Computing F-Matrices for Modular Tensor Categories

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Abstract

In this paper, we find algorithms to compute the F-matrices for modular tensor categories, and in particular relevant anyon theories. Moreover, we implement this algorithm in SageMath.

Contents

1 Introduction

Topological quantum computing [1] is a proposal for a quantum computer that is theoretically distinct from (but computationally equivalent [2] to) the usual gate/circuit model of quantum computing. This proposal deals with topological states of matter. Roughly speaking, this means that the states are invariant under small perturbations, such as noise. This makes them naturally resilient to decoherence. To compute, one changes the topology of the state. One can visualize this in Fig. 1.

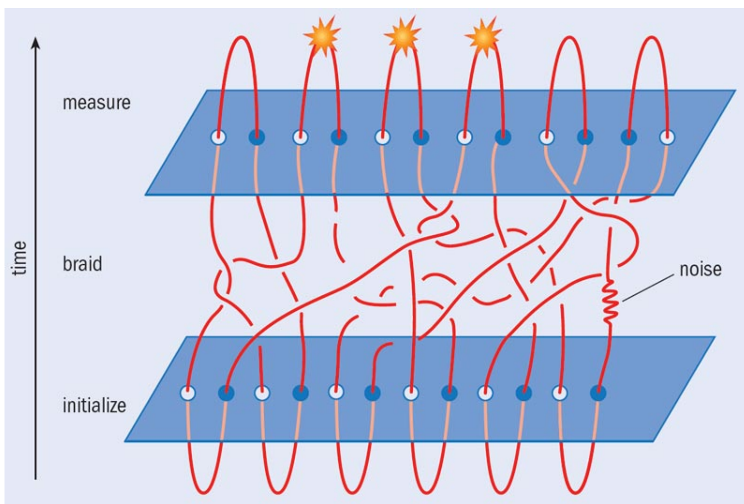

Figure 1: Visualization of TQC from [3]

The particles in TQC are the anyons, which are two dimensional particles unaffected by diffeomorphisms. There might be several kinds of anyons, with different creation and destruction rules. From creation time until measurement time, the only data we care about is how these particles twist and braid each other, since that is the topological data that exists. Physically, this hinges on the existence of non-abelian anyons. Recently some evidence of their existence has been found [4]. Mathematically, this is formalized using (Unitary) Modular Tensor Categories.

The topological manipulations on anyons induce a linear operation on the state they represent. The first challenge is to discover what are these linear operations, and in particular representing the so-called F and R-matrices. In this paper, we propose and implement and algorithm to do this. Unless mentioned otherwise, the presented results have already been known in literature.

In section 2.1 we formally define modular tensor categories, and the objects we are interested in. In section 2.2, we define a Groebner basis, which is the main algorithmic tool we use. In section 3 we outline the algorithm we propose, which is our main result. In section 3.1 and 3.2 we provide implementation and optimization details needed to make the algorithm workable. Finally, in section 4 we conclude with the current outlook of future work. Additionally, in the appendix we include several F matrices found by our algorithm. Our contributions are the optimizations on the algorithm in section 3 and the open source implementation of it.

2 Background

2.1 Modular Tensor Categories

In this section we will introduce the axioms regarding a Modular Tensor Category (MTC). According to nLab, it is fusion category that is also a ribbon category such that the "modularity operation" is non-degenerate [5]. We will assume familiarity with category theory, and in particular monoidal and abelian categories. We will expand this definition following [5–8], but in particular [6].

Through this section, there are three views we should keep in mind. We can view a MTC through the axiomatic definition, through the physical interpretation, and through the graphical notation. With this last one, we emphasize that the diagrams are actually representing elements, and we can operate on them.

2.1.1 Fusion category

A fusion category $\mathcal{C}$ is a monoidal category such that:

1. There is a finite number of simple objects.
    - (a) We label these simple objects by a finite set $\mathcal{L}$ (label set). Graphically, these will be labelled dots. Physically, these are the anyons.
    - (b) There is a distinguished element $1 \in \mathcal{L}$. Graphically, nothing. Physically, this is the vaccum.
    - (c) If $a \not\cong b$ are simple objects, then $\mathrm{Hom}(a, b)$ only has the trivial morphism.
2. It is a semi-simple category:
    - (a) There is addition bifunctor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
    - (b) Any object can be decomposed as the sum of simple objects. If a, b are simple, we can think of $a \oplus b$ as the particle in superposition with a or b.
3. It is a $\mathrm{Vect}_{\mathbb{C}}$-enriched category. That is, for any two objects a, b, the set $\mathrm{Hom}(a, b)$ is a $\mathbb{C}$-vector space. Graphically, this means we consider "formal sums" of diagrams. This is used to represent the quantum state.
4. The endomorphisms of the unit object form the field of complex numbers. that is, $\mathrm{Hom}(1, 1) = \mathbb{C}$.

Essentially, it is a monoidal category that is "finite" in an appropriate sense.

We will now explore some properties and structures implied by a fusion category.

Since $\mathcal{C}$ is semisimple and monoidal, we can decompose the tensor product of two simple objects into the direct sum of simple objects. That is, $i \otimes j = \oplus_{k \in \mathcal{L}} N_k^{ij} k$ for some coefficients N_c^{ab}. We call these the **fusion coefficients**, and a collection of label set and fusion coefficients a **fusion rule**. From a physical perspective, this means that if we interact i with j, there are N_k^{ij} ways to create k.

