U_n has the same asymptotic half-straight lines as those of φ when $y \to +\infty$. We denote the a part of the scattering data of U_n by $\hat{a}(\lambda, y)$.

Lemma 3.5. Suppose $\lambda \in \mathbb{R}\backslash\{0\}$. We have $a(\lambda, y) = \hat{a}(\lambda, y)$ and $b(\lambda, y) = 0$.

Proof. By equation (3.8),

$$\Phi_{+,1}(x, y, \lambda) = a(\lambda, y)\Phi_{-,1}(x, y, \lambda) - b(-\lambda, y)\Phi_{-,2}(x, y, \lambda). \tag{3.13}$$

Recall that $\phi_+ = \exp -\frac{Ki\sigma_3}{4} x\Phi^*$ and $\partial_x \Phi_+ = A\Phi_+$, we then get

$$\partial_x \Phi_+^* = \partial_x \left(\exp\left(-\frac{Ki\sigma_3}{4}x\right)\Phi_+\right) = \exp\left(-\frac{Ki\sigma_3}{4}x\right)\left(-\frac{Ki\sigma_3}{4} + A\right)\Phi_+ = \exp\left(-\frac{Ki\sigma_3}{4}x\right)A^* \exp\left(\frac{Ki\sigma_3}{4}x\right)\Phi_+^*. \tag{3.14}$$

Consider the norm $\|M\| := \sqrt{\sum_{j,k}|m_{jk}|^2}$, where m_{jk} are entries of a matrix M.

Then, equation (3.14) gives

$$\partial_x \|\Phi_+^*\| = \frac{\Phi^* \partial_x \Phi_+^*}{\|\Phi_+^*\|} \le \|\partial_x \Phi_+^*\| \le C_0 \|A^*\| \|\Phi_+^*\|. \tag{3.15}$$

By Theorem 2.1 in [6], we know that A^* decays to 0 exponentially fast with respect to the distance from each end to the nodal set, which in our case is linearly related to the x-coordinate.

Thus, by equation (3.15) and Gronwall inequality, we have $\|\Phi^*\| \le C$ for some universal constant C.

It then follows that $\|\Phi_+\| \le C_1$, and similarly $\|\Phi_-\| \le C_2$.

By equation (3.13), we have that

$$|b(-\lambda, y)| = \frac{|\Phi_{+,1} - a(\lambda, y)\Phi_{-,1}|}{|\Phi_{-,2}|}.$$

Sending $x \to -\infty$ and recall from lemma 3.4 that $a(\lambda, y) = a(\lambda, 0)$, we have that

$$|b| \le |e^{\frac{Kix}{4}}||C_1 + a(\lambda, 0)C_2| \le C_3.$$

So, for fixed λ, $|b(\lambda, y)|$ is uniformly bounded with respect to y. By lemma 3.4, this implies that $|b(\lambda, y)| = 0$ since $|b|$ grows exponentially fast.

Next, we let $\hat{A}$ to be the matrix obtained from replacing u by U_n in the matrix A, and let $\hat{\Phi_\pm}$ be the solutions to $\partial_x \hat{\Phi_\pm} = \hat{A}\hat{\Phi_\pm}$ with the same asymptotic behavior as that of $\Phi_\pm$. Notice that a is the ratio between $\Phi_\pm$, our goal is to compare $\Phi_\pm$ and $\hat{\Phi_\pm}$.

Write

$$\partial_x \Phi_+ = \hat{A}\Phi_+ + (A - \hat{A})\Phi_+.$$

Then,

$$(\hat{\Phi}_+)^{-1}\Phi_+ = (\hat{\Phi}_+)^{-1}\Phi_+\Big|_{+\infty} + \int_{+\infty}^{x} \partial_x\left((\hat{\Phi}_+)^{-1}\Phi_+\right) = I + \int_{+\infty}^{x} (\hat{\Phi}_+)^{-1}(A - \hat{A})\Phi_+.$$

Here, we use the following formulae in the last step:

$$\partial_x\left((\hat{\Phi}_+)^{-1}\hat{\Phi}_+\right) = \partial_x\left((\hat{\Phi}_+)^{-1}\right)\Phi_+ + (\hat{\Phi}_+)^{-1}\hat{A}\hat{\Phi}_+ = 0,$$

and thus

$$\partial_x \hat{\Phi}_+^{-1} = \hat{\Phi}_+^{-1}\hat{A}.$$

The above equations give

$$\Phi_+ = \hat{\Phi}_+\left(I + \int_{+\infty}^{x} (\hat{\Phi}_+)^{-1}(A - \hat{A})\Phi_+\right). \tag{3.16}$$

Since φ and U_n have the same asymptotic behavior, we can choose $\delta > 0$ such that

$$|\varphi - U_n| \le C_4 \exp\left(-\delta\sqrt{x^2 + y^2}\right). \tag{3.17}$$

Similar estimates can be done on the derivatives of $\varphi - U_n$. Notice that for fixed λ, A is a linear function of the solution and its first derivative, we get

$$\|A - \hat{A}\| \le C_5 \exp\left(-\delta\sqrt{x^2 + y^2}\right).$$

By equation (3.16), we have

$$\|\Phi_+ - \hat{\Phi}_+\| \le C_6 \exp\left(-\delta\sqrt{x^2 + y^2}\right).$$

Similarly,

$$\|\Phi_- - \hat{\Phi}_-\| \le C_7 \exp\left(-\delta\sqrt{x^2 + y^2}\right).$$

Recall that

$$\begin{cases} \Phi_{+,1}(x, y, \lambda) = a(\lambda, y)\Phi_{-,1}(x, y, \lambda), \\ \hat{\Phi}_{+,1}(x, y, \lambda) = \hat{a}(\lambda, y)\hat{\Phi}_{-,1}(x, y, \lambda). \end{cases}$$

The above implies

$$\|a - \hat{a}\| = \left\|\frac{\Phi_{+,1}}{\Phi_{-,1}}\right\| = \left\|\frac{\Phi_{+,1}(\hat{\phi}_{-,1} - \Phi_{-,1}) + \Phi_{-,1}(\Phi_{+,1} - \hat{\Phi}_{+,1})}{\Phi_{-,1}\hat{\phi}_{-,1}}\right\|.$$

By the above estimates, we know that for fixed λ, we get

$$\lim_{y \to +\infty}\left(a(\lambda, y) - \hat{a}(\lambda, y)\right) = 0.$$

By lemma 3.4, we know that $a(\lambda, y) = \hat{a}(\lambda, y)$ for any $y \in \mathbb{R}$.

□

By lemma 3.1 and 3.2, we know that $\Phi_{+,1}$, $\Phi_{-,2}$ are analytic in the upper half λ-plane $\mathbb{R}^{2,+} = \{\lambda \in \mathbb{C} : Im\lambda \geq 0\}$; while $\Phi_{+,2}$,$\Phi_{-,1}$ are analytic in the lower half plane $\mathbb{R}^{2,-} = \{\lambda \in \mathbb{C} : Im\lambda \leq 0\}$.

Let $W(\Phi_{+,1}, \Phi_{-,2}) = \det|\Phi_{+,1}, \Phi_{-,2}|$ be the Wronskian determinant of $\Phi_{+,1}$ and $\Phi_{-,2}$. For $\lambda \in \mathbb{R}\backslash\{0\}$, we have by lemma 3.3 that $\Phi_{+,1} = a(\lambda)\Phi_{-,1} - b(-\lambda)\Phi_{-,2}$. Hence,

$$\begin{aligned} W(\Phi_{+,1}, \Phi_{-,2}) =& W(a(\lambda)\Phi_{-,1}, \Phi_{-,2}) - W(b(-\lambda)\Phi_{-,2}, \Phi_{-,2}) \\ =& aW(\Phi_{-,1}, \Phi_{-,2}). \end{aligned}$$

By lemma 3.1 and 3.2, we get the following asymptotic behaviors of $\Phi_{-,1}$ and $\Phi_{-,2}$ as $x \to -\infty$:

$$\begin{aligned} \lim_{x\to-\infty} \Phi_{-,1} =& i\sigma_2 \lim_{x\to-\infty} \Phi_{+,1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} e^{\frac{Kix}{4}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\frac{Kix}{4}} \\ 0 \end{bmatrix}, \\ \lim_{x\to-\infty} \Phi_{-,2} =& e^{-\frac{Kix}{4}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-\frac{Kix}{4}} \end{bmatrix}. \end{aligned}$$

So, we have

$$\det|\Phi_{-,1}, \Phi_{-,2}| = \det \begin{bmatrix} e^{\frac{Kix}{4}} & 0 \\ 0 & e^{-\frac{Kix}{4}} \end{bmatrix} = 1.$$

Note that a is independent of x, this implies that for $\lambda \in \mathbb{R}\backslash\{0\}$,

$$a(\lambda, y) = W(\Phi_{+,1}, \Phi_{-,2}). \tag{3.18}$$

By equation (3.18), we know that $a(\lambda, y)$ can be analytically extended into $\mathbb{R}^{2,+}$ where both $\Phi_{+,1}$ and $\Phi_{-,2}$ are analytic. Again by the asymptotic behaviors of $\Phi_{+,1}$ and $\Phi_{-,2}$ as $\lambda \to 0$, we know that a is continuous up to the boundary of $\mathbb{R}^{2,+}$.

