The Courant Nodal Domain Theorem

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Abstract

This report will discuss the Courant Nodal Domain Theorem. We will build results and terminology leading up to the theorem. We will investigate a careful proof of the theorem, taking into account the generality of the nodal domains. We will then discuss the application of the Courant Nodal Domain Theorem to various domain geometries and to graphs.

Subject: Analysis and PDEs

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1 Introduction

In a 1923 paper and subsequent book in 1924 [3], Courant proved a result regarding the zeros of Laplacian eigenfunctions. This result is now referred to as the "Courant Nodal Domain Theorem" and will be the main focus of this report.

Theorem 1. (Courant Nodal Domain Theorem) *The Dirichlet eigenfunction ψ_k has at most k nodal domains.*

The terminology in this theorem will be discussed later; however, a remarkable part of the theorem is its generality, which is reflected by the relatively simple statement. It can apply to a wide variety of elliptic eigenvalue problems and even has extensions in graph theory. This generality not only makes this theorem very powerful but also makes it slightly difficult to prove, in that one must be careful of one's assumptions.

The main goal of this report is to take into account this generality and provide a thorough proof of the Courant nodal domain theorem without making any implicit assumptions on the space and nodal domains.

This report starts off with discussing the Dirichlet Laplacian eigenvalue problem. We will define the problem, the spaces we will be working in, and prove some basic facts about the eigenvalue problem. We will then introduce the powerful Rayleigh quotient and min-max theorem.

The main theorem we will attempt to provide a proof of is Courant's Nodal Domain Theorem. After the basic facts, we will start off with the build-up to the theorem by proving lemmas on domain monotonicity and the resulting corollaries. We will then present and prove Courant's nodal domain theorem, taking into account the generality of the nodal domains. The first proof will be a specific version of the theorem (specified to normal nodal domains). The second proof will consider the general case.

After the proof of Courant's nodal domain theorem, we discuss how often the number of nodal domains reach equality with the upper bound. Pleijel's theorem will show us that the bound in Courant's theorem is only achieved a finite number of times.

After Pleijel's theorem, we will have a few sections exploring the applications of Courant's theorem. We will first explore the square case as a case study and show how we can use bounds on the eigenvalue number to apply the Courant nodal domain theorem and do a quick check to see if we get reasonable results. Then, we will also discuss the torus, sphere, and the quantum harmonic oscillator.

Finally, we will see that the results we proved in the continuous case have analogs in the discrete case. We will see that the eigenvalue problem on graphs has analogous properties. We will prove some of these results and investigate their connections to the continuous case.

The last section is the appendix that covers important definitions and theorems we use in this report.

2 Preliminaries

Before we prove the Courant Nodal Domain Theorem, we will begin with a quick discussion of relevant facts and definitions that will be useful for building the context of our problem and the tools we will use to solve it.

We start off by defining the type of domain we will be using. We will be working with a domain M that is **bounded and open** in $\mathbb{R}^n$ with a smooth boundary ∂M. The boundedness and boundary conditions will be incredibly useful when applying Green's identity. We will be working in the space $L^2(M)$. We let $L^2(M)$ be the space of measurable functions f on M for which $\int_M |f|^2 \, \mathrm{dV} < +\infty$. On $L^2(M)$ we have the usual inner product. With the inner product, $L^2(M)$ is a Hilbert space.

2.1 The Laplacian

In this report, the primary operator of interest is the Laplacian. The Laplacian is defined as

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i}.$$

The Laplacian also has a discrete, matrix form for graphs (discussed in later sections). We will now focus on the continuous case. We are interested in the eigenvalue problem

$$\lambda f = -\Delta f.$$

The eigenvalue problem in its current form is very general. We must further specify the type of functions we will work with, boundary conditions, and more, which we will do in the coming sections.

2.2 Sobolev Spaces

The first step in specifying our eigenvalue problem is to define the space of functions we wish to work with. Sobolev spaces provide a perfect balance of manageable function properties while also being general enough to encompass functions of interest (dense in the set of functions of interest).

Definition. (Sobolev Space) *The Sobolev spaces on an open set $M \subset \mathbb{R}^n$ are defined for $m \in \mathbb{N}$ by*

$$H^m(M) := \{u \in L^2(M): D^\alpha u \in L^2(M) \quad \textit{for} \quad |\alpha| \leq m\}.$$

Moreover, the Sobolev spaces have their own inner products

$$\langle u, v \rangle_{H^m} := \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle,$$

and norms

$$\|u\|_{H^m} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2 \right)^{1/2}.$$

We are interested specifically in the space $H^1(M)$. This space consists of functions such that the first derivative of functions are in L^2. The Sobolev spaces are just another way of imposing regularity on the set of functions we are working on. Sobolev spaces are commonly used in cases where we must take a derivative of a function we are dealing with and want the output function to be manageable.

To define the viable functions for the Dirichlet case, $H^1(M)$ is not enough, we must impose a few more restrictions to ensure Dirichlet boundary conditions. For example, there exists $f \in H^1([0,1])$ such that $f(0) \neq 0 \neq f(1)$. This is an issue. Instead we want functions such that $f \in H^1([0,1])$ and $f(0) = f(1) = 0$. It turns out such a space can be defined nicely, and it is denoted as $H_0^1(\Omega)$. The 0 subscript is a reference to zero boundary values.

More formally, one can define $H_0^1(M) := \overline{C_0^\infty(M)} \subset H^1(M)$ where the closure is taken with respect to the H^1 norm. All such functions in $H_0^1(M)$ weakly fulfill the Dirichlet boundary conditions on ∂M. This H_0^1 domain will be used to define the Dirichlet Laplacian below. The functions in H_0^1 vanish on $\partial\Omega$ in some weak sense.

Remark. The motivation for weak derivatives is to eventually use them in Green's identities without over-restricting our function space. One can define the weak derivative in many ways (depending on desired generality), but since we are dealing with the Dirichlet case and smooth boundary, it is the standard multivariable definition of a function that matches the inner product of divergence with the desired function and an arbitrary vector field.

2.3 The Dirichlet Laplacian

As discussed earlier, the Laplacian is defined as $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$. We are interested in the eigenvalue problem $\lambda f = -\Delta f$. On M, the Dirichlet Laplacian is on the domain $D(-\Delta) := \{u \in H_0^1(M)$: $-\Delta u \in L^2(M)$ in the weak sense$\}$ [1]. From now on, we will assume that the Laplacian is defined on this domain. We will now investigate some important preliminary facts about the Dirichlet Laplacian eigenvalue problem.

Theorem 2. *For the Dirichlet eigenvalue problem, the problem has a discrete spectrum. The eigenvalues are positive and have finite multiplicity. Moreover, there exists an orthonormal basis for $L^2(M)$ consisting of eigenfunctions.*

For the Dirichlet case, the first eigenvalue is simple. The first eigenfunction can be chosen positive on M. It is usual to enumerate the eigenvalues in increasing order, counting multiplicities.

$$\lambda_1(M) < \lambda_2(M) \leq \lambda_3(M) \ldots$$

To show the orthogonality of the eigenfunctions, we will need Green's identity. Recall that Green's identity gives us that

$$\int_M (u\,\Delta v - \Delta u\, v\, d)\,\mathrm{dV} = \int_{\partial M} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \mathrm{dS},$$

for functions on a domain M with sufficiently smooth boundary ∂M, which is the case with our domain M.

Since we assume that the boundary is smooth, orthogonality comes very easily since we can apply Green's formula assuming that $\Delta\psi = -\lambda\psi$ and $\Delta\phi = -\tau\phi$. For proving orthogonality, we start off by writing the simple identity by definition of eigenfunctions

$$-(\lambda - \tau) \int_M \phi\, \psi\, \mathrm{dV} = \int_M (\phi\, \Delta\psi - \psi \Delta\phi)\, \mathrm{dV},$$

where $\lambda - \tau$ is non-zero since we choose λ, τ to be distinct eigenvalues. By Green's identity, we have that

$$\int_M \phi\, \Delta\psi\, \mathrm{dV} = \int_M \psi\, \Delta\phi\, \mathrm{dx} + \int_{\partial M} \phi \frac{\partial \psi}{\partial \nu} \mathrm{dS} - \int_{\partial M} \psi \frac{\partial \phi}{\partial \nu} \mathrm{dS},$$

recall however that the the boundary terms are 0 because of the Dirichlet conditions, $\phi, \psi \in H_0^1(M)$; therefore $\phi, \psi = 0$ on ∂M in a weak sense and the last two terms vanish. Thus, we have that

$$\int_M \phi\, \Delta\psi\, \mathrm{dV} = \int_M \psi\, \Delta\phi\, \mathrm{dx},$$

plugging this in, we get that

$$-(\lambda - \tau) \int_M \phi\, \psi\, \mathrm{dV} = \int_M (\phi\, \Delta\psi - \psi \Delta\phi)\, \mathrm{dV} = 0.$$

Since $\lambda - \tau$ is non-zero, we are left with $\int_M \phi\, \psi\, \mathrm{dV} = 0$. Thus, we have shown orthogonality of the Dirichlet eigenfunctions corresponding to distinct eigenvalues.

Finally, one last observation (from the assertion of completeness in the theorem) is that *Parseval's identities* apply. Considering some $f \in L^2(M)$ we have that

$$f = \sum_{j=1}^{\infty} (f, \phi_j)\, \phi_j,$$

in $L^2(M)$ and

$$\|f\|^2 = \sum_{j=1}^{\infty} (f, \phi_j)^2.$$

The completeness argument will turn out to be very useful in future proofs and is quite a remarkable property given the restraints we put on our domain.