For our purposes we will assume that for each i, j, k, there is at most one way to do this. That is, $N_k^{ij} \in \{0, 1\}$. Graphically, we represent this as Fig. 2.

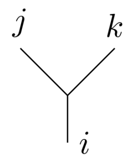

Figure 2: Representation of interacting j and k, resulting in i.

Note that from the presented structure, we can conclude that if x is an object and a is simple, then $\mathrm{Hom}(x, a)$ has a dimension equal to the number of components a in x. In particular, the dimension of $\mathrm{Hom}(a \otimes b, c)$ is N_c^{ab}.

Recall that the tensor is associative only up to isomorphism. This means there is a change of basis isomorphism $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$. Translating this into the label set, this data is the same as a family transformations $F_l^{ijk} : \mathrm{Hom}((i \otimes j) \otimes k, l) \to \mathrm{Hom}(i \otimes (j \otimes k), l)$. We can understand $\mathrm{Hom}((i \otimes j) \otimes k, l)$ as the processes that from i and j create m, and from m and k create l. Thus we can label the basis vectors of $\mathrm{Hom}((i \otimes j) \otimes k, l)$ by such m. Note that each m as a multiplicative scalar degree of freedom.

Assume we fix a basis for each space $\mathrm{Hom}(a \otimes b, c)$. This means we can understand F_l^{ijk} as a matrix with row and column indices also in the label set. Figure 3 represents this graphically. We emphasize that we interpret the diagrams as literally representing elements of the Hom spaces.

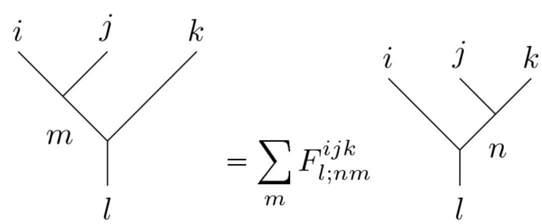

Figure 3: Change of basis, F-matrix. From [6]

We can prove $((a \otimes b) \otimes c) \otimes d \cong a \otimes (b \otimes (c \otimes d))$ in two distinct ways.

- $((a \otimes b) \otimes c) \otimes d \cong (a \otimes b) \otimes (c \otimes d) \cong a \otimes (b \otimes (c \otimes d))$, by pivoting over c then b.
- $((a \otimes b) \otimes c) \otimes d \cong (a \otimes (b \otimes c)) \otimes d \cong a \otimes ((b \otimes c) \otimes d) \cong a \otimes (b \otimes (c \otimes d))$, by pivoting over b, then $b \otimes c$, then c.

The pentagon axiom of a monoidal category impplies that both maps are the same. In terms of simple objects, we can rewrite this graphically as in Fig. 4.

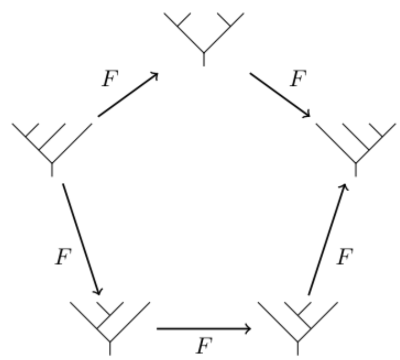

Figure 4: Visualization of the pentagon axiom

Recall that we understand the diagrams as elements of the Hom sets and F as a linear transformation. We call this the pentagon axiom, and it will be the main tool to calculate the F-matrices.

2.1.2 Braided Monoidal Category

A braided monoidal category $\mathcal{C}$ is a monoidal category such that there exist a natural isomorphisms $x \otimes y \to y \otimes x$, for every pair of objects x, y.

Note that if we have $(x \otimes y) \otimes z$ there are two ways to ways to rotate it to $(y \otimes z) \otimes x$. We can either

1. $(x \otimes y) \otimes z \cong x \otimes (y \otimes z) \cong (y \otimes z) \otimes x$, applying associativity then exchanging x with $y \otimes z$.
2. $(x \otimes y) \otimes z \cong (y \otimes x) \otimes z \cong y \otimes (x \otimes z) \cong y \otimes (z \otimes x) \cong (y \otimes z) \otimes x$, by exchanging y with x then z with x, together with the correct associativity.

The first hexagon axiom says both maps are the same.

A similar argument works with the opposite rotation, obtaining a pair of maps from $x \otimes (y \otimes z)$ to $z \otimes (x \otimes y)$. The second hexagon axiom says both maps are the same.

If we assume that $\mathcal{C}$ is additionally a fusion category, we can express this braiding as a change of basis $R : \mathrm{Hom}(j \otimes k, i) \to \mathrm{Hom}(k \otimes j, i)$, as in figure 5. In our case, this is just a complex scalar. Note that k goes *over* j. In the same train of thought, we can think of placing j over j as R^{-1}.

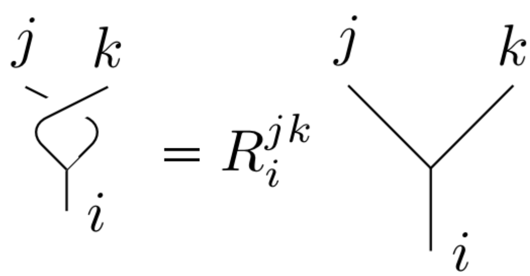

Figure 5: Change of basis for braiding

Furthermore, we can express one of the hexagon axioms graphically as in Fig. 6. The other can be expressed by replacing R with R^{-1}. Graphically, this means that