With the above argument, we want to generalize lemma 3.5 to $\lambda \in \mathbb{R}^{2,+}$ by the following lemma.

Lemma 3.6. Suppose $Im\lambda \geq 0$ and $\lambda \neq 0$. Let a be defined by equation (3.18), then $a(\lambda, y) = \hat{a}(\lambda, y)$.

Proof. Recall from lemma 3.1 that the function $\varphi_{+,1} = \Phi_{+,1} \exp(-\frac{Kix}{4})$ satisfies the equations:

$$\begin{cases} \varphi_{+,11}(x, y, \lambda) = 1 + \int_{+\infty}^{x} [A_{11}^* \varphi_{+,11} + A_{12}^* \varphi_{+,21}](s, y, \lambda) ds, \\ \varphi_{+,21}(x, y, \lambda) = \int_{+\infty}^{x} \exp\left(\frac{Ki}{2}(s-x)\right) [A_{21}^* \varphi_{+,11} + A_{22}^* \varphi_{+.21}](s, y, \lambda) ds. \end{cases}$$

The solution is analytic in the upper half λ-plane.

Similarly, for $\hat{\varphi}_{+,1}$ corresponds to U_n, we have:

$$\begin{cases} \hat{\varphi}_{+,11}(x, y, \lambda) = 1 + \int_{+\infty}^{x} [\hat{A}_{11}^* \hat{\varphi}_{+,11} + \hat{A}_{12}^* \hat{\varphi}_{+,21}](s, y, \lambda) ds, \\ \hat{\varphi}_{+,21}(x, y, \lambda) = \int_{+\infty}^{x} \exp\left(\frac{Ki}{2}(s-x)\right) [\hat{A}_{21}^* \hat{\varphi}_{+,11} + \hat{A}_{22}^* \hat{\varphi}_{+.21}](s, y, \lambda) ds. \end{cases}$$

Set $\rho_j := \varphi_{+,j1}(x, y, \lambda) - \hat{\varphi}_{+,j1}(x, y, \lambda)$ for $j = 1, 2$, then

$$\begin{cases} \rho_1 = \int_{+\infty}^{x} [\hat{A}_{11}^* \rho_1 + \hat{A}_{21}^* \rho_2 + f_1](s, y, \lambda) ds, \\ \rho_2 = \int_{+\infty}^{x} \exp\left(\frac{Ki}{2}(s-x)\right) [\hat{A}_{21}^* \rho_1 + \hat{A}_{22}^* \rho_2 + f_2] ds, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} f_1 &:= (A_{11}^* - \hat{A}_{11}^*)\varphi_{+,11} + (A_{12}^* - \hat{A}_{12}^*)\varphi_{+,21}, \\ f_2 &:= (A_{21}^* - \hat{A}_{21}^*)\varphi_{+,11} + (A_{22}^* - \hat{A}_{22}^*)\varphi_{+,21}. \end{aligned}$$

By equation (3.5), we know that $\varphi_{+,1}, \hat{\varphi}_{+,1}$ are uniformly bounded on the whole $x-y$ plane. Using the decay estimate in equation (3.17), we get $\|A-\hat{A}\| \to 0$ as $x \to +\infty$. Observing the expression for f_1, f_2, we know that this implies $|f_1|, |f_2| \to 0$ as $x \to +\infty$. Thus, by expression of ρ_1, ρ_2, we have $|\rho_1|, |\rho_2| \to 0$ as $x \to +\infty$.

Now we can use Picard iteration starting from $(\rho_1, \rho_2) = (0,0)$ at $+\infty$. By the same argument in lemma 3.1, we have that

$$\begin{cases} |\rho_1(x)| \le \left(\int_{+\infty}^{x} (|f_1(s)| + |f_2(s)|) ds\right) \exp(Q'(x)), \\ |\rho_2(x)| \le \left(\int_{+\infty}^{x} (|f_1(s)| + |f_2(s)|) ds\right) \exp(Q'(x)), \end{cases} \tag{3.20}$$

where

$$Q'(x) := \int_x^{+\infty} \sum_{i,j=1}^{2} \left(|A^*_{ij}(s)| + |\hat{A}^*_{ij}(s)|\right) ds.$$

Notice that the decay estimate also suggests $|f_1|, |f_2| \to 0$ as $y \to +\infty$. The estimates in (3.20) and the dominant convergence theorem imply that $|\rho_1|, |\rho_2| \to 0$ as $y \to +\infty$. It follows that

$$\lim_{y \to +\infty} [\varphi_{+,1}(x,y,\lambda) - \hat{\varphi}_{+,1}(x,y,\lambda)] = 0.$$

Similarly, let $\varphi_{-,2} = \Phi_{-,2} \exp(\frac{Kix}{4})$, we have

$$\lim_{y \to +\infty} [\varphi_{-,2}(x,y,\lambda) - \hat{\varphi}_{-,2}(x,y,\lambda)] = 0.$$

Recall that

$$a = \left[e^{\frac{Kix}{4}} \varphi_{+,1} | e^{-\frac{Kix}{4}} \varphi_{-,2}\right],$$

$$\hat{a} = \left[e^{\frac{Kix}{4}} \hat{\varphi}_{+,1} | e^{-\frac{Kix}{4}} \hat{\varphi}_{-,2}\right],$$

we then have

$$\lim_{y \to +\infty} [a(\lambda, y) - \hat{a}(\lambda, y)] = 0.$$

By the remark of lemma 3.4, we also have $\partial_y a(\lambda, y) = 0$. Therefore, we conclude that $a(\lambda, y) = \hat{a}(\lambda, y)$.

□

Let λ_j, $j = 1, 2, \cdots, m$ be the zeroes of a in $\mathbb{R}^{2,+}$. By definition, $W(\Phi_{+,1}, \Phi_{-,2}) = a(\lambda_j, y) = 0$ at these points, so the vectors $\Phi_{+,1}$ and $\Phi_{-,2}$ are linear dependent. Define c_j by the following equation:

$$\Phi_{+,1}(x,y,\lambda_j) = c_j(y) \Phi_{-,2}(x,y,\lambda_j). \tag{3.21}$$

Then, by the same calculation in lemma 3.4, we have $c_j' = \frac{1}{2}(\lambda_j + \frac{1}{\lambda_j}) c_j$. Note that here, c_j corresponds to $-b(-\lambda_j)$ is the ratio between $\Phi_{+,1}$ and $\Phi_{-,2}$ by equation (3.13). So, $c_j(y) = c_j(0) \exp \frac{1}{2}(\lambda_j + \frac{1}{\lambda_j}) y$.

Let $\hat{c}_j(y)$ be the corresponding function for U_n, and it is natural to ask whether $c_j = \hat{c}_j$. To prove this fact, we can directly analyze the precise asymptotic behavior of $\Phi_{+,1}$ as $y \to \infty$ which is hard to compute. To bypass this difficulty, we choose to first prove $u = U_n + \pi$ and then state it as a corollary.

Now we have all the necessary scattering data, i.e. a, b, λ_j, c_j, and we are able to prove the main theorem of this section with an additional condition on the zeroes of a.