2.4 Rayleigh Quotient

When investigating eigenvalue problems in the discrete case with finite matrices on graphs, a very useful theorem commonly used in discrete mathematics is Rayleigh quotients and Courant's Min-Max Theorem (see the appendix for a discussion on these linear algebraic theorems) . It turns out that such theorems have almost exact analogs in the continuous case, and they will prove very useful in helping us decompose this eigenvalue problem.

Theorem 3. (Max-Min Theorem - as given by Chavel [2]) *Given $v_1, \dots v_{k-1} \in L^2(M)$, let*

$$\mu = \inf \frac{\|\nabla f\|^2}{\|f\|^2},$$

where f varies over the subspace of functions in $H_0^1(M)$ orthogonal to $v_1, \dots v_{k-1}$ in $L^2(M)$. Then, for ordered Dirichlet Laplace eigenvalues λ_k we have that

$$\mu \leq \lambda_k.$$

If $v_1, \dots v_{k-1}$ are eigenfunctions corresponding $\lambda_1, \dots, \lambda_{k-1}$, then $\mu = \lambda_k$.

Note that this version is equivalent to the more standard form [1] below:

Theorem 4. (Max-Min Theorem - Borthwick [1] - Alternative Statement) *Let $\{\lambda_k\}$ be the Dirichlet eigenvalues of a bounded open set $M \subset \mathbb{R}^n$, written in increasing order and repeated according to multiplicity. Define Λ_k as the set of subspaces of $H_0^1(M)$ of dimension k. Then:*

$$\lambda_k = \min_{W \in \Lambda_k} \left\{ \max_{f \in W/\{0\}} \frac{\|\nabla f\|^2}{\|f\|^2} \right\},$$

for each $k \in \mathbb{N}$

The quotient $\frac{\|\nabla u\|^2}{\|u\|^2}$ is also called Rayleigh's quotient. These theorems are useful tools that will give us a chance to compare eigenvalues to each other over various domains and to connect functions to their eigenvalues.

2.5 Weyl's Asymptotic Formula

A problem of principle concern that we will see more of later is determining the respective ranking of an eigenvalue based on its value. The ranking of an eigenvalue will be incredibly important in helping us determine properties of the eigenfunctions and help us make theorems about the eigenfunctions.

It is slightly difficult to determine such a ranking straight from the magnitude of an eigenvalue; however, we can come up with bounds and asymptotics that can help us approximate a ranking. In this section, we present Weyl's asymptotic formula which gives us the limiting behavior of the ranking of an eigenvalue in terms of its value as well as the geometry of the domain. Apart from the fact that Weyl's asymptotic formula gives this very important relationship and helps us define the ranking of eigenvalues, the formula also illustrates an important relationship between the geometry of the domain and the eigenvalues. We will use this later in the proof of Pleijel's theorem.

Let $N(\lambda)$ be the number of eigenvalues $\leq\lambda$ (counted with multiplicity). Then Weyl's asymptotic formula gives us that:

$$N(\lambda)\sim\omega_n(\mathrm{vol}\,M)\lambda^{n/2}/(2\,\pi)^n,$$

where ω_n is the volume of the unit disk in $\mathbb{R}^n$ and vol M is the volume of M, which exists since M is bounded *[2]*. Substituting in $N(\lambda_k)=k$ and $\lambda=\lambda_k$ and rearranging the equation, we get that as $k\to\infty$ that:

$$(\lambda_k)^{n/2}\sim\left\{\frac{(2\pi)^n}{\omega_n}\right\}\frac{k}{\mathrm{vol}\,M}.$$

We will use both asymptotic expressions for future proofs. Before we proceed, we will give a simple one dimensional example, in which many of the terms simplify and the asymptotic behavior turns out to exactly predict the growth of the eigenvalues.

2.6 Example in Dimension One

We consider the Dirichlet Laplacian on $M=(0,L)$ we get:

$$\phi_k(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{\pi\,k\,x}{L}\right),$$

and

$$\lambda_k=\left(\frac{\pi\,k}{L}\right)^2.$$

We see that this is exactly what Weyl's formula predicts since $\omega_n=2$ and vol $M=L$ and $n=1$. We have that Weyl's formula gives us that $\lambda_k\sim\left(\frac{\pi\,k}{L}\right)^2$, which is the exact value of the eigenvalue.

Here are a few eigenfunctions to pictorially illustrate what the eigenfunctions look like on our domain :

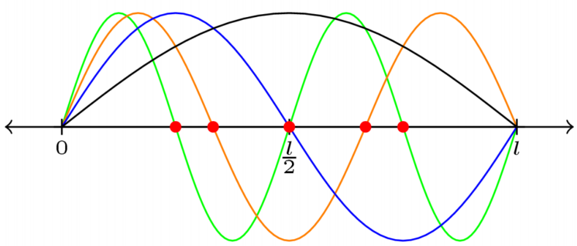


Figure 1. One Dimensional Eigenfunctions of Dirichlet Laplacian

In higher dimensions, Weyl's formula gives the asymptotic behavior as $\lambda\to\infty$. It is not always exact as in the one-dimensional case.

3 The Nodal Domain Theorem

3.1 Domain Monotonicity of Eigenvalues

The first eigenvalue of the Dirichlet Laplacian is sometimes called the *fundamental tone* of the domain Ω, we have that $\lambda_1(\Omega)=\lambda^*(\Omega)=\inf\frac{\|\nabla f\|^2}{\|f\|^2}$ where f ranges over non-zero functions in $H_0^1(\Omega)$. In some papers [2], the first eigenvalue is referred to as $\lambda^*(\Omega)$, we will refer to it simply as $\lambda_1(\Omega)$.

Let $\Omega_1,\ldots,\Omega_m$ be pairwise disjoint domains with smooth boundaries in M, whose boundaries intersect transversally with ∂M when they do intersect. We will now consider an eigenvalue problem on M and consider what it means for Ω_k.

Given an eigenvalue problem on M consider the same eigenvalue problem on Ω_r be requiring vanishing Dirichlet data on $\partial\Omega_r \cap M$ and by leaving $\partial\Omega_r \cap \partial M$ with Dirichlet boundary conditions (i.e. 0).

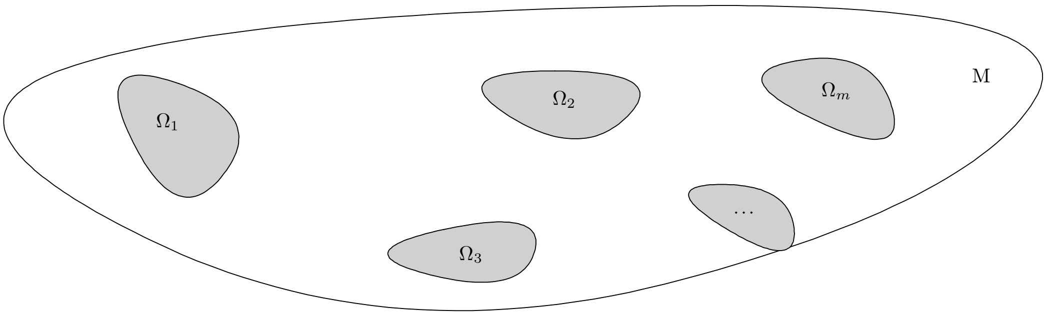


Figure 2. Subdomains on M

Arrange all the eigenvalues of $\Omega_1, \ldots, \Omega_m$ in an increasing sequence:

$$0 \leq v_1 \leq v_2 \leq \ldots$$

with each eigenvalue repeated according to its multiplicity and let the eigenvalues of M be λ_k in sorted order. Then, we have for all k that

$$\lambda_k \leq v_k.$$

Proof. We will use the max-min theorem. Pick functions in L^2, $\phi_1, \ldots \phi_{k-1}$. For $j = 1, \ldots k$ let ψ_j: $\bar{M} \to \mathbb{R}$ be an eigenfunction of v_j when restricted to the appropriate sub-domain, and identically zero everywhere else. Then we have that $\psi_j \in H_0^1(M)$ (i.e. the set of interest) and $\psi_1, \ldots \psi_k$ may be chosen orthonormal in $L^2(M)$.

There exists $\alpha_1, \ldots \alpha_k$, not equal to zero, satisfying

$$\sum_{j=1}^{k} \alpha_j (\psi_j, \phi_l) = 0,$$

for $l = 1, \ldots, k-1$. Therefore, the function

$$f = \sum_{j=1}^{k} \alpha_j \psi_j$$

is orthogonal to $\phi_1, \ldots \phi_{k-1}$ in $L^2(M)$ which implies that

$$\lambda_k \, \| f \|^2 \leq \| \nabla f \|^2 = \sum_{j=1}^{k} v_j \, \alpha_j^2 \leq v_k \, \| f \|^2.$$

Thus, showing that $\lambda_k \leq v_k$. □

Corollary 5. *If $\Omega \subset M$, then for the Dirichlet eigenvalue problem on Ω and on M we have*

$$\lambda_k(M) \leq \lambda_k(\Omega).$$

If $M - \bar{\Omega}$ is not empty set, then the inequality is strict.

