Solutions to The Elliptic Sine-Gordon Equation in The Plane

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Introduction

The elliptic sine-Gordon equation

1. is a semilinear elliptic PDE with a special double well potential.
2. originated from the classical (hyperbolic) sine-Gordon equation.
3. is an integrable PDE so we can use the method of inverse scattering transform **[4]** to analyze its solutions.
4. has solutions with a nice explicit formula, and the nodal set of these solutions behaves like multiple straight-half lines at infinity away from the origin, and we call these solutions the unique multi-end solutions.

A Family of Multi-end Solutions to the Elliptic sine-Gordon Equation

Our goal is to write down a family of explicit, real-valued, nonsingular solutions of the elliptic sine-Gordon equation:

$$-\partial_x^2 u - \partial_y^2 u = \sin u. \tag{1}$$

Following Liu and Wei's work **[1]**, we take the bi-logarithmic transformation:

$$u = 2i \ln \frac{\bar{F}}{F}. \tag{2}$$

Then, using the bilinear derivative operator D **[3]**, we find the bilinear form of the elliptic sine-Gordon equation:

$$(D_x^2 + D_y^2)(F \cdot F) + \frac{1}{2}(\bar{F}^2 - F^2) = \lambda F^2. \tag{3}$$

Now, the key idea is to seek solutions with formal expansion of powers of ε:

$$F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots . \tag{4}$$

Comparing the $O(\varepsilon^k)$ terms, we can solve F_k inductively and give the explicit expression for the function F.

Denote the real and imaginary part of F by f_n, g_n, we get a family of solutions to the elliptic sine-Gordon equation by equation (2):

$$U_n := 2i \ln \frac{\bar{F}}{F} = 4 \arctan \frac{g_n}{f_n}. \tag{5}$$

Moreover, we can analyze the asymptotic behavior of these solutions by directly investigating the explicit expression. And it turns out that

$$\lim_{j \to +\infty} U_n(x_j, y_j) = \begin{cases} 4\arctan(\exp(\eta_k + \beta_k)) - \pi, & k \text{ odd,} \\ 4\arctan(\exp(-\eta_k - \beta_k)) - \pi, & k \text{ even.} \end{cases}$$

Here β_k is a constant, and $\eta_j = p_j x + q_j y + \eta_j^0$ for some real constant p_j, q_j, η_j^0, $j = 1, 2, \cdots, n$. So, we can see that the nodal set of U_n is indeed $2n$ straight-half lines.

In the special case $n = 2$, if we choose $p_1 = p_2 = q_1 = -q_2 = \frac{\sqrt{2}}{2}$, then the solution is the classical saddle solution:

$$4 \arctan \left(\frac{\cosh\left(\frac{y}{\sqrt{2}}\right)}{\cosh\left(\frac{x}{\sqrt{2}}\right)} \right). \tag{6}$$

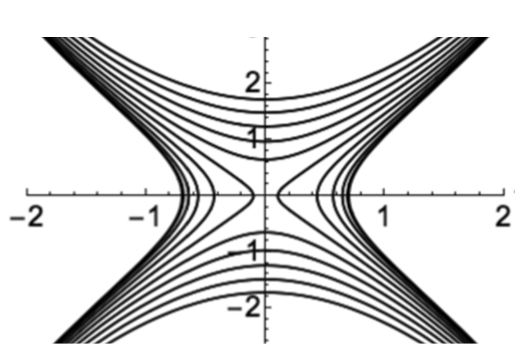

Figure 1: Nodal set of the "saddle solution" to the elliptic sine-Gordon equation.

Inverse Scattering Transform and The Classification of Multiple-End Solutions

Recall the correspondence $\varphi + \pi \longleftrightarrow u$, the multi-end solutions to $\Delta u = \sin u$ corresponds to solutions to $-\Delta\varphi = \sin\varphi$ whose π level sets are asymptotic to $2n$ half-straight lines. We prove a classification theorem of the solutions to the elliptic sine-Gordon equation (1) by using the inverse scattering transform developed in **[4]**. The main result is given by the following theorem:

Theorem 1. Suppose φ is an $2n$-ended solution of the equation $-\Delta\varphi = \sin\varphi$. Then there exist parameters p_j, q_j, η_j^0, $j = 1, \cdots, n$, such that $\varphi = U_n$, where U_n is defined in equation (5).

Now, we give the sketch of the proof:

- The equation (1) have a Lax pair:

$$\begin{cases} \Phi_x = A\Phi, \\ \Phi_y = B\Phi. \end{cases} \tag{7}$$

- The compatibility of equations in (7) gives

$$A_y + AB = B_x + BA. \tag{8}$$

  This is equivalent to equation (1).
- Using Picard iteration and certain assumptions on parameters, we derive the Jost solutions $\Phi_\pm$.
- For each fixed $y \in$, Φ_+, Φ_- are solutions to the same ODE system. Hence, they are related by

$$\Phi_+(x, y, \lambda) = \Phi_-(x, y, \lambda) \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ b^*(\lambda, y) & a^*(\lambda, y) \end{bmatrix}, \tag{9}$$

  for some functions a, a^*, b, b^* independent of x.
- Similarly define the a, b functions for U_n, and denote them by $\hat{a}, \hat{b}$. We prove that the scattering data for φ and U_n are the same, i.e. $a = \hat{a}$ and $b = \hat{b}$.
- Assuming that a only has simple zeroes, we use the inverse scattering transform to derive the same Hirota form in the equation (5) for φ as that of U_n.
- Finally, we compute the explicit formula for the scattering data a to show that the zeroes are indeed simple. In fact,