Lemma 3.7. Suppose all zeroes of a in $\mathbb{R}^{2,+}$ are simple, then $u = U_n + \pi$. Moreover,

$$a(\lambda) = \hat{a}(\lambda) = \prod_{j=1}^{m} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}$$

for $\lambda \in \mathbb{C} \backslash \{0\}$.

Proof. We use the inverse scattering method to construct u from its scattering data.

For fixed $y \in \mathbb{R}$, we know by equation (3.13) that for $\lambda \in \mathbb{R}$,

$$\Phi_{-,1}(x,y,\lambda) = \frac{\Phi_{+,1}(x,y,\lambda)}{a(\lambda,y)}. \tag{3.22}$$

Consider the operator

$$(\mathscr{P}f)(\xi) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(\lambda)}{\lambda-\xi}d\lambda.$$

Let us first rewrite equation (3.22) as

$$\Phi_{-,1}(x,y,\lambda)\exp\left(-\frac{K(\lambda)i}{4}x\right)-(1,0)^T = \frac{\Phi_{+,1}(x,y,\lambda)}{a(\lambda,y)}\exp\left(-\frac{K(\lambda)i}{4}x\right)-(1,0)^T. \tag{3.23}$$

The left-hand side is analytic in the lower half λ-plane, and the right-hand side is meromorphic in the upper half λ-plane with simple zeroes λ_j, $j=1,\cdots,m$. Here $Im\lambda_j>0$. Note that the function $\exp\left(-\frac{K(\lambda)i}{4}x\right)$ has two essential singularities at $\lambda=0$ and $\lambda=\infty$. However, the issues regarding the essential singularities are discussed in [7] for the hyperbolic sine-Gordon equation. In fact, we have that

$$\begin{cases}\Phi_{-,1}(x,y,\lambda)\exp\left(-\frac{K(\lambda)i}{4}x\right)-(1,0)^T = O(\frac{1}{|\lambda|}), as\lambda\to\infty,\\ \Phi_{-,1}(x,y,\lambda)\exp\left(-\frac{K(\lambda)i}{4}x\right)-(1,0)^T = O(|\lambda|), as\lambda\to 0.\end{cases} \tag{3.24}$$

For fixed $\xi\in\mathbb{C}$ such that $Im\,\xi<0$, we apply the operator $\mathscr{P}$ to both sides of equation (3.23). Denote the left-hand side and the right-hand side of the equation by $F(\lambda)$ and $G(\lambda)$.

Let $\gamma_R := \{\lambda : Im\lambda = 0\}\bigcup\{\lambda : |\lambda| = R \; and \; Im\lambda < 0\}$, and $\gamma'_R := \{\lambda : Im\lambda = 0\}\bigcup\{\lambda : |\lambda| = R \; and \; Im\lambda > 0\}$. Then, we have

$$LHS = \frac{1}{2\pi i}\lim_{R\to\infty}\oint_{\gamma_R}\frac{F(\lambda)}{\lambda-\xi}d\lambda - \oint_{\gamma_R\cap\{|\lambda|=R\}}\frac{F(\lambda)}{\lambda-\xi}d\lambda.$$

By equation (3.24), we know the integrand of the second term is of order $O(\frac{1}{|\lambda|^2})$, and thus the second term vanishes. Thus, by the Cauchy integral formula, we have

$$LHS = F(\xi) = \Phi_{-,1}(x,y,\xi)\exp\left(-\frac{K(\lambda)i}{4}x\right)-(1,0)^T.$$

Similarly, we get

$$RHS = -\frac{1}{2\pi i}\lim_{R\to\infty}\oint_{\gamma'_R}\frac{G(\lambda)}{\lambda-\xi}d\lambda.$$

Note that the "$-$" sign is because the boundaries γ_R and γ'_R are of opposite orientation. Then, by residue theorem, we have that

$$\begin{aligned}RHS &= -\lim_{R\to\infty}\sum_{j=1}^{m}Res(\lambda_j,\gamma'_R)\\ &= -\sum_{j=1}^{m}\lim_{\lambda\to\lambda_j}\left[\frac{(\lambda-\lambda_j)}{\lambda-\xi}\frac{c_j(y)}{a(\lambda,y)}\Phi_{-,2}(x,y,\lambda)\exp\left(-\frac{K(\lambda)i}{4}x\right)\right]\\ &= -\sum_{j=1}^{m}\frac{c_j(y)\Phi_{-,2}(x,y,\lambda_j)\exp\left(-\frac{K(\lambda_j)i}{4}x\right)}{a(\lambda_j,y)+\partial_\lambda a(\lambda_j,y)(\lambda_j-\xi)}\\ &= \sum_{j=1}^{m}\left[\frac{\tilde{c}_j(y)}{\xi-\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\Phi_{-,2}(x,y,\lambda_j)\right],\end{aligned}$$

where

$$\tilde{c}_j(y) = \frac{c_j(y)}{\partial_\lambda a(\lambda_j, y)}.$$

Note that we use the formula for the residue of simple poles and the equation $\Phi_{+,1}(\lambda_j) = c_j \Phi_{-,2}(\lambda_j)$ in the third "=", and we use L'Hôpital's rule in the 4-th "=".

Hence, we obtain

$$\Phi_{-,1}(x,y,\xi)\exp\left(-\frac{K(\lambda)i}{4}x\right) - (1,0)^T = \sum_{j=1}^{m}\left[\frac{\tilde{c}_j(y)}{\xi-\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\Phi_{-,2}(x,y,\lambda_j)\right]. \tag{3.25}$$

By lemma 3.2, we have $\Phi_{-,2}(x,y,-\xi) = -i\sigma_2\Phi_{-,1}(x,y,\xi)$. Taking $\xi = -\lambda_l$, equation (3.25) becomes

$$i\sigma_2\Phi_{-,2}(x,y,\lambda_l)\exp\left(\frac{K(\lambda_l)i}{4}x\right) - (1,0)^T = -\sum_{j=1}^{m}\left[\frac{\tilde{c}_j(y)}{\lambda_l+\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\Phi_{-,2}(x,y,\lambda_j)\right]. \tag{3.26}$$

Let $M = (m_{lj})$ with entries

$$m_{lj} := \frac{\tilde{c}_j(y)}{\lambda_l+\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{2}x\right).$$

Let $\eta := (\eta_1, \cdots, \eta_{2m})^T$, where

$$\eta_l = \begin{cases} \exp\left(\frac{K(\lambda_l)i}{4}x\right)\Phi_{-,22}(x,y,\lambda_l), l = 1,2,\cdots,m, \\ \exp\left(\frac{K(\lambda_{l-m})i}{4}x\right)\Phi_{-,12}(x,y,\lambda_l), l = m+1,\cdots,2m. \end{cases}$$

Then, equation (3.26) gives

$$\exp\left(\frac{K(\lambda_l)i}{4}x\right)\begin{bmatrix}\Phi_{-,22}(\lambda_l)\\ -\Phi_{-,12}(\lambda_l)\end{bmatrix} + \begin{bmatrix}\sum_{j=1}^m m_{lj}\Phi_{-,12}(\lambda_l)\\ \sum_{j=1}^m m_{lj}\Phi_{-,22}(\lambda_l)\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

This implies

$$\begin{pmatrix} I & M \\ -M & I \end{pmatrix}\eta = e_1, \tag{3.27}$$

where I is the m by M identity matrix and $e_1 := (1,\cdots,1,0,\cdots,0)$ with first m entries being 1 and last m entries being 0.