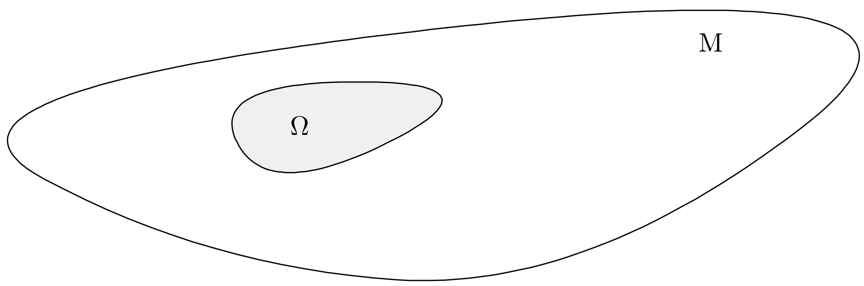


Figure 3. Subsets of the Domain

Note. The Neumann case is when the derivative on the boundary is zero. The same progression of theorems can also be accomplished with the Neumann case even though all the results above were for Dirichlet eigenvalue problems. The Neumann case, however, is slightly more difficult to prove; in the proof, we must deal with more complicated nodal domains that cover our domain of interest. Look at the appendix for the proof of the domain monotonicity of the Neumann case. The proof is a good example of the important nuances that must be considered with the Neumann case.

3.2 Courant's Nodal Domain Theorem

The focus of this section will be on nodal domains:

Definition 6. (Nodal Domains) *The nodal domain of a continuous function* $f\colon \Omega \rightarrow \mathbb{R}$ *is a connected component of the set* $\{f \neq 0\} \subset \Omega$

For example, on one dimension, we have:

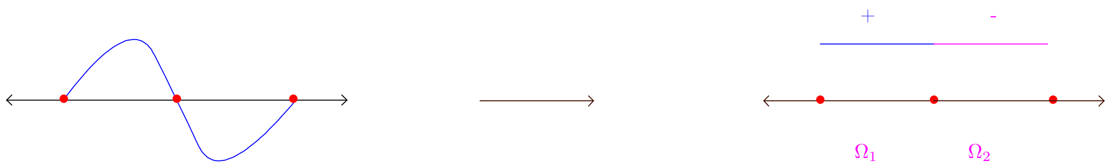


Figure 4. (A) One Dimensional Example of Nodal Domains

where we split up the set $\{f \neq 0\}$ for some function into two domains Ω_1 and Ω_2. In two dimensions, we have:

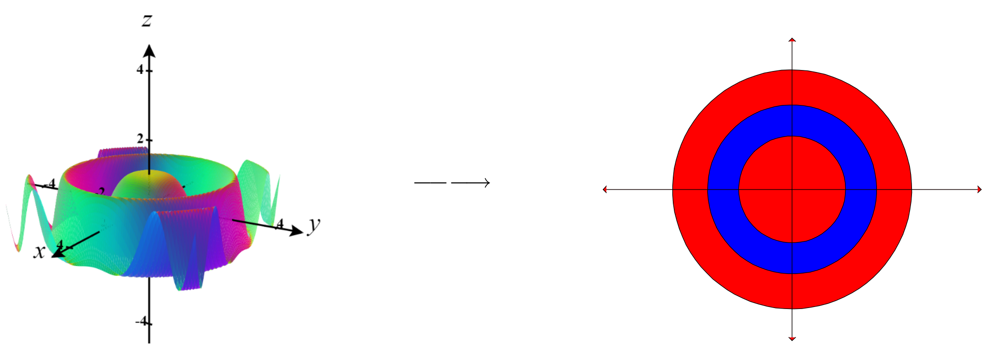


Figure 4. (B) Two Dimensional Example of Nodal Domains

We can first start off with a special case of the Courant Nodal Domain Theorem when the nodal sets are normal. The Courant Nodal Domain Theorem is more general than this and will be presented after this theorem; however, this more-specific theorem has a very intuitive proof that helps build the conceptual foundation for the proof of Courant's Nodal Domain Theorem in full generality. This argument also goes back to the original work of Courant.

Theorem 7. *Given that the nodal sets are normal (i.e. Green's identity can be applied on to each boundary of the nodal domains), then the Dirichlet eigenfunction ψ_k has at most k nodal domains.*

Proof. Suppose that ψ_k has at least k nodal domains. We label these as $V_1, \dots V_k$ and define the restrictions

$$\psi_k^{(j)}(x) := \begin{cases} \psi_k(x) & x \in V_j \\ 0 & x \notin V_j \end{cases}$$

for $j = 1, \dots, k$. Note that each $\psi_k^{(j)} \in H_0^1(M)$. We can see this is true by splitting ψ into a positive and negative component $\psi^+(x) := \max\{\psi(x), 0\}$ and similarly for $\psi^-(x)$. Now, one can check that these components lie in $H_0^1(M)$.

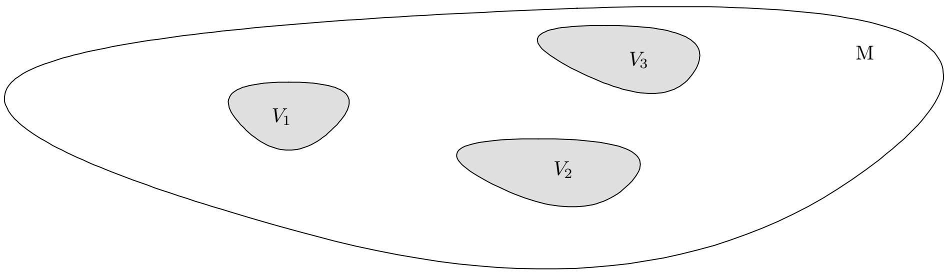


Figure 5. Nodal Domains on Domain M

Counting dimensions shows that there exists an nonzero element $u \in \operatorname{span}\{\psi_k^{(1)}, \dots, \psi_k^{(k)}\}$ such that u is orthogonal to each ψ_i for $i = 1, \dots, k-1$. The eigenfunction basis decomposition for u thus has the form

$$u = \sum_{i=k}^{\infty} \langle \psi_i, u \rangle \psi_i.$$

This implies that

$$\|\nabla u\|^2 = \sum_{i=k}^{\infty} \lambda_i |\langle \psi_i, u \rangle|^2.$$

We can see this via:

$$\begin{aligned} \|\nabla u\|^2 &= -\sum_{i=1}^{k} \int \nabla(\langle \psi_i, u \rangle \psi_i) \, \nabla(\langle \psi_i, u \rangle \psi_i) \, \mathrm{dx} \\ &= -\sum_{i=1}^{k} \int \langle \psi_i, u \rangle \psi_i \, \Delta(\langle \psi_i, u \rangle \psi_i) \, \mathrm{dx} \\ &= \sum_{i=k}^{\infty} \lambda_i |\langle \psi_i, u \rangle|^2. \end{aligned}$$

On the other hand, since u is a linear combination of disjoint components of ψ_k we have that

$$u = \sum_{i=1}^{k} \langle \psi_k^{(i)}, u \rangle \psi_k^{(i)}.$$

We have that using Green's formula

$$\int_U \nabla v \cdot \nabla u \, \mathrm{dx} = -\int_U u \, \Delta v \, \mathrm{dx} + \int_{\partial U} u (\nu \cdot \nabla v) \, \mathrm{dS},$$

that

$$\begin{aligned}\|\nabla u\|^2 &= \int_\Omega \nabla u\,\nabla u\,\mathrm{dx}\\ &= -\int_\Omega u\,\Delta u\,\mathrm{dx} + \int_{\partial\Omega} u(\nu\cdot\nabla u)\,\mathrm{dS}\\ &= -\int_\Omega u\,\Delta u\,\mathrm{dx}.\end{aligned}$$

We were able to remove the boundary term due to the Dirichlet boundary conditions. Now, we can plug in $u=\sum_{i=1}^k \langle\psi_k^{(i)},u\rangle\psi_k^{(i)}$ and moreover, we know that for each $\psi_k^{(i)}$ that $\psi_k^{(i)}$ and $\Delta\psi_k^{(i)}$ are non-zero on V_j. Thus, we can split up the integral to

$$\|\nabla u\|^2 = -\sum_{i=1}^k \int_{V_i} \langle\psi_k^{(i)},u\rangle\psi_k^{(i)}\,\Delta\big(\langle\psi_k^{(i)},u\rangle\psi_k^{(i)}\big)\,\mathrm{dx}.$$

Note that $-\Delta\psi_k^{(i)}$ on V_i is $\lambda_k\psi_k^{(i)}$ by definition. Therefore, we get that

$$\begin{aligned}\|\nabla u\|^2 &= -\sum_{i=1}^k \int_{V_i} \langle\psi_k^{(i)},u\rangle\psi_k^{(i)}\,\Delta\big(\langle\psi_k^{(i)},u\rangle\psi_k^{(i)}\big)\,\mathrm{dx}\\ &= \sum_{i=1}^k \lambda_k\big|\langle\psi_k^{(i)},u\rangle\big|^2\int_{V_i}\big|\psi_k^{(i)}\big|^2\,\mathrm{dx}\\ &= \lambda_k\sum_{i=1}^k \big|\langle\psi_k^{(i)},u\rangle\big|^2\int_{V_i}\big|\psi_k^{(i)}\big|^2\,\mathrm{dx}\\ &= \lambda_k\|u\|^2.\end{aligned}$$

Written out, we get

$$\|\nabla u\|^2=\lambda_k\|u\|^2=\sum_{i=k}^\infty \lambda_i|\langle\psi_i,u\rangle|^2.$$

Thus, it must be the case that $\langle\psi_i,u\rangle=0$ unless $\lambda_i=\lambda_k$ implying that u is itself an eigenfunction with eigenvalue λ_k.

