The elliptic sine-Gordon equation
1. is a semilinear elliptic PDE with a special double well potential.
2. originated from the classical (hyperbolic) sine-Gordon equation.
3. is an integrable PDE so we can use the method of inverse scattering transform [4] to analyze its solutions.
4. has solutions with a nice explicit formula, and the nodal set of these solutions behaves like multiple straight-half lines at infinity away from the origin, and we call these solutions the unique multi-end solutions.

A Family of Multi-end Solutions to the Elliptic sine-Gordon Equation

Our goal is to write down a family of explicit, real-valued, nonsingular solutions of the elliptic sine-Gordon equation:

\[ -\partial_{xx} u - \partial_{yy} u = \sin u. \]  

(1)

Following Liu and Wei’s work [1], we take the bi-logarithmic transformation:

\[ u = 2 \ln \sqrt{2} \frac{F(y)}{F(x)}. \]

(2)

Then, using the bilinear derivative operator \( D \) [3], we find the bilinear form of the elliptic sine-Gordon equation:

\[ (D^2 - D_0^2)(F \cdot \bar{F}) + \frac{1}{2}(F^2 - F_0^2) = \lambda F^2. \]

(3)

Now, the key idea is to seek solutions with formal expansion of powers of \( \varepsilon \):

\[ F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots. \]

(4)

Comparing the \( O(\varepsilon^3) \) terms, we can solve \( F_2 \) inductively and give the explicit expression for the function \( F \).

Denote the real and imaginary part of \( F \) by \( u, v \), so we get a family of solutions to the elliptic sine-Gordon equation by equation (2):

\[ U_0 = -2 \ln \sqrt{2} \frac{F(y)}{F(x)} \]

(5)

Moreover, we can analyze the asymptotic behavior of these solutions by directly investigating the explicit expression. And it turns out that

\[ \lim_{y \to \pm \infty} U_0(x; y) = \begin{cases} \arctan(\eta_0 + \beta_0) = \pi, & k \text{ odd,} \\ \arctan(\eta_0 - \beta_0) = 0, & k \text{ even.} \end{cases} \]

Here \( \beta_0 \) is a constant, and \( \eta_0 = p_j + g_j y + q_j \) for some real constant \( p_j, q_j, g_j \), \( j = 1, \ldots, n \). So, we can see that the nodal set of \( U_0 \) is indeed \( 2n \) straight-half lines.

In the special case \( n = 2 \), if we choose \( p_1 = p_2 = q_1 = -q_2 = \frac{\pi}{2} \), then the solution is the classical saddle solution:

\[ 4 \arctan \left( \frac{\sinh \left( \frac{\pi}{4} \right)}{\sinh \left( \frac{\pi}{2} \right)} \right). \]

(6)

Figure 1: Nodal set of the “saddle solution” to the elliptic sine-Gordon equation.

Future Work

I mainly investigate section 2 and section 5 of Liu and Wei’s paper on the elliptic sine-Gordon equation. But, there are still sections 3, 4, and 6 remain to be studied.

• Section 3: The Bäcklund transformation is used in the study of the hyperbolic sine-Gordon equation, where we can derive the multi-soliton solutions in an algebraic way.
• Section 4: Besides the uniqueness of the multi-end solutions, it turns out that these solutions \( U_\alpha \) are \( L^\infty \) nondegenerate.
• Section 6: Another important fact about the solutions \( U_\alpha \) is that they have a finite Morse index.

Future goal: It is a natural question to ask whether we can obtain these beautiful results of the elliptic sine-Gordon equation in high dimensions. My goal is to generalize some results using Liu and Wei’s method.

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References