Consider

$$\eta_+^* = \begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix}, e_+^* = \begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix}, Z_+ = \begin{pmatrix} I+iM & M \\ 0 & I-iM \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} I & M \\ -M & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ iI & I \end{pmatrix}\begin{pmatrix} I+iM & M \\ 0 & I-iM \end{pmatrix}\begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix},$$

we can rewrite equation 3.27 into

$$Z_+\eta_+^* = e_+^*.$$

Let $H_{+,j}$ be the matrix obtained from replacing the j-th column of Z_+ by e_+^*, then Cramer's rule gives

$$\eta_j = \frac{\det H_{+,j}}{\det Z_+}, j = 1,2,\cdots,m. \tag{3.28}$$

Similarly, let

$$e_-^* = \begin{pmatrix} I & 0 \\ iI & i \end{pmatrix}, Z_- = \begin{pmatrix} I-iM & M \\ 0 & I+iM \end{pmatrix},$$

and $H_{-,j}$ be the matrix obtained by replacing the j-th column of Z_- by e_-^*. Then, we get

$$\eta_j = \frac{\det H_{-,j]}}{\det Z_-}, j = 1, 2, \cdots, m.$$

Next, we insert equation (3.25) to $\partial_x \Phi_{-,1} = A\Phi_{-,1}$. First, we write $\Phi_{-,1}$ as

$$\Phi_{-,1}(x, y, \xi) = e^{\frac{K(\xi)i}{4}x} \left[\sum_{j=1}^{m} \frac{\tilde{c}_j}{\xi - \lambda_j} e^{-\frac{K(\lambda_j)i}{4}x} \Phi_{-,2}(x, y, \lambda_j) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right].$$

Then,

$$\partial_x \Phi_{-,1} = \frac{K(\xi)i}{4}\Phi_{-,1} + e^{\frac{K(\xi)i}{4}x} \left[\sum_{j=1}^{m} \frac{\tilde{c}_j}{\xi - \lambda_j} \left(-\frac{K(\lambda_j)i}{4} + A \right) e^{-\frac{K(\lambda_j)i}{4}x} \Phi_{-,2} \right].$$

Multiplying both sides of $\partial_x \Phi_{-,1} = A\Phi_{-,1}$ by $e^{-\frac{K(\lambda_j)i}{4}x}$ and expand both sides in terms of ξ for ξ large. Also, re call that $A = \frac{i}{4}\left[\left(\xi - \frac{\cos u}{\xi}\right) - (u_x - iu_y)\sigma_2 - \frac{\sin u}{\xi}\sigma_1\right]$, and we can compare the $O(1)$ terms of the bottom entry as following:

For RHS, the middle term of A gives: $\frac{1}{4}(u_x - iu_y)\Phi_{-,11} e^{-\frac{K(\xi)i}{4}x}$ *has* $O(1)$ *terms* $\frac{1}{4}(u_x - iu_y)$,

For LHS, the first term gives: $\frac{i}{4}\left(\xi - \frac{1}{\xi}\right) \sum_{j=1}^{m} \frac{1}{\xi}(1 + O(\frac{1}{\xi}))\tilde{c}_j e^{-\frac{K(\lambda_j)i}{4}x}\Phi_{-,22} = \frac{i}{4}\sum_{j=1}^{m} \tilde{c}_j e^{-\frac{K(\lambda_j)i}{4}x}\Phi_{-,22}$,

and the second term gives: $\frac{1}{\xi}\xi\sigma_3 \tilde{c}_j e^{-\frac{K(\lambda_j)i}{4}x}\Phi_{-,22} = \frac{i}{4}\tilde{c}_j e^{-\frac{K(\lambda_j)i}{4}x}\Phi_{-,22}$.

Thus, we have

$$u_x - iu_y = 2i \sum_{j=1}^{m} \left[\tilde{c}_j(y) \exp - \frac{K(\lambda_j)i}{4} x \Phi_{-,22} \right],$$

By equation (3.28) and the definition of η_j, we have

$$u_x - iu_y = 2i \sum_{j=1}^{m} \left[\tilde{c}_j(y) \exp\left(-\frac{K(\lambda_j)i}{2} x \right) \frac{\det H_{+,j}}{\det Z_+} \right].$$

Our next goal is to simplify this expression. First, define

$$v_j := \tilde{c}_j(y) \exp\left(-\frac{K(\lambda_j)i}{2} x \right).$$

We observe that $M = \left(\frac{v_j}{\lambda_l + \lambda_j}\right)_{lj}$. Then, let $\tilde{Z}_\pm$ be the matrix obtained by multiplying the l and $l+m$-th rows of $Z_\pm$ by v_l, $l = 1, 2, \cdots, m$. For each $j = 1, 2, \cdots, m$, we get the corresponding $\tilde{H}_{\pm,j}$ by the same operation. Then, we get

$$u_x - iu_y = 2i \sum_{j=1}^{m} \frac{v_j \det \tilde{H}_{+,j}}{\det \tilde{Z}_+}.$$

$$u_x - iu_y = 2i \sum_{j=1}^{m} \frac{v_j \det \tilde{H}_{-,j}}{\det \tilde{Z}_-}.$$

Notice that

$$\begin{cases} \partial_x v_l v_j = \frac{i}{2}\left(K(\lambda_l) + K(\lambda)j\right) v_l v_j, \\ \partial_y v_l v_j = \frac{1}{2}\left(\lambda_l + \lambda_j + \frac{1}{\lambda_l} + \frac{1}{\lambda_j}\right) v_l v_j, \end{cases}$$

we get

$$(\partial_x - i\partial_y)(v_l v_j) = -i(\lambda_l + \lambda_j) v_l v_j.$$

Now, we define

$$\tilde{M} := \left(\frac{v_l v_j}{\lambda_l + \lambda_j} \right),$$

and $\tilde{I}$ to be the diagonal matrix with entries $v_j,\ j = 1, 2, \cdots, m$. Using these notations, we can calculate the right-hand side of equation (3) to recover the Hirota form of our solution u.

Proposition 3.2. Denote $\partial_x - i\partial_y$ by ∂,then we have

$$\sum_{j=1}^{m} \frac{v_j \det \tilde{H}_{+,j}}{\det \tilde{Z}_+} = \frac{\partial(\det(\tilde{I} + i\tilde{M}))}{\det(\tilde{I} + i\tilde{M})} - \frac{\partial \det(\tilde{I} + i\tilde{M})}{\det(\tilde{I} + i\tilde{M})}.$$

Proof. First, we recall the Jacobi formulae:

$$\partial \det A = \det A \operatorname{Tr}(A^{-1} \partial A).$$

This implies

$$\frac{\partial \det(\tilde{I} - i\tilde{M})}{\det(\tilde{I} - i\tilde{M})} = \operatorname{Tr}\left[(\tilde{I} - i\tilde{M})^{-1} (\partial \tilde{I} - i\partial \tilde{M}) \right].$$

We then have

$$\begin{aligned}
\frac{\partial \det(\tilde{I} - i\tilde{M})}{\det(\tilde{I} - i\tilde{M})} - \frac{\partial \det(\tilde{I} + i\tilde{M})}{\det(\tilde{I} + i\tilde{M})} &= \operatorname{Tr}\left[(\tilde{I} - i\tilde{M})^{-1} (\partial \tilde{I} - i\partial \tilde{M}) - (\tilde{I} + i\tilde{M})^{-1} (\partial \tilde{I} + i\partial \tilde{M}) \right] \\
&= \operatorname{Tr}\left[\left((\tilde{I} - i\tilde{M})^{-1} - (\tilde{I} + i\tilde{M})^{-1} \right) \partial \tilde{I} \right] - \operatorname{Tr}\left[\left((\tilde{I} - i\tilde{M})^{-1} + (\tilde{I} + i\tilde{M})^{-1} \right) i\partial \tilde{M} \right] \\
&= \operatorname{Tr}\left[2i(\tilde{I} - i\tilde{M})^{-1} \tilde{M} (\tilde{I} + i\tilde{M})^{-1} \partial \tilde{I} \right] - \operatorname{Tr}\left[\left((\tilde{I} - i\tilde{M})^{-1} + (\tilde{I} + i\tilde{M})^{-1} \right) i\partial \tilde{M} \right].
\end{aligned}$$

Consider the matrix $V := (v_1 \eta, \cdots, v_m \eta)$, and recall that

$$Z_+ \eta_+^* \begin{pmatrix} I + iM & M \\ 0 & I - iM \end{pmatrix} \eta_+^* = e_+^*.$$

Note that the first m entries of η and η_+^* are identical, we have that $V = (I - iM)^{-1}(I + iM)^{-1}K$ where $K = (v_1, \cdots, v_m)$. Thus, we get

$$\sum_{j=1}^{m} (v_j \eta_j) = \operatorname{Tr} V = \operatorname{Tr}\left[(I - iM)^{-1} (I + iM)^{-1} K \right].$$