By construction, u vanishes outside $V_1\cup V_2\ldots\cup V_k$, but unique continuation property (see appendix for discussion on the unique continuation principle) implies that u cannot vanish on an open set. It follows that ψ_k cannot have more than k nodal domains. □

As discussed, the more general case, with the possibility of "non-normal" nodal domains is a little bit trickier to prove, and is proven in Courant's general theorem below.

Theorem. (Courant Nodal Domain Theorem) *The Dirichlet eigenfunction ψ_k has at most k nodal domains.*

To prove this, we again suppose that ψ_k has at least k nodal domains. We label these as $V_1,\ldots V_k$ and define the restrictions

$$\psi_k^{(j)}(x):=\begin{cases}\psi_k(x) & x\in V_j\\ 0 & x\notin V_j\end{cases},$$

for $j=1,\ldots,k$. Note that each $\psi_k^{(j)}\in H_0^1(M)$. If we were proceeding as in our last proof, at this point we would have used Green's identity on the nodal domains; however, since we make no assumption about the form of the nodal domains, we cannot use Green's identity.

At the same time, we must make the same important step we made before to show that $\lambda_k = \lambda_1(V_j)$. Thus, we come up with the following lemma to prove exactly this without explicitly using Green's identity on the nodal domains.

Lemma 8. *Let ψ_k be an eigenfunction with eigenvalue λ_k and let V_j be a nodal domain of ψ_k. Then $\psi_k \in H_0^1(V_j)$ and*

$$\lambda_k = \lambda_1(V_j).$$

Proof. Assume $\psi_k > 0$ on V_j and for each $\epsilon > 0$, set

$$V_{j,\epsilon} = \{x \in V_j \colon \psi_k^{(j)}(x) > \epsilon\},$$

and

$$\psi_{k,\epsilon}^{(j)} = \begin{cases} \psi_k^{(j)} - \epsilon & V_{j,\epsilon} \\ 0 & M / V_{j,\epsilon} \end{cases}.$$

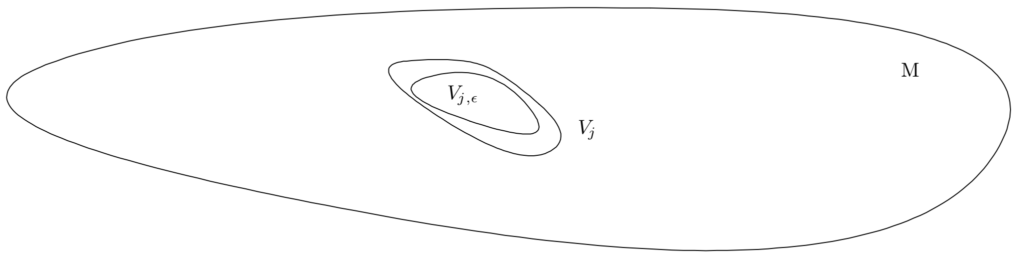


Figure 6. $V_{j,\epsilon}$ on Domain M

By Sard's theorem [9] there exists a sequence ϵ_i of regular values of u, decreasing to 0 as $i \to \infty$, and define

$$V_{j,i} = V_{j,\epsilon_i},$$

and

$$\psi_{k,i}^{(j)} = \psi_{k,\epsilon_i}^{(j)}.$$

Now, since $\partial V_{j,i}$ is C^∞ we now know that Green's formula can be applied to get

$$\begin{aligned} \lambda_k \int\int_{V_{j,i}} \psi_{k,i}^{(j)} \psi_k^{(j)} \,\mathrm{dV} &= -\int\int_{V_{j,i}} \psi_{k,i}^{(j)} \Delta \psi_{k,i}^{(j)} \,\mathrm{dV} \\ &= \int\int_{V_{j,i}} |\nabla \psi_{k,i}^{(j)}|^2 \mathrm{dV} \\ &\geq \lambda_1(V_j) \int\int_{V_{j,i}} |\psi_{k,i}^{(j)}|^2 \mathrm{dV}. \end{aligned}$$

By letting $i \to \infty$, we get that

$$\lambda_k \int\int_{V_{j,i}} \psi_k^{(j)2} \,\mathrm{dV} \geq \lambda_1(V_j) \int\int_{\Omega} \psi_k^{(j)2} \,\mathrm{dV}.$$

Therefore, we have that:

$$\lambda_k \geq \lambda_1(V_j).$$

Now, we will show the opposite inequality, let $\epsilon > 0$ be a regular value of $\psi_k^{(j)}$, and let $v_\epsilon > 0$ be the eigenfunction of the Dirichlet eigenvalue $\lambda_1(V_{j,\epsilon}) = \lambda^*(V_{j,\epsilon})$. Then

$$\begin{aligned}
\lambda_k \int\int_{V_{j,\epsilon}} v_\epsilon \psi_k^{(j)}\,\mathrm{dV} &= -\int\int_{V_{j,\epsilon}} v_\epsilon \big(\Delta \psi_k^{(j)}\big)\,\mathrm{dV} \\
&= -\int\int_{V_{j,\epsilon}} (\Delta v_\epsilon)\psi_k^{(j)}\,\mathrm{dV} + \int\int_{\partial V_{j,\epsilon}} \psi_k^{(j)}\left(\frac{\partial v_\epsilon}{\partial v}\right)\mathrm{dA} \\
&\leq -\int\int_{V_{j,\epsilon}} (\Delta v_\epsilon)\psi_k^{(j)}\,\mathrm{dV} \\
&= \lambda_1(V_j)\int\int_{V_{j,\epsilon}} v_\epsilon\,\psi_k^{(j)}\,\mathrm{dV},
\end{aligned}$$

which implies that

$$\lambda_k \leq \lambda_1(V_{j,\epsilon}),$$

for all regular values $\epsilon > 0$. We now show that

$$\lim_{\epsilon \to 0} \lambda_1(V_{j,\epsilon}) = \lambda_1(V_j).$$

Given any $\delta > 0$, there exists $f \in C^\infty(V_j)$ compactly supported on V_j, such that

$$D[f, f] / \|f\|^2 \leq \lambda_1(V_j) + \delta.$$

But there certainly exists $\epsilon > 0$ for which:

$$\mathrm{supp}\, f \subset V_j,$$

so we have that

$$\lambda_1(V_j) \leq D[f, f] / \|f\|^2.$$

We therefore have , for given $\delta > 0$, the existence of $\epsilon > 0$ for which

$$\lambda_1(V_j) \leq \lambda_1(V_{j,\epsilon}) \leq \lambda_1(V_j) + \delta.$$

Since $\lambda^*(V_j)$ is increasing with respect to ϵ, we get that $\lim_{\epsilon \to 0} \lambda_1(V_{j,\epsilon}) = \lambda_1(V_j)$. □

Now, from this lemma we know that $\lambda_k = \lambda_1(V_j)$ where λ_k is the eigenvalue corresponding to ψ_k and V_j is a nodal domain.

Now, for each nodal domain V_j, consider the first eigenfunction f_j in $H_0^1(V_j)$, we know by definition of eigenfunction that:

$$\int_{V_j} \|\nabla f_j\|^2\,\mathrm{dx} = \lambda_k \int_{V_j} \|f_j\|^2\,\mathrm{dx}.$$

Then, using this function instead of ψ_j, we find a linear combination of f_j orthogonal to the first $k-1$ eigenfunctions $\psi_1, \ldots, \psi_{k-1}$:

$$g = \sum_{j=1}^{k} \alpha_j f_j.$$

For function g, we now have that the energy ratio is equal to the λ_k and that g is an eigenfunction itself. But it is zero on one of the nodal domains. Therefore, we have a contradiction. This completes the proof of the generalized Courant Nodal Domain Theorem.

We will now explore how often the upper bound in Courant's theorem is achieved. This will require a little more machinery to be developed.

3.3 Pleijel's Theorem

Pleijel's theorem is an important result that tells us how often the upper bound in Courant's Nodal Domain Theorem is achieved. This is an important indicator of how the number of nodal domains actually behaves. In order to prove Pleijel's theorem, we will need to start off with a foundational inequality, the Faber-Krahn inequality.

Theorem 9. (Faber-Krahn Inequality) *Suppose that M is a bounded open set $\mathbb{R}^n$, and let $B\subset\mathbb{R}^n$ be a ball with $\mathrm{vol}(B)=\mathrm{vol}(M)$. The lowest Dirichlet eigenvalues satisfy,*

$$\lambda_1(M)\geq\lambda_1(B)$$

with equality only if M is a ball.

The proof for this theorem can be seen in both Borthwick [2] and in the original paper [10]. Note that we can further specify this inequality for our case specifically and show that $\lambda_1\geq\frac{\pi j^2}{V}$ where j is the smallest positive zero of the Bessel function J_0 for dimension 2 and V is the area/volume of the domain. A discussion of Bessel functions is given in the appendix.

The Faber-Kahn inequality is a consequence of the isoperimetric inequality on $\mathbb{R}^n$. If we were dealing with more complicated domains, we would need the isoperimetric inequality. There is also a specific analog for Neumann eigenvalues (Szego-Weinberger Theorem), and we will see later a version for quantum harmonic oscillators (discussed in future section).

Given the Faber-Krahn inequality, we can now state and prove Pleijel's theorem for the Dirichlet case.