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}. \tag{10}$$

Graphs of the Solutions

- For $n = 1$, we have the most simple Heteroclinic solution given by

$$U_1 = 4\arctan(e^x) - \pi.$$

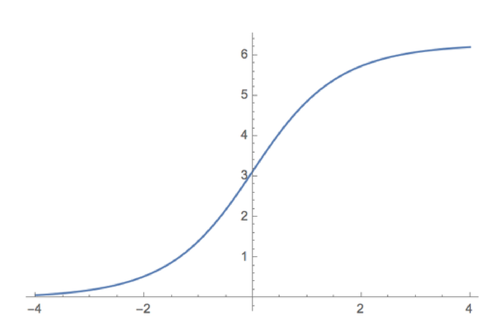

Figure 2: The Heteroclinic solution to the elliptic sine-Gordon equation in one dimension.

  We can see that the nodal set behaves like 2 half-straight lines away from the origin, i.e. it is a 2-end solution.
- For $n = 2$, we use numerical methods to give approximate solutions to the elliptic sine-Gordon equations by considering the system of second-order ODEs(Lax Pair). Depending on the choices of parameters, the solutions can vary largely in shape. But their nodal sets all behave like 4 half-straight lines at infinity.

Graphs of the Solutions

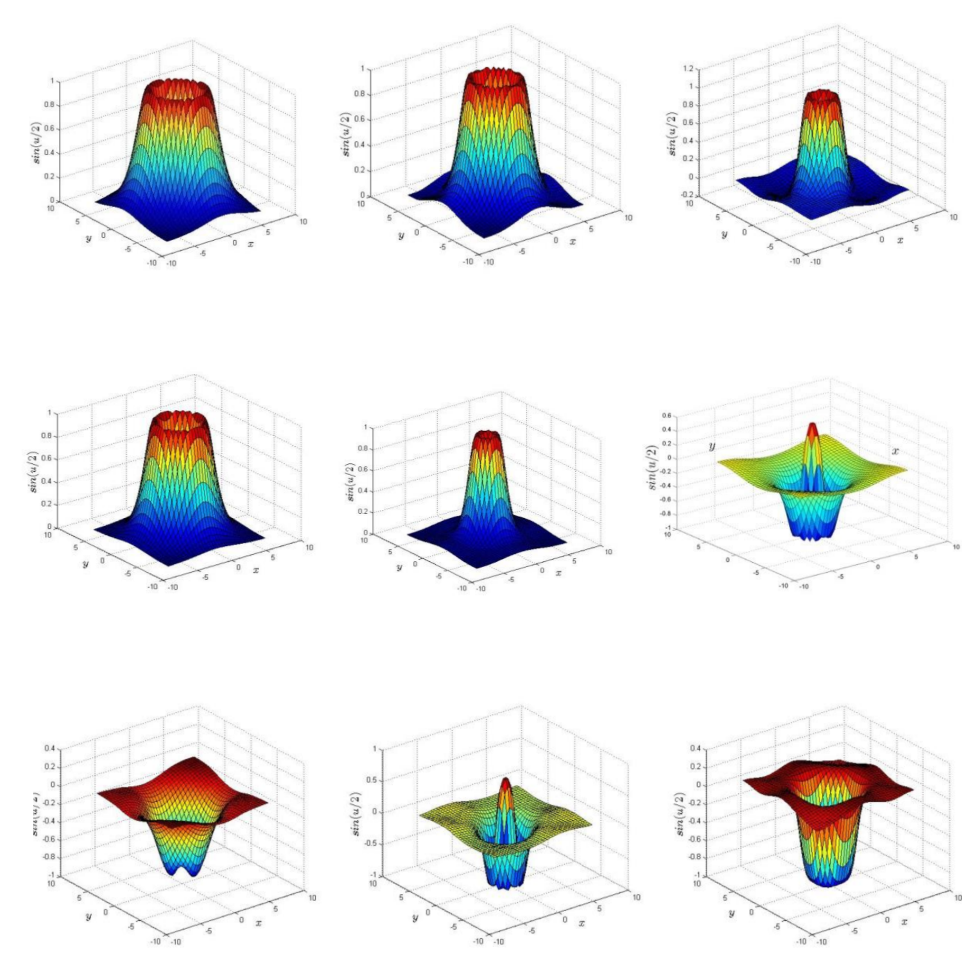

Figure 3: Graphs of different 4-end solutions to the elliptic sine-Gordon equation in the plane.

Future Work

I mainly investigate section 2 and section 5 of Liu and Wei's paper on the elliptic sine-Gordon equation. But, there are still sections $3, 4,$ and 6 remain to be studied.

- Section 3: The Bäcklund transformation is used in the study of the hyperbolic sine-Gordon equation, where we can derive the multi-soliton solutions in an algebraic way.
- Section 4: Besides the uniqueness of the multi-end solutions, it turns out that these solutions U_n are L^∞ nondegenerate.
- Section 6: Another important fact about the solutions U_n is that they have a finite Morse index.
- Future goal: It is a natural question to ask whether we can obtain these beautiful results of the elliptic sine-Gordon equation in high dimensions. My goal is to generalize some results using Liu and Wei's method.

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