We calculate that $\partial \tilde{M} = -i(v_l v_j)_{lj}$. So, we have $\tilde{I}K = i\partial \tilde{M}$ and $\tilde{M} = \tilde{I}M$. Thus,

$$\sum_{j=1}^{m} .(v_j \eta_j) = \operatorname{Tr}\left[(\tilde{I} - i\tilde{M})^{-1} \tilde{I} (\tilde{I} + i\tilde{M})^{-1} i\partial \tilde{M} \right].$$

By above, we have that

$$\begin{aligned}
R := &\sum_{j=1}^{m} (v_j \eta_j) + \left(\frac{\partial \det(\tilde{I} - i\tilde{M})}{\det(\tilde{I} - i\tilde{M})} - \frac{\partial \det(\tilde{I} + i\tilde{M})}{\det(\tilde{I} + i\tilde{M})} \right) \\
&= \operatorname{Tr}\left[(\tilde{I} - i\tilde{M})^{-1} \tilde{I} (\tilde{I} + i\tilde{M})^{-1} i\partial \tilde{M} \right] - \operatorname{Tr}\left[\left((\tilde{I} - i\tilde{M})^{-1} + (\tilde{I} + i\tilde{M})^{-1} \right) i\partial \tilde{M} \right] + \operatorname{Tr}\left[2i(\tilde{I} - i\tilde{M})^{-1} \tilde{M} (\tilde{I} + i\tilde{M})^{-1} \partial \tilde{I} \right] \\
&= \operatorname{Tr}\left[2i(\tilde{I} - i\tilde{M})^{-1} \tilde{M} (\tilde{I} + i\tilde{M})^{-1} \partial \tilde{I} \right] - \operatorname{Tr}\left[(\tilde{I} - i\tilde{M})^{-1} \tilde{I} (\tilde{I} + i\tilde{M})^{-1} i\partial \tilde{M} \right].
\end{aligned}$$

We next compute

$$\begin{aligned}
&\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{M}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{I}\right]-\frac{1}{2}\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{M}\right]\\
=&\operatorname{Tr}\left[M(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{I}\right]-\frac{1}{2}\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{M}\right]\\
=&\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{I}M\right]-\frac{1}{2}\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\partial\tilde{M}\right].
\end{aligned}$$

Here, we use the fact that $\operatorname{Tr}(AB)=\operatorname{Tr}(BA)$. Notice that $\partial\tilde{I}M=-i\left(\frac{\lambda_l v_l v_j}{\lambda_l+\lambda_j}\right)_{lj}$ and $\partial\tilde{M}=-i(v_l v_j)_{lj}$, we have

$$R=\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\tilde{I}G\tilde{I}\right],$$

where

$$G:=-i\left(\frac{\lambda_l}{\lambda_l+\lambda_j}\right)_{lj}+i(\frac{1}{2})_{lj}.$$

We observe that both G and $\tilde{I}G\tilde{I}$ are antisymmetric.

By the same computation, we can similarly obtain

$$R=\operatorname{Tr}\left[(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\tilde{I}G\tilde{I}\right].$$

Hence,

$$2R=\operatorname{Tr}\left[\left((\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}+(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}\right)\tilde{I}G\tilde{I}\right].$$

Since $(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}+(\tilde{I}-i\tilde{M})^{-1}\tilde{I}(\tilde{I}+i\tilde{M})^{-1}$ is symmetric and $\tilde{I}G\tilde{I}$ is antisymmetric, we have $R=0$. Here, we use the fact that for A symmetric and B antisymmetric, we get

$$\operatorname{Tr}(AB)=\operatorname{Tr}(B^TA^T)=-\operatorname{Tr}(BA)=-\operatorname{Tr}(AB)=0.$$

Finally, recall that $\eta_j=\frac{\det H_{+,j}}{\det Z_+}$, we arrive at

$$\sum_{j=1}^m\frac{v_j\det\tilde{H}_{+,j}}{\det\tilde{Z}_+}=\frac{\partial(\det(\tilde{I}+i\tilde{M}))}{\det(\tilde{I}+i\tilde{M})}-\frac{\partial\det(\tilde{I}+i\tilde{M})}{\det(\tilde{I}+i\tilde{M})}.$$

□

By the proposition above, we know from equation (3) that

$$\begin{aligned}
u_x-iu_y&=2i(\partial_x-i\partial_y)\ln\frac{\det(\tilde{I}+i\tilde{M})}{\det(\tilde{I}-i\tilde{M})}\\
&=2i(\partial_x-i\partial_y)\ln\frac{\det(-iI+M)}{\det(iI+M)}.
\end{aligned}$$

This gives

$$u=2i\ln\frac{\det(-iI+M)}{\det(iI+M)}.$$

Note that here, the equation should be correct up to a constant, but we will fix this issue later.

Now, our goal is to recover the Hirota form in section 2. Define $v_j^*=\lambda_j^{-1}v_j$, then $M=\left(\frac{\lambda_j v_j^*}{\lambda_l+\lambda_j}\right)$. Consider the following polynomial:

$$g(r):=\det(irI+M).$$

We first prove the following proposition to compute $g(0)$.

Proposition 3.3.

$$\det\left(\frac{2\lambda_j}{\lambda_i+\lambda_j}\right)_{ij} = \prod_{1\le\alpha\le\beta}\left(\frac{\lambda_\alpha-\lambda_\beta}{\lambda_\alpha+\lambda_\beta}\right)^2.$$

Proof. In fact, we have

$$\det\left(\frac{1}{a_i+b_j}\right)_{ij} = \frac{\prod_{1\le i<j\le m}(a_i-a-j)(b_i-b_j)}{\prod_{i,j=1}^m(a_i+b_j)}. \tag{3.29}$$

This is because when we multiply the determinant by the denominator on RHS, we must obtain a polynomial on a_i, b_j by the definition of the determinant. Moreover, the polynomial must have degree n^2-n by simple calculation. Also, it must have divisors (a_i-a_j), (b_i-b_j) since the determinant vanish when $a_i=a_j$ or $b_i=b_j$. This gives equation (3.29) after noticing that the leading coefficient of $\prod_{i=1}^m a_i b_i$ is 1. Next, plugging in $a_i=b_i=\lambda_i$ in equation (3.29), we have our desired equation. Note that here the product $\prod_{i=j=1}^m(a_i+b_j)$ cancels out $\prod_{j=1}^m 2\lambda_j$ in the expression of the determinant. □

Let

$$b(l_1,\cdots,l_j) = \prod_{1\le\alpha\le\beta\le j}\left(\frac{\lambda_{l_\alpha}-\lambda_{l_\beta}}{\lambda_{l_\alpha}+\lambda_{l_\beta}}\right)^2,$$

then the above proposition implies

$$g(0) = \det M = b(1,2,\cdots,m)v_1^*\cdots v_m^*.$$

Note that here the factor "2"s are absorbed by the phase parameter in v_j^*, i.e. $2e^{ax+by+c}=e^{ax+by+c'}$.

The coefficient contributed by terms with ir is then the sum of all the $m-1$-th order principal minors of M:

$$\sum_{l_1<\cdots<l_{m-1}}[b(l_1,\cdots,l_{m-1})v_{l_1}^*\cdots v_{l_{m-1}}^*].$$

Therefore, we have

$$\det(iI+M) = \sum_{j=1}^m\left(\sum_{l_1<\cdots<l_j}\left[i^{m-j}b(l_1,\cdots,l_j)v_{l_1}^*\cdots v_{l_j}^*\right]\right).$$

Notice that here $b(l_1,\cdots,l_j)$ corresponds to $\alpha(l_1,\cdots,l_j)$ up to a sign, so we have recovered the Hirota form. So, it remains to compare the choice of parameters for u and $U_n+\pi$ with the same Hirota form.