Theorem 10. (Pleijel's Theorem) *In dimension 2 for the Dirichlet Laplacian eigenvalue problem on M, we have that for the sequence of eigenvalues λ_n with corresponding eigenfunctions ϕ_n, we have that:*

$$\limsup_{n\to\infty}\frac{\mu(\phi_n)}{n}\leq\frac{4\pi}{\lambda(\mathbb{D}_1)}=\left(\frac{2}{j}\right)^2<1,$$

where $\mathbb{D}_1$ is the disk of unit area and j is the least positive zero of J_0, the Bessel function.

Remark. Chavel [2] gives a more general version of Pleijel's theorem, expanded to compact n-dimensional manifolds with boundary (in which the isoperimetric inequality is assumed).

Proof. The proof of the two dimensional special case comes straight from upper bound indicated in the theorem. We use Faber-Krahn inequality with respect to the first eigenvalues, hence the use of Bessel's function. More specifically, if we let $\Omega_1,\Omega_2,\ldots\Omega_m$ be the nodal domains (which we know that $m\leq n$ of due to Courant's nodal domain theorem), we know that in each Ω_i, the function ϕ is non-zero and by Lemma 9 that $\lambda_1(\Omega_i)=\lambda_n$. By the Faber-Krahn inequality, we have that:

$$\lambda_1(\Omega_i)\geq\frac{\pi j^2}{\mathrm{vol}(\Omega_i)}.$$

Thus, putting it together we get that:

$$\frac{\mathrm{vol}(\Omega_i)}{\pi j^2} \geq \frac{1}{\lambda_n}.$$

Thus, adding up these inequalities for all Ω_i, we get:

$$\frac{\mathrm{vol}(M)}{\pi j^2} \geq \frac{m}{\lambda_n},$$

where $m = \mu(\phi_n)$ is the number of nodal domains of ϕ_n. Using Weyl's asymptotic law from section 2.5, we get that:

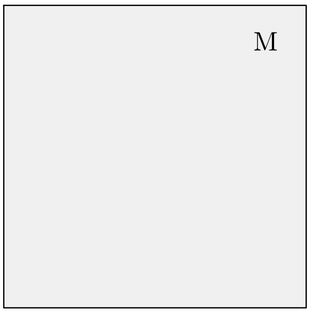
$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{\mathrm{vol}(M)}.$$

Finally,

$$\begin{aligned} \limsup_{n \to \infty} \frac{\mu(\phi_n)}{n} &\leq \limsup_{n \to \infty} \frac{\mu(\phi_n)}{\lambda_n} \frac{4\pi}{\mathrm{vol}(M)} \\ &\leq \frac{4}{j^2}. \end{aligned}$$

As discussed earlier, here j represents the the first zero of the Bessel function. The Bessel function is further discussed in the appendix. Since j of J_0 is strictly greater than 2, we have that $\left(\frac{2}{j}\right)$ is less than 1, thus giving us our limit inequality in the theorem. □

4 Square Case



The eigenfunctions of the the Laplacian with the Dirichlet boundary conditions on the square are

$$\phi_{n,m} = \sin(n\,x)\sin(m\,y),$$

with corresponding eigenvalue $\lambda_{n,m} = n^2 + m^2$. By inspection, we see that $\phi_{n,m}$ has at least $n\,m$ nodal domains since the sine functions partitions the square into a grid. The Courant Nodal Domain Theorem tells us that given that λ_{nm} is the k^{th} smallest eigenvalue that the number of nodal domains is bounded from above by k.

In this section, we will verify Courant's Nodal Domain Theorem on the square case. Through the process of verifying the theorem, we will explore important concepts and patterns in nodal domains. The first goal is to be able to order the eigenfunctions in order based on the magnitude of the eigenvalue. This is a complicated problem, since even given that $\lambda_{n,m} = n^2 + m^2$, it is difficult to determine where this eigenvalue falls within the ranking of other eigenvalues. We will work in the coming sections to develop an upper and lower bound for this ranking, and in the final section put it together with Courant's theorem to complete the verification.

One of the reasons this problem is complicated is because there are multiple eigenfunctions per eigenvalue. This forces us to consider linear combinations. Here are some examples:

Rank	Eigenvalue	Eigenfunction	CNDT
1	2	$c_1\,\phi_{1,1}$	≤ 1
3	5	$c_1\,\phi_{2,1}+c_2\,\phi_{1,2}$	≤ 3
4	8	$c_1\,\phi_{2,2}$	≤ 4
6	10	$c_1\,\phi_{3,1}+c_2\,\phi_{1,3}$	≤ 6

In the case that we have a single $\phi_{n,m}$ as our eigenfunction, the determination of the number of nodal domains is simple. It is $n\,m$, as discussed above. Otherwise, it is difficult to determine since it depends significantly on the constants $c_1,\ldots c_n$; however, we can give a limit based on the ranking using Courant's Nodal Domain Theorem. For example, we know that from Courant's theorem that the maximum number of nodal domains for eigenvalue 5 and 10 is 2 and 4 respectively.

We can try to generalize this by thinking about the number of lattice points in a disk. If we are given that our eigenvalue is n^2+m^2, then we are interested in the number of eigenvalues below it. We will first develop an overestimate, then we will make an underestimate.

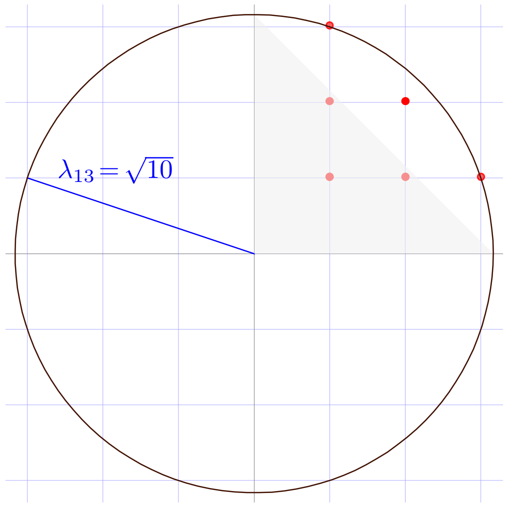


Figure 7. Positive Lattice Points Within a Disk of Radius $\sqrt{10}$

4.1 Upper Bound

To develop the overestimate, we notice that all other eigenvalues $\lambda_{n'm'}$ live on lattice points such that $n'^2+m'^2\leq n^2+m^2$. Therefore we are interested on all the lattice points within the disk of radius n^2+m^2 from the origin. A first estimate would be the area of the disk divided by the size of a unit square (since each unit square contains one unique lattice point).

We can designate the the outer corner as the unique lattice point for each square . Pictorially we have that:

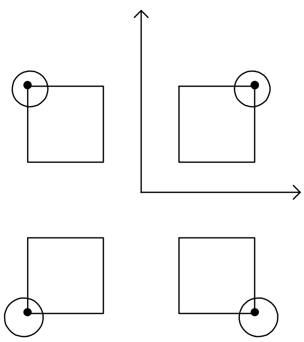


Figure 8. Upper-Diagonal Lattice Point Unique Identification

By doing this, we also avoid counting the lattice points on the axes (which are not of interest to us, since $n=0$ and $m=0$ do not fulfill the differential equation). Now, our goal is to count the number of unique unit squares that are within the disk. Notice that since we chose the lattice points to be the top corner, we have automatically turned the area of the disk into an overestimate (since it also includes parts of unit squares that are not completely inside the disk). Thus, we have that the overestimate is:

$$\frac{1}{4}\left(\frac{\text{Area of Disk}}{\text{Area of Unit Square}}\right)=\frac{(n^2+m^2)\pi}{4},$$

where we divide by 4 to indicate we care about the first quadrant (positive, n,m only).

4.2 Lower Bound

Next, we will try to work out a lower bound. We will try to purposely underestimate the number of lattice points within the circle. In order to do this, we will again follow a similar procedure as we did with the upper bound as choose a unique lattice point for each unit circle as follows:

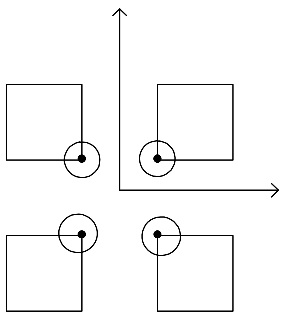


Figure 9. Bottom-Diagonal Lattice Point Unique Identification

We know that any unit circle that is include is most definitely a lower bound since any unit circle that is only partially covered by the disk will correspond to a lattice point that is truly inside the disk.

The only thing we must account for is the axes points which must not be counted. Interestingly, all the axes points are double counted and the origin is counted 4 times. Therefore, if we get an overestimate of the number of the number of axes points, multiply it by 2 and add 4, that will give us a good overall lower bound.

The number of axes points can be overestimated with $\sqrt{(n^2+m^2)}$. Therefore, we have the overestimate of all axes points as $8\sqrt{n^2+m^2}+4$. Thus, our overall underestimate is

$$\frac{(n^2+m^2)\pi-8\sqrt{n^2+m^2}-4}{4}=\frac{(n^2+m^2)\pi}{4}-2\sqrt{n^2+m^2}-1.$$

4.3 Applying Courant's Nodal Domain Theorem

Putting the upper bound and lower bound together, we get that

$$\frac{(n^2+m^2)\pi}{4}-2\sqrt{n^2+m^2}-1\leq N(\lambda_{\text{nm}})\leq\frac{(n^2+m^2)\pi}{4}.$$

We can make some quick observations about this bound. We know that the number of nodal domains of $\phi_{n,m}=m\,n$. Thus, to integrate this result with Courant's Nodal Domain Theorem, we want to show that:

$$m\,n<\frac{(n^2+m^2)\pi}{4}-2\sqrt{n^2+m^2}-1.$$

In other words, that our lower bound on eigenvalue number is an upper bound for the number of nodal domains. We do not expect it to be true for all n, m since $\frac{(n^2+m^2)\pi}{4} - 2\sqrt{n^2+m^2} - 1$ is a lower bound, but we do expect that there exists some point such that after (n^2+m^2) is greater than a certain number that the bound always holds.

To proceed, we already know via algebraic inequality that

$$m\,n \leq \frac{m^2+n^2}{2}.$$

Thus, we want to check that:

$$\frac{m^2+n^2}{2} \leq \frac{(n^2+m^2)\pi}{4} - 2\sqrt{n^2+m^2} - 1.$$

We can show this by introducing a new variable $t = \sqrt{n^2+m^2}$. Thus, in terms of t, we wish to show that:

$$\frac{t^2}{2} \leq \frac{\pi\, t^2}{4} - 2t - 1.$$

If we rearrange the equation, we get that:

$$0 \leq \left(\frac{\pi}{4} - \frac{1}{2}\right) t^2 - 2t - 1.$$

This is only true for some $t \geq t_0$. We can find this t_0 by solving for the roots of the quadratic equation to get that

$$r_{\pm} = \frac{2}{(\pi-2)}(\pm 2 + \sqrt{2+\pi}).$$

A quick testing of values shows that for $[r_+, \infty)$ the polynomial is positive, thus, we get that $t_0 = \frac{2}{(\pi-2)}(2+\sqrt{2+\pi}) \approx 7.47$. Therefore for n, m that fulfill $\sqrt{n^2+m^2} > 7.47$, we get that the Courant Nodal Domain Theorem holds. Therefore, we have reassurance that our lower bound is a valid and a relatively tight lower bound. For the other values of $\sqrt{n^2+m^2} < 7.47$, we can manually check and confirm that the Courant Nodal Domain Theorem holds. Thus, we have verified the Courant Nodal Domain Theorem for the square case.

Finally, it is important to note that our analysis here can be formed on many different geometry types, such as torus and spheres. The torus case is remarkably similar to the square case since the boundary conditions in the torus require that the "boundaries" have the same value. The more difficult case is the sphere case in which one must consider a more complicated Laplacian and eigenfunctions. These cases will be discussed in future sections.

4.4 Stern's Results on Nodal Domains

This section will give a summary of Antoine Stern's results. This section is based off the paper by Berard and Helffer [6] where they summarize and expand Stern's work. They discuss the important question of how "low" the number of nodal domains can get for eigenfunctions with large eigenvalues. The Courant Nodal Domain Theorem tells us how large the number of nodal domains can get but nothing about a possible lower bound. In general, the number of nodal domains can be quite small (i.e. it can be 2 even for eigenvalues that are far from the first). The paper [6] goes over many important findings from Stern about such sequences of eigenfunctions with small number of nodal domains and expands on them.

The main focus of the paper is the sparsely-published work of Antoine Stern (1924) in her PhD thesis under R. Courant. Stern showed that in the square (Dirichlet) and 2-sphere that there exists an infinite sequence of eigenvalues and corresponding eigenfunctions with exactly two nodal domains. Stern even provided an exact sequence of eigenvalues. She showed that on the square, the sequence of eigenfunctions with 2 nodal domains have eigenvalues $\lambda_{2m,1} = 1 + 4m^2$. Her proof technique was via a checkerboard argument (based on the sign of the function +/-) and analysis of critical zeros of the eigenfunctions on the square. The case of $\lambda_{4,1}$ is discussed in-depth in the paper [6].

The remarkable feature of Stern's results is that it can be applied to different geometries and domains very easily. For example, even though Stern herself did not consider these geometries, infinite sequences with as low as two nodal domains on the the torus and sphere can be discovered and proved using her technique.

Stern's technique can be dramatically visualized by looking at eigenfunctions of the form $u_{2r,1} + \mu u_{1,2r}$. As discussed above the key eigenvalue is $\lambda_{2r,1} = 1 + 4r^2$ and the eigenfunctions are of the form $u_{2r,1} + \mu u_{1,2r}$ where μ is a real parameter close to 1. When $\mu = 1$, there are a lot of nodal domains, but right when μ departs from 1, it "splits" up these nodal domains and connects all the previously separate regions. This is nicely depicted for the case $r = 3$.

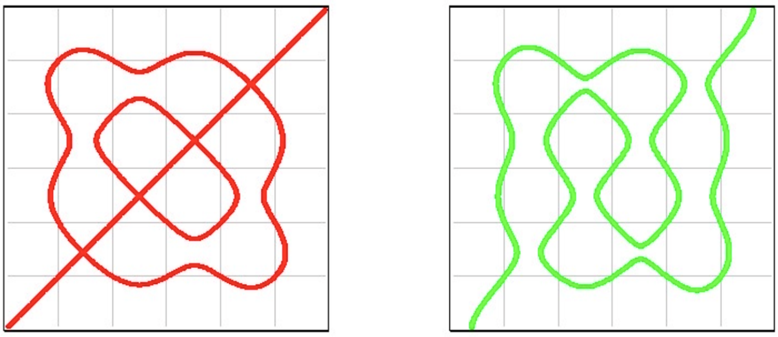
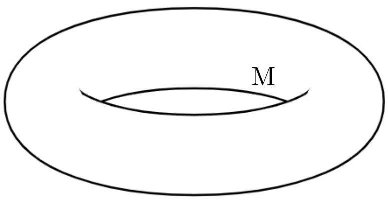


Figure 10. Nodal Set of λ_{37} ($\lambda_{6,1}$) [6] with Different Constants

5 Other Geometries and Extensions

5.1 Torus

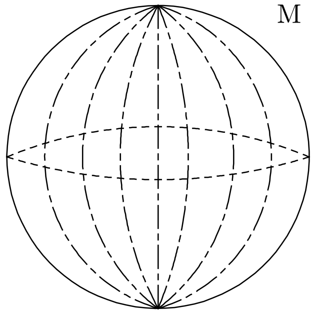


The torus case shares great similarity to the square case. Given that we are working with Dirichlet conditions, the eigenfunctions on the torus case can be simply identified by replacing the sine functions with cosine functions in order to incorporate the "even" aspect of the torus boundaries.

Given that sine and cosine have similar periodic nodal points, the analysis of eigenvalue number turns out to be very similar in torus case as the square case. The eigenfunctions correspond to lattice points and we can develop similar bounds to determine in $N_{\text{torus}}(\lambda)$.

As with $N(\lambda)$, the similarity between the square and torus case also carries over with respect to the possibility of having only two nodal domains even in eigenfunctions with large eigenvalues. In fact, as shown in a paper by Jakobson and Nadirashvili [12], we see that it is easier to find such a sequence of eigenfunctions with 2 nodal domains on the Torus case without using Stern's method [6].

5.2 Sphere



The sphere is a departure from the simpler cases of the square and the torus. The eigenfunctions are no longer simple combinations of trigonometric functions but instead the spherical harmonics. While working in the spherical case, we work with spherical coordinates and the spherical laplacian. This allows us to nicely deal with the boundary conditions.

In the sphere case, we again get a "checkerboard" like pattern with the nodal curves in polar coordinates. Although Stern did not discuss the the Sphere case, Berard and Helffer [6] show that one can attain two nodal domains for a sequence of eigenfunctions on the 2-sphere. Specifically, they showed that:

Theorem 11. (Berard and Helffer [6]) *There exists an infinite sequence of eigenvalues of the 2-sphere, tending to infinity, and associated spherical harmonics with exactly 2 nodal domains.*

Berard and Helffer have an even more general version of this theorem in which they show that one can find a sequence of eigenfunctions with a specified number of integer domains greater than three (albeit with some more restrictions on the integer of choice).

5.3 The Quantum Harmonic Oscillator

The two-dimensional isotropic quantum harmonic oscillator is defined as

$$\hat{H} := -\Delta + |x|^2,$$

acting on $L^2(\mathbb{R}^2, \mathbb{R})$.

Our analysis on the square, torus, and sphere of the lower bound of nodal domains is not limited to the simple Laplacian. Similar results can be shown for the 2D quantum harmonic oscillator. Moreover, the method of proof for the Schrodinger operator is very similar and heavily inspired from the methods that Stern used in the Laplacian case on the square.

It is quite fortuitous that the quantum harmonic oscillator can be put under a similar analysis. Although surpassing, many theorems, like the Faber-Krahn inequality, have almost exact analogs in the quantum harmonic oscillator case as seen in Charron [15]. These analogous results provide the foundation for the analysis of nodal domains on the quantum harmonics oscillator and allow us to use the same methods in both cases.

Leydold [13] showed that given H_n as the n^{th} eigenspace of the isotropic quantum harmonic oscillator, with eigenvalue $2(n+1)$. If $n = 4\,k$ for $k \geq 1$, there exists an eigenfunction in H_n with exactly three nodal domains and that it is the best possible lower bound.

Moreover, Berard and Helffer in [14] have shown using a very similar perturbation argument as Stern to identify such eigenfunctions discussed in Leydold and many other interesting sequences in the quantum harmonic oscillator case. In the same paper, they have also shown the rigorous existence of a sequence of eigenfunctions with "many" eigenfunctions.

This analysis of the quantum harmonic oscillators shows another potential application of the Courant Nodal Domain Theorem and how the theorems and methods built for analyzing the Dirichlet Laplacian can be applied further to more general operators. Quantum harmonic oscillators are an important are of research in mathematical physics and partial differential equations, and these analyses can provide important insights into related problems.

6 The Discrete Laplacian on Graphs

6.1 Courant Nodal Domain on Graphs

The Courant Nodal Domain Theorem has significant applications to graphs. The famous results of Fielder used for spectral partitioning are an example of its success and capabilities.

A discrete eigenvector on a graph is defined only on the vertex set V of a graph Γ. We assume that the graph is simple, undirected, and loop-free. We assume that it's matrix representation is A and that it's Laplacian is L with associated quadratic form

$$\mathfrak{L} = u^T L\, u = \sum_{P_i \sim P_j} (u_i - u_j)^2.$$

The discrete Laplacian eigenvalue problem similarly has a discrete spectrum; the eigenvalues are positive and have finite multiplicity. Additionally, the eigenvalues are orthogonal. Although we will not prove these properties explicitly here, the results follow directly from well-known linear algebra results on Hermitian matrices.

Since we want to make analogous results as in the continuous case in the discrete case, we need to define corresponding concepts in the discrete case. Firstly, we will define the discrete counterpart of nodal domains. Nodal domains are slightly more difficult in the discrete case since the nodes of the graph need not be vertexes (for example, a vertex does not need to be 0 and instead the sign of the vertexes can switch across one edge). This motivates the following definitions, which are defined in terms of an eigenfunction u_i defined on the graph Γ:

Definition 12. *A* ***strong*** *positive (negative) sign graph S is a maximal, connected subgraph of Γ, on vertexes $P_i \in V$ with $u_i > 0$ $(u_i < 0)$*

Definition 13. *A* ***weak*** *positive (negative) sign graph S is a maximal, connected subgraph of Γ, on vertexes $P_i \in V$ with $u_i \geq 0$ $(u_i \leq 0)$ and with at least one $P_i \in V$ having $u_i > 0$ $(u_i < 0)$.*

In the graph below, we have an example of what "nodal domains" will look in the discrete case. In this case, they are both strong and weak sign graphs.

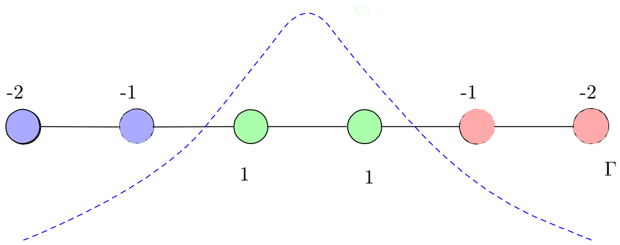


Figure 11. Nodal Domains on Graphs

There are many interesting analogous theorems we can prove corresponding to our previous results in the continuous case with the Courant nodal domain theorem. One with remarkable similarity is the following:

Theorem 14. (Weak Graphs) *If Γ is connected, then any eigenvector corresponding to λ_n has at most n weak sign graphs.*

The theorem on weak graphs looks very similar to the Courant Nodal Domain Theorem. In order to prove this theorem, we need to bring in some important definitions and lemmas.

Definition. (Adjacent Graphs) *Two different strong or weak sign graphs S_1, S_2 are said to be adjacent if there exist $P_1 \in S_1$ and $P_2 \in S_2$ such that $P_1 \sim P_2$, where P_1 and P_2 are vertexes and $\sim$ represents sharing an edge.*

Now, we will set up some notation and definitions for the rest of this section. Assume that u has m strong sign graphs, S_i for $1 \le i \le m$, we will define w_i such that $w_i = u$ on S_i but 0 everywhere else. We then can see that by construction

$$u = \sum_{i=1}^{m} w_i,$$

we now form the following function:

$$v = \sum_{i=1}^{m} c_i w_i,$$

where c_i are the weights of v.

We can identify some basic facts about the zero vertexes of u.

Lemma 15. *The zero vertex belongs to exactly one weak sign graph and exactly one weak negative sign graph. Moreover, if two different weak sign graphs S_1, S_2 have a non-empty intersection, then they must have opposite signs.*

We will now use these facts to prove a lemma about adjacent weak sign graphs.

Lemma 16. *Suppose S_1, S_2 are adjacent weak sign graphs. There is a pair of vertexes P_1, P_2 such that $P_1 \in S_1$, and $P_2 \in S_2 / S_1$ and $P_1 \sim P_2$*

Proof. Assume that S_1 is weak positive and S_2 is weak negative. If S_1, S_2 are disjoint, then there must exist $P_1 \sim P_2$ since the two graphs are adjacent. If they have a non-empty intersection, then $S_1 \cap S_2$ is a subgraph of Γ. Not all the vertexes $P_1 \in S_1 \cap S_2$ can be interior vertexes. Any boundary vertex P_1 will have the required property such that $P_2 \sim P_1$ □

We suppose that u has $m \ge n$ weak sign graphs S_i, and we have

$$v = \sum_{i=1}^{m} c_i w_i,$$

and choose c_i not all zero. Moreover, we choose the c_i to make v orthogonal to the first $m-1$ eigenvectors of L.

Just like we had a unique continuation principle for the coefficients in the continuous case, we will again have a unique continuation principle reference (but for the discrete case).

Lemma 17. (analog of unique continuation) *Suppose $m > n$ and two of the weak sign graphs of S_1 and S_2 of u are adjacent. Without loss of generality, we may suppose that S_1 is weak positive and S_2 is weak negative. Then $c_1 = c_2$, where c_i are the corresponding relative weights in v.*

We will not prove this lemma here, but it is proven by Davies [5]. Now, this lemma gives us the key component to complete our proof for Theorem 15 on weak graphs. We now present a proof of this theorem:

Proof. Assume for contradiction that u has m weak sign graphs, denoted S_i for $i = 1, \dots m$ and $m > n$ consider v defined as above. At least one of the coefficients (without loss of generality, the first coefficient) c_1 is nonzero since we required that v be non-zero. Then, since $n \geq 1$, we have that $m \geq 2$ since $m > n$.

Since Γ is connected, S_1 must be adjacent to at least one other sign graph, S_2. Now, using the analog of the unique continuation principle (the previous lemma) we have that it must be the case that $c_1 = c_2$. If $m \geq 3$, again due to the connectivity of Γ, one of the S_1 or S_2 must be adjacent to another sign graph S_3, thus $c_1 = c_2 = c_3$ again by the lemma.

We repeat this $m - 1$ more times until we get that $c_1 = c_2 = \cdots = c_m$. Therefore, we have that it must be the case:

$$v = \sum_{i=1}^{m} c_i\, w_i = c_1 \sum_{i=1}^{m} w_i = c_1\, u.$$

However, this contradicts the assumption v is orthogonal to the eigenvectors of L. Therefore $m \leq n$ $\square$

Thus, we have shown this analog of Courant's nodal domain theorem in the graph case.

6.2 Fiedler's Results

The analysis of the Laplacian on graphs is a very fruitful area of research that has been explored deeply. Results like Theorem 15 are built on the foundation of many results about the Laplacian eigenvalue problem on graphs. A combination of linear algebra and graph theory results come together to provide theorems and lemmas that we implicitly assumed in the previous section.

Results like Theorem 15 contribute to the much bigger field of eigenvalue graph problems. Some of the most famous and applicable results are by Miroslav Fiedler. His analysis on Laplacian eigenvalues has revealed the connections between the Laplacian eigenvalue problem and the algebraic connectivity properties of graphs. Fiedler's insights are especially useful in spectral graph partitioning algorithms. Seen below is an example pseudocode for a partitioning algorithm:

Algorithm

1. Compute the second eigenvector, $\boldsymbol{x}$, of the laplacian
2. Sort the vertices so that **x**'s elements are arranged as:
$$x_1 \geq x_2 \dots \geq x_n$$
3. Let $S_i = \begin{cases} \{1, \dots i\} & i \leq n/2 \\ \{i+1, \dots n\} & i > n/2 \end{cases}$ be subgraphs
4. return $\min_i \{\psi(S_i)\}$

where ϕ is the conductance of a subgraph, defined on a graph $G = (V, E)$ with vertices and edges V and E, as

$$\phi(G) = \min_{S \subset V, \mathrm{vol}(S) \leq |E|} \psi(S), \text{where} \quad \psi(S) := \frac{|E(S, \bar{S})|}{\mathrm{vol}(S)},$$

that is, the ratio of the number of edges cut to the volume of the set (sum of degrees). Many problems in computer science revolve around determining how connected a graph is and determining optimal ways to partition graphs into subgraphs with the goal of having the number of cuts being minimized and the number of vertices in each subgraph as equal as possible.

One of the most important results Fiedler has shown toward this goal is that the two subgraphs resulting from the spectral partitioning algorithm are always connected. This ties very closely to our analysis that the second Laplacian eigenvector has two weak sign graphs, which we showed in Lemma 17. Fiedler also has results on giving tight bounds on the number of weak sign graphs for specific types of graphs.

This field is still very active and current research still tries to find better and better algorithms to partition a graph. Furthermore, spectral partitioning in itself has applications to Markov chain mixing, high dimensional expanders, and many more important applied problems.

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8 Appendix

8.1 Linear Algebra Results

Hermitian matrices have remarkably nice properties. These properties are explored thoroughly in Matrix Analysis [11]. These properties give rise to two important theorems about Hermitian matrices that are especially related to our analysis of the Dirichlet Laplacian eigenvalue problem.

The first theorem is the Rayleigh Quotient theorem, very similar to the theorem presented in this report.

Theorem 18. *Let $A \in M_n$be Hermitian, let the eigenvalues of A be ordered as in:*

$$\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}.$$

Let $i_1, \ldots i_k$ be given integers with

$$1 \leq i_1 < \ldots < i_k \leq n,$$

let $x_{i_1}, \ldots, x_{i_k}$ be orthonormal and such that

$$A\, x_{i_p} = \lambda_{i_p} x_{i_p},$$

for each $p = 1, \ldots, k$ and let $S = \operatorname{span}\{x_{i_1}, \ldots x_{i_k}\}$ then we have that:

a)

$$\begin{aligned} \lambda_{i_1} &= \min_{\{x : 0 \neq x \in S\}} \frac{x^* A\, x}{x^* x} \\ &\leq \max_{\{x : 0 \neq x \in S\}} \frac{x^* A\, x}{x^* x} \\ &= \lambda_{i_k}. \end{aligned}$$

b) For any unit vector x,

$$\lambda_{i_1} \leq \ x^* A\, x \ \leq \lambda_{i_k}.$$

c) For any unit vector x,

$$\lambda_{\min} \leq \ x^* A\, x \ \leq \lambda_{\max},$$

and

$$\begin{aligned} \lambda_{\min} &= \min_{x \neq 0} \frac{x^* A\, x}{x^* x} \\ \lambda_{\max} &= \max_{x \neq 0} \frac{x^* A\, x}{x^* x}. \end{aligned}$$

Proof. The intuition of the proof stems from the decomposition of A into eigenvectors. We assume that x is unit length in S. Then we can write x as

$$x = \alpha_1\, x_{i_1} + \cdots + \alpha_k\, x_{i_k}.$$

In this form, we see that:

$$1 = x^* x = |\alpha_1|^2 + \cdots + |\alpha_k|^2.$$

This implies that:

$$x^* A\, x = |\alpha_1|^2 \lambda_{i_1} + \cdots + |\alpha_k|^2\, \lambda_{i_k},$$

from the convexity of the square function, it is clear that the optimization will extract the minimum or maximum eigenvalue. □

Remark. We can also see a geometric interpretation of this theorem (specifically part c). We can think of $\lambda_{\max}$ as the maximum of the continuous real-valued function $f(x) = x^* A\, x$ on the unit sphere in $\mathbb{C}^n$ (a compact set) since we are restricting to unit length.

From this theorem, we can go further to get almost exactly the same theorem as we had in this report (Theorem 3 and 4). It is called the Min-Max theorem or the Courant-Fischer Theorem:

Theorem 19. (Courant-Fischer Theorem)

Let $A \in M_n$ be a Hermitian and let $\lambda_1 \leq \ldots \leq \lambda_n$ be the algebraically ordered eigenvalues. Let $k = \{1, \ldots n\}$ and let S denote a subspace of $\mathbb{C}^n$. Then

$$\lambda_k = \min_{\{S:\dim(S)=k\}} \max_{\{x:0\neq x\in S\}} \frac{x^* A\, x}{x^*\, x},$$

and

$$\lambda_k = \max_{\{S:\dim(S)=k\}} \min_{\{x:0\neq x\in S\}} \frac{x^* A\, x}{x^*\, x}.$$

The Courant-Fischer theorem follows very nicely from the Rayleigh Quotients theorem. In the Rayleigh quotients, we only had one optimization (a min **or** a max). This was because the set that we were searching in was limited and we were only expecting to get out the maximum or minimum eigenvalue. The Courant-Fischer theorem generalizes to any eigenvalue λ_k. The way that it is able to generalize is by limiting the search domain S behind the scenes with the second optimization. For example, by taking the minimum of all the maximums, it ensures that its picking up the λ_k eigenvalue instead of λ_{k+1} since the minimization chooses the correct search space S that has dimension i and contains the k^{th} eigenvector.

8.2 Domain Monotonicity of Eigenvalues (Neumann Data)

Let $\Omega_1, \ldots \Omega_m$ be as above, and also assume:

$$\bar{M} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \ldots \bar{\Omega}_m.$$

For each $r = 1, \ldots m$ add Neumann data to $\partial\Omega_r \cap M$ and leave original data on $\partial\Omega_r \cap M$ unchanged. Arrange all the eigenvalues of $\Omega_1, \ldots, \Omega_m$ in increasing order, with repetition according to multiplicity:

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots$$

Then for each $k = 1, 2, \ldots$ we have

$$\mu_k \leq \lambda_k.$$

Proof. Let $\psi_i: \bar{M} \to \mathbb{R}$ be the eigenfunction of μ_i when ψ_i is restricted to the appropriate subdomain and ψ_i be 0 identically everywhere else. Now if f is any function in $\mathbb{H}(M)$, then $f \in \mathbb{H}(\Omega_r)$ for every $r = 1, \dots m$. We can therefore argue that if f is orthogonal to $\psi_1, \dots \psi_{k-1}$ in $L^2(M)$, then:

$$D[f, f] = \sum_{r=1}^{m} \int_{\Omega_r} \|\nabla f\|^2 \, \mathrm{dV} \geq \sum_{r=1}^{m} \mu_k \int \int_{\Omega_r} f^2 \, \mathrm{dV} = \mu_k \|f\|^2,$$

but there exists a nontrivial

$$f = \sum_{j=1}^{k} \alpha_j \phi_j,$$

orthogonal to $\psi_1, \dots \psi_{k-1}$ in $L^2(M)$. Then,

$$D[f, f] \leq \lambda_k \|f\|^2. \qquad \square$$

8.3 The Unique Continuation Principle

We use the unique continuation principle at the end of our proof of the Courant Nodal Domain Theorem to prove our contradiction. We will provide here the statement of the theorem and some discussion.

The unique continuation principle was proven in remarkable generality by N. Aronszajn [7]. There are many forms of the unique continuation principle, and we will just provide a specific form that is useful for our analysis.

If we consider a general operator of the form $P = -\Delta + q$ where $q \in L^\infty(\Omega)$. Then, we have that:

Theorem 20. (Unique Continuation Principle) *If $u \in H^2(\Omega)$ satisfies*

$$Pu = 0 \quad \text{in} \quad \Omega,$$

and we have that:

$$u = 0 \quad \textit{in some ball } B \textit{ contained in } \Omega,$$

then $u = 0$ in Ω.

This theorem is very powerful and useful especially in the situation we set up at the end of the Courant Nodal Domain Theorem proof. It is able to deliver the final blow for contradiction.

A quick corollary follows from this theorem:

Corollary 21. *Since each eigenfunction is $C^\infty(M)$ and has the unique continuation property, it cannot vanish on an open set.*

This corollary was used in our contradiction argument in the Courant Nodal Domain Theorem.

8.4 Bessel Functions

Bessel functions are solutions to the differential equation:

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - n^2)\, y = 0,$$

for an arbitrary complex number n, which is called the "order" of the Bessel function. We will concentrate on the case that n is an integer or half-integer.

The solution of interest to us are called Bessel functions *(of the first kind)*. One can write out a series expansion as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma\,(m+n+1)} \left(\frac{x}{2}\right)^{2m+n},$$

where Γ is the gamma function (factorial). We are often interested in the roots of these function. Further discussion on Bessel's functions can be found in Borthwick [1].

Bibliography

1. Borthwick, David. *Spectral Theory: Basic Concepts and Applications*. Vol. 284. Springer Nature, (2020).

2. Chavel, Isaac. *Eigenvalues in Riemannian geometry*. Academic press, (1984).

3. Courant, Richard, and Hilbert, David *Methods of Mathematical Physics* Vol. 1. (1966)

4. Horn, Roger A., and Charles R. Johnson. *Matrix analysis*. Cambridge university press, (2012).

5. Davies, E. Brian, Gladwell, Graham M. L, Leydold, Josef, Stadler, Peter F. Discrete nodal domain theorems. Linear Algebra Appl. 336 (2001), 51–60.

6. Bérard, Pierre, and Helffer, Bernard. "Nodal sets of eigenfunctions, Antonie Stern's results revisited." *Séminaire de théorie spectrale et géométrie* 32 (2014): 1-37.

7. Aronszajn, Nachman. "A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order". Kansas University, Lawrence, (1956).

8. A.M. Dval, V. Reiner, "Perron–Frobenius type results and discrete versions of nodal domain theorems", Linear Algebra Appl. 294 (1999) 259–268.

9. Sard, Arthur. "The measure of the critical values of differentiable maps." *Bulletin of the American Mathematical Society* 48.12 (1942): 883-890.

10. Faber, G. "Beweis, dass unter allen homogenen Membranen von gleicher Flache und gleicher Spannung die kreisformige den tiefsten Grundton gibt", Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu Munchen Jahrgang (1923), 169–172

11. Horn, Roger A., and Charles R. Johnson. *Matrix analysis*. Cambridge university press, (2012)

12. Jakobson, Dmitry, and Nikolai Nadirashvili. ''Eigenfunctions with few critical points." *Journal of Differential Geometry* 53.1 (1999): 177-182.

13. Leydold, Josef. *Knotenlinien und Knotengebiete von Eigenfunktionen*. Diss. uniwien, (1989)

14. Bérard, Pierre, and Helffer, Bernard. "Some nodal properties of the quantum harmonic oscillator and other Schrödinger operators in $\mathbb{R}^2$." *Geometric and Computational Spectral Theory* 700 (2017): 87-116.

15. Charron, P. " On Pleijel's theorem for the isotropic harmonic oscillator". Memoire de Maıtrise, Universite de Montreal (2015). 5, 6.