Notice that in v_j^*, the coefficient before x after taking the ln is $-\frac{K(\lambda_j)}{2}i$, which equals

$$\frac{Im\lambda_j}{2}\left(1+\frac{1}{(Re\lambda_j)^2+(Im\lambda_j)^2}\right)-i\frac{Re\lambda_j}{2}\left(1-\frac{1}{(Re\lambda_j)^2+(Im\lambda_j)^2}\right).$$

The coefficient before y is $-\frac{1}{2}(\lambda_j+\frac{1}{\lambda_j})$, which equals

$$\frac{Re\lambda_j}{2}\left(1+\frac{1}{(Re\lambda_j)^2+(Im\lambda_j)^2}\right)+i\frac{Im\lambda_j}{2}\left(1-\frac{1}{(Re\lambda_j)^2+(Im\lambda_j)^2}\right).$$

Since u and $U_n+\pi$ have same asymptotic behavior as $y\to+\infty$, and u is real-valued, we must have the following:

1. $(Re\lambda_j)^2+(Im\lambda_j)^2=1$ to make sure u is real,
2. $m=n$ so that u and $U_n+\pi$ has same number of ends,
3. $Im\lambda_j=p_j$ and $Re\lambda_j=q_j$ so that u and $U_n+\pi$ has same asymptotic behavior at infinity,

4. the constant issue is resolved by asymptotic behavior.

Therefore, $u = U_n + \pi$.

Next, we want to prove $a = \hat{a} = \prod_{j=1}^{m} \frac{\lambda-\lambda_j}{\lambda+\lambda_j}$ for λ with $Im\lambda \geq 0$.
For $\lambda \in \mathbb{R}\backslash\{0\}$, we recall that $\det\Phi_{\pm} = 1$ and $b = 0$. By the relation between $\Phi_{\pm}$, we obtain

$$\det\Phi_{+} = \det\Phi_{-} \det \begin{bmatrix} a(\lambda) & 0 \\ 0 & a(-\lambda) \end{bmatrix}.$$

This gives $a(-\lambda)a(\lambda) = 1$. Now, we define

$$\beta(\lambda) := a(\lambda) \prod_{j=1}^{m} \frac{\lambda+\lambda_j}{\lambda-\lambda_j}.$$

Since a only has simple zeroes λ_j, we know that $\beta(\lambda)$ is analytic in the upper half λ-plane. By equations (3.22) and (3.25), and using the asymptotic behavior of Φ_+ as $x \to +\infty$, we obtain

$$\frac{1}{a(\lambda)} = \frac{\Phi_{-,1}}{\Phi_{+,1}} = 1 + \sum_{j=1}^{m} \frac{\tilde{c}_j}{\lambda-\lambda_j}\frac{1}{c_j},$$

where we use $\Phi_{-,2} = \frac{\Phi_{+,1}}{c_j}$ in the last step.

Thus, we have

$$\frac{1}{a(\lambda)} = 1 + \sum_{j=1}^{m} \frac{d_j}{\lambda-\lambda_j}$$

for some constant d_j. We also have

$$\frac{1}{a(-\lambda)} = 1 - \sum_{j=1}^{m} \frac{d_j}{\lambda+\lambda_j}.$$

Combining these two equations, we have

$$1 = \frac{1}{a(\lambda)}\frac{1}{a(-\lambda)} = \frac{P(\lambda)}{\prod(\lambda-\lambda_j)}\frac{Q(\lambda)}{\prod(\lambda+\lambda_j)},$$

for some polynomial $P(\lambda), Q(\lambda)$.

Notice that $P(\lambda)$ has no zeros among λ_js and $Q(\lambda)$ has no zeros among $-\lambda_j$s, we must have

$$\prod(\lambda+\lambda_j)\Big|P(\lambda), \prod(\lambda-\lambda_j)\Big|Q(\lambda).$$

Since $\deg P = \deg Q = m$ and P, Q both have leading coefficient 1, we conclude that $P(\lambda) = \prod(\lambda+\lambda_j)$ and $Q(\lambda) = \prod(\lambda-\lambda_j)$.

In other words, we deduce that

$$a(\lambda) = \prod_{j=1}^{m} \frac{\lambda-\lambda_j}{\lambda+\lambda_j}$$

for $\lambda \in \mathbb{R}\backslash\{0\}$. This means that $\beta(\lambda) = 1$ on the real line except for the origin. Let us write $\beta(\lambda) = a(\lambda)R(\lambda)$ where $R(\lambda)$ is the product $\prod \frac{\lambda-\lambda_j}{\lambda+\lambda_j}$. Then, we observe that $\partial_\lambda R = 0$ on the real axis and $\partial_\lambda R < 0$ along the vertical direction in the upper λ-half plane. This implies that β is bounded in the upper λ-half plane. By Liouville's theorem, we have $\beta = 1$ in the entire upper λ-half plane, which concludes the form of $a(\lambda) = \hat{a}(\lambda)$ we want.

□

Next, we compute the scattering data of $U_n+\pi$. By lemma 3.7, it suffices to show that the zeros of the scattering data $\hat{a}$ are simple to conclude that $u=U_n+\pi$, which is exactly theorem 3.1. The computation is divided into two steps. First, we compute the scattering data of the 2-ended solution. Then, for $n>1$, we observe that for a suitable choice of parameters, as $y\to\infty$, $U_n+\pi$ splits into n heteroclinic solutions, and passing each one of these solutions along x-direction, we gain a factor $\frac{\lambda-\lambda_j}{\lambda+\lambda_j}$. These factors ensure that the zeroes of a are always simple when we build up the product inductively.

Lemma 3.8. Let p_j,q_j be parameters in the solution $U_n+\pi$ and $\lambda_j=-q_j+p_j$, then, for $\lambda\in\mathbb{R}^{2,+}$, the scattering data $\hat{a}$ of $U_n+\pi$ is given by

$$\hat{a}(\lambda)=\prod_{j=1}^{n}\frac{\lambda-\lambda_j}{\lambda+\lambda_j}.$$

Proof. Our first goal is to compute $\hat{a}$ for $U_1+\pi$. Inspired by equation (3.25), we define $\hat{\Phi}_{-,1}$ by:

$$\hat{\Phi}_{-,1}(x,y,\lambda)=\exp\left(\frac{K(\lambda)i}{4}x\right)(1,0)^T+\exp\left(\frac{K(\lambda)i}{4}x\right)\sum_{j=1}^{n}\left[\frac{\tilde{c}_j(y)}{\lambda-\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\hat{\Phi}_{-,2}(x,y,\lambda_j)\right].$$

Here

$$\tilde{c}_j(y):=\frac{\hat{c}_j(0)\exp\left(-\frac{1}{2}(\lambda_j+\frac{1}{\lambda_j})y\right)}{2\lambda_j}\prod_{l\neq j}\frac{\lambda_j+\lambda_l}{\lambda_j-\lambda_l},\tag{3.30}$$

where $\hat{c}_j(0)$ is a real parameter and $\hat{\Phi}_{-,2}=(\hat{\Phi}_{-,12},\hat{\Phi}_{-,22})^T$ is given by

$$\hat{\Phi}_{-,12}(x,y,\lambda_j)=\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\frac{\det H_{+,j+n}-i\det H_{+,j}}{\det Z_+},$$

$$\hat{\Phi}_{-,22}=\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\frac{\det H_{+,j}}{\det Z_+}.$$

With the definition of $\tilde{c}_j$, we define

$$m_{lj}:=\frac{\tilde{c}_j(y)}{\lambda_l+\lambda_j}\exp\left(-\frac{K(\lambda_j)i}{2}x\right).$$

We underline that the $\tilde{c}_j$ here is only defined by equation (3.30), which doesn't require the assumption that a only has simple zeros.

Now, we want to show $\partial_x\hat{\Phi}_{-,1}=A\hat{\Phi}_{-,1}$. For $n=1$, it turns out that this can be done by direct calculation.

First recall that in the previous lemma, the fact that $\det(iI+M)=F$ doesn't require the assumption that a has simple zeros since the calculation only depends on equation (3.25) itself. So, in our case, we get

$$U_1+\pi=2i\ln\frac{-i+m_{11}}{i+m_{11}}.$$

This gives

$$e^{i(U_1+\pi)}=\left(\frac{-i+m_{11}}{i+m_{11}}\right)^2,e^{-i(U_1+\pi)}=\left(\frac{i+m_{11}}{-i+m_{11}}\right)^2.$$

So, we have

$$\sin U_1=\frac{1}{2i}\left[\left(\frac{-i+m_{11}}{i+m_{11}}\right)^2-\left(\frac{i+m_{11}}{-i+m_{11}}\right)^2\right]\tag{3.31}$$

$$\cos U_1=\frac{1}{2}\left[\left(\frac{i+m_{11}}{-i+m_{11}}\right)^2+\left(\frac{-i+m_{11}}{i+m_{11}}\right)^2\right].\tag{3.32}$$

Moreover,

$$\hat{\Phi}_{-,1}(x,y,\lambda) = \exp\left(\frac{K(\lambda)i}{4}x\right)(1,0)^T + \exp\left(\frac{K(\lambda)i}{4}x\right)\left[\frac{\hat{c}_1(y)}{2(\lambda-\lambda_1)\lambda_1}\exp\left(-\frac{K(\lambda_j)i}{4}x\right)\hat{\Phi}_{-,2}(x,y,\lambda_1)\right],$$

where

$$\hat{\Phi}_{-,12}(x,y,\lambda_1) = \exp\left(-\frac{K(\lambda_j)i}{4}x\right)\frac{m_{11}}{1+m_{11}^2},$$

$$\hat{\Phi}_{-,22}(x,y,\lambda_1) = \exp\left(-\frac{K(\lambda_j)i}{4}x\right)\frac{1}{1+m_{11}^2}.$$

Here, we use $\det H_{+,1} = 1$ and $\det H_{+,2} = -i - m_{11}$.

Recall that

$$\hat{A}\hat{\Phi}_{-,1} = \frac{i}{4}\left[\left(\lambda + \frac{\cos U_1}{\lambda}\right)\sigma_3 - [(\partial_x - i\partial_y)U_1]\sigma_2 + \frac{\sin U_1}{\lambda}\sigma_1\right]\hat{\Phi}_{-,1}.$$

We compute the first component J_1 of $\hat{A}\hat{\Phi}_{-,1}$:

$$\begin{aligned} J_1 = \frac{i}{4}\left(\lambda + \frac{\cos U_1}{\lambda}\right)\exp\left(\frac{K(\lambda)i}{4}x\right)\left[1 + \frac{\hat{c}_1(y)}{2(\lambda-\lambda_1)\lambda_1}\exp\left(-\frac{K(\lambda_1)i}{2}x\right)\frac{m_{11}}{1+m_{11}^2}\right] \\ + \frac{1}{4}[(\partial_x - i\partial_y)U_1]\left[1 + \frac{\hat{c}_1(y)}{2(\lambda-\lambda_1)\lambda_1}\exp\left(-\frac{K(\lambda_1)i}{2}x\right)\frac{1}{1+m_{11}^2}\right] \\ + \frac{i}{4}\frac{\sin U_1}{\lambda}\left[1 + \frac{\hat{c}_1(y)}{2(\lambda-\lambda_1)\lambda_1}\exp\left(-\frac{K(\lambda_1)i}{2}x\right)\frac{1}{1+m_{11}^2}\right]. \end{aligned}$$

Recall that

$$m_1 1 = \frac{\hat{c}_1(y)}{4\lambda_1^2}\exp\left(-\frac{K(\lambda_1)i}{2}x\right).$$

So, we obtain

$$J_1\exp\left(-\frac{K(\lambda)i}{4}x\right) = \frac{i}{4}\left(\lambda + \frac{\cos U_1}{\lambda}\right)\left(1 + \frac{2\lambda}{\lambda-\lambda_1}\frac{m_{11}^2}{1+m_{11}^2}\right) + \frac{1}{4}[(\partial_x - i\partial_y)U_1]\left(\frac{2\lambda}{\lambda-\lambda_1}\frac{m_{11}}{1+m_{11}^2}\right) + \frac{i}{4}\frac{\sin U_1}{\lambda}\left(\frac{2\lambda}{\lambda-\lambda_1}\frac{m_{11}}{1+m_{11}^2}\right).$$

On the other hand, the first component J_1^* of $\partial_x\hat{\Phi}_{-,1}$ is

$$J_1^* = \frac{K(\lambda)i}{4}\exp\left(\frac{K(\lambda)i}{4}x\right)\left(1 + \frac{2\lambda}{\lambda-\lambda_1}\frac{m_{11}^2}{1+m_{11}^2}\right) + \exp\left(\frac{K(\lambda)i}{4}x\right)\frac{2\lambda_1}{\lambda-\lambda_1}\frac{-K(\lambda_1)im_{11}^2}{(1+m_{11}^2)^2},$$

where we use $m_{11}' = -\frac{K(\lambda_1)i}{2}m_1 1$ and $(\frac{m_{11}^2}{1+m_{11}^2})' = \frac{2m_{11}m_{11}'}{(1+m_{11}^2)^2}$ for the second term.

Finally, we compute

$$4(J_1 - J_1^*)\exp\left(-\frac{K(\lambda)i}{4}x\right) = i\frac{1+\cos U_1}{\lambda}\left(1 + \frac{2\lambda_1}{\lambda-\lambda_1}\frac{m_{11}^2}{1+m_{11}^2}\right) + i\frac{\sin U_1}{\lambda}\frac{2\lambda_1}{\lambda-\lambda_1}\frac{m_{11}}{1+m_{11}^2} - \frac{8i}{\lambda-\lambda_1}\frac{m_{11}^2}{(1+m_{11}^2)^2},$$

where we use $(\partial_x - i\partial_y)U_1 = -\frac{4i\lambda_1 m_{11}}{1+m_{11}^2}$.

Inserting equations (3.31),(3.32) into the above equation, we see that RHS is identically zero. This proves that the first entry of $\partial_x\hat{\Phi}_{-,1} - \hat{A}\hat{\Phi}_{-,1}$ vanishes. Similarly, the second entry of $\partial_x\hat{\Phi}_{-,1} - \hat{A}\hat{\Phi}_{-,1}$ is also zero by direct computation. Thus, we have $\partial_x\hat{\Phi}_{-,1} = \hat{A}\hat{\Phi}_{-,1}$. Moreover, by the asymptotic behavior of $\hat{A}$, we observe that $\hat{\Phi}_{-,1}$ satisfy the required asymptotic behavior as $x \to -\infty$:

$$\lim_{x\to-\infty}\hat{\Phi}_{-,1}\exp\left(-\frac{K(\lambda)i}{4}x\right) = (1,0)^T.$$

Using the explicit expression of $\hat{\Phi}_{-,1}$, and recall that $\hat{\Phi}_{+,1} = \hat{a}\hat{\Phi}_{-,1}$ for $\lambda \in \mathbb{R}\backslash\{0\}$, we use the same argument in lemma 3.7 to conclude

$$\hat{a}(\lambda)^{-1} = 1 + \frac{d_1}{\lambda-\lambda_1}. \tag{3.33}$$

By lemma 3.7, we know that $d_1 = \frac{\hat{c}_1}{c_1} = 2\lambda_1$ in our case, so we obtain

$$\hat{a}(\lambda) = \frac{\lambda-\lambda_1}{\lambda+\lambda_1}, \lambda \in \mathbb{R}\backslash\{0\}. \tag{3.34}$$

However, we don't know whether λ_1 is a zero of $\hat{a}$ at this point. This means that we cannot copy the proof in lemma 3.7 to conclude that $\hat{a} = \frac{\lambda-\lambda_1}{\lambda+\lambda_1}$ on the upper λ-half plane. Instead, we would like to show that $\hat{a}$ cannot have repeated zeros so that we can then apply lemma 3.7 to conclude that

$$a(\lambda) = \hat{a}(\lambda) = \frac{\lambda-\lambda_1}{\lambda+\lambda_1}.$$

Assume that we have the opposite such that λ_j^* is a zero of $\hat{a}$ on $\mathbb{R}^{2,+}$ with multiplicity $\kappa > 1$. Applying the operator $\mathscr{P}$ and the residue theorem to the following equation:

$$\hat{\Phi}_{-,1}e^{-\frac{K(\lambda_j^*)i}{4}x} - (1,0)^T = \frac{\hat{\Phi}_{+,1}}{\hat{a}}e^{-\frac{K(\lambda_j^*)i}{4}x} - (1,0)^T,$$

then there are terms on LHS of the form

$$\lim_{\lambda-\lambda_j^*}(\frac{d}{d\lambda})^{\kappa-1}\frac{(\lambda-\lambda_j)^\kappa}{\hat{a}}\frac{\hat{\Phi}_{+,1}}{\xi-\lambda}e^{-\frac{K(\lambda_j^*)i}{4}x} = \frac{\hat{\Phi}_{+,1}e^{-\frac{K(\lambda_j^*)i}{4}x}}{(\xi-\lambda_j^*)^\kappa}.$$

Then, dividing LHS by $\hat{\Phi}_{+,1}$, we know that $\hat{a}^{-1}$ cannot have form $\frac{\lambda-\lambda_1}{\lambda+\lambda_1}$ on the real line since $\kappa > 1$, which is a contradiction.

Next, we would like to consider the scattering data $\hat{a}$ of $U_n + \pi$ for $n > 1$. We first consider the $\hat{a}$ for the 4-end solution $U_2 + \pi$. Let L_1, L_2 be the two half-straight lines of the two ends of $U_2 + \pi$ on the upper half $x - y$ plane. By lemma 2.8, along each end, as $y \to +\infty$, the solution converges to the 1-dimentional solution $U_1 + \pi$ with suitable choice of p_j, q_j, η_j^0. Let $U_{1,\alpha} + \pi$ be the 1-dimensinoal solution along L_1 on the left and $U_{1,\beta} + \pi$ be the one along L_2 on the right.

Let $\hat{\Phi}_{-,1,\alpha}$ and $\hat{\Phi_{-,1,\beta}}$ be the corresponding Yost solutions. Then,

$$\partial_x\hat{\Phi}_{-,1,\alpha} = \hat{A}\hat{\Phi}_{-,1,\alpha}, \partial_x\hat{\Phi}_{-,1,\beta} = \hat{A}\hat{\Phi}_{-,1,\beta}.$$

Let $\hat{\Phi}_{-,1,\alpha}$ and $\hat{\Phi_{-,1,\beta}}$ be the corresponding Yost solutions. Then,

$$\partial_x\hat{\Phi}_{-,1,\alpha} = \hat{A}_\alpha\hat{\Phi}_{-,1,\alpha}, \partial_x\hat{\Phi}_{-,1,\beta} = \hat{A}_\beta\hat{\Phi}_{-,1,\beta}.$$

Moreover, we have

$$\lim_{x\to-\infty}\hat{\Phi}_{-,1,\alpha}e^{-\frac{K(\lambda)i}{4}x} = \lim_{x\to-\infty}\hat{\Phi}_{-,1,\beta}e^{-\frac{K(\lambda)i}{4}x} = (1,0)^T.$$

Let us also denote $\hat{\Phi}_{-,1}$ be the Yost solution of $U_2 + \pi$ which satisfies

$$\partial_x\hat{\Phi}_{-,1} = \hat{A}\hat{\Phi}_{-,1}$$

and

$$\lim_{x\to-\infty}\hat{\Phi}_{-,1}\exp-\left(\frac{K(\lambda)i}{4}x\right)=(1,0)^T.$$

Recall that for $\lambda\in\mathbb{R}\backslash\{0\}$, we have

$$\hat{\Phi}_{+,1}=\hat{a}\hat{\Phi}_{-,1}$$

where $\hat{\Phi}_{+,1}$ is the another Yost solution satisfies

$$\lim_{x\to+\infty}\hat{\Phi}_{+,1}\exp-\left(\frac{K(\lambda)i}{4}x\right)=(1,0)^T.$$

To obtain the formulae for $\hat{a}$, we should analyze the asymptotic behavior of $\hat{\Phi}_{-,1}$ as $x\to+\infty$.

Let us denote L^* be the bisector of lines L_1 and L_2. Since $U_2+\pi$ approaches $U_{1,\alpha}$ exponentially fast along L_1, we use the argument in lemma 3.5 to get

$$|\hat{\Phi}_{-,1}-\hat{\Phi}_{1,\alpha}|\leq C\exp\left(-\delta_1\sqrt{x^2+y^2}\right),\ \textit{if } y>0 \textit{ and } (x,y) \textit{ is at left of } L^*.$$

Here $\delta_1>0$ is some constant.

Moreover, by the scattering data $\hat{a}(\lambda)=\frac{\lambda-\lambda_1}{\lambda+\lambda_1}$ of $U_{1,\alpha}$, we know that for (x,y) at the left of L^*, we have

$$|\hat{\Phi}_{-,1,\alpha}(x,y)\exp\left(-\frac{K(\lambda)i}{4}x\right)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}(1,0)^T|\leq C\exp\left(-\delta_2 d(x,y)\right),$$

where $\delta_2>0$ and $d(x,y)$ is the distance from (x,y) to L_1.

Combining the above equations, we obtain

$$|\hat{\Phi}_{-,1}(x,y)\exp\left(-\frac{K(\lambda)i}{4}x\right)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}(1,0)^T|\leq C\exp\left(-\delta d(x,y)\right)$$

for some small enough $\delta>0$.

Next, we define

$$\hat{\Phi}^*_{-,1,\beta}:=\frac{\lambda+\lambda_1}{\lambda-\lambda_1}\hat{\Phi}_{-,1,\beta}.$$

Similarly, we have

$$|\hat{\Phi}^*_{-,1,\beta}\exp\left(-\frac{K(\lambda)i}{4}x\right)-\frac{\lambda+\lambda-1}{\lambda-\lambda_1}(1,0)^T|\leq C\exp\left(-\tilde{\delta}\tilde{d}(x,y)\right),\ \textit{if } (x,y)\in L^*.$$

Reducing δ if necessary, we have

$$|\hat{\Phi}_{-,1,}(x,y)-\hat{\Phi}^*_{-,1,\beta}(x,y)|\leq C\exp\left(-\delta y\right),\ \textit{if } y>0 \textit{ and } (x,y)\in L^*.$$

Again by the proof in lemma 3.5, we obtain that for (x,y) at the left of L^*

$$|\hat{\Phi}_{-,1}(x,y)-\hat{\Phi}^*_{-,1,\beta}(x,y)|\leq C\exp\left(-\delta y\right)+C\exp\left(-\delta\sqrt{x^2+y^2}\right). \tag{3.35}$$

Notice that as $x\to+\infty$,

$$\hat{\Phi}^*_{-,1,\beta}e^{-\left(\frac{K(\lambda)i}{4}x\right)}=\frac{\lambda+\lambda_1}{\lambda-\lambda_1}\hat{\Phi}_{-,1,\beta}e^{-\left(\frac{K(\lambda)i}{4}x\right)}=\frac{\lambda+\lambda_1}{\lambda-\lambda_1}\frac{\lambda+\lambda_2}{\lambda-\lambda_2}\hat{\Phi}_{+,1,\beta}e^{-\left(\frac{K(\lambda)i}{4}x\right)},$$

by the asymptotic behavior of $\hat{\Phi}_{+,1,\beta}$, we have

$$\lim_{x\to+\infty} |\hat{\Phi}_{-,1}(x,y)e^{-\left(\frac{K(\lambda)i}{4}x\right)} - \frac{\lambda+\lambda_1}{\lambda-\lambda_1}\frac{\lambda+\lambda_2}{\lambda-\lambda_2}(1,0)^T| \leq C\exp(-\delta y).$$

Recall that $\hat{a}$ is the ratio of $\hat{\Phi}_{+,1}$ and $\hat{\Phi}_{-,1}$, we can send $y \to +\infty$, we deduce

$$\hat{a}(\lambda) = \frac{\lambda+\lambda_1}{\lambda-\lambda_1}\frac{\lambda+\lambda_2}{\lambda-\lambda_2}, if\, \lambda \in \mathbb{R}\backslash\{0\}.$$

Finally, for general $U_n + \pi$ where $n \geq 0$, we repeat the above argument to get an additional factor $\frac{\lambda-\lambda_j}{\lambda+\lambda_j}$ one at a time. This implies

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda-\lambda_j}{\lambda+\lambda_j}, if\, \lambda \in \mathbb{R}\backslash\{0\}.$$

Using the argument for $n = 1$, we can show that all these zereos λ_j are simple, and conclude by lemma 3.7 that

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda-\lambda_j}{\lambda+\lambda_j}, if\, \lambda \in \mathbb{R}^{2,+}.$$

□

With all the lemmas above, we have all the scattering data necessary to conclude the main theorem of this section.

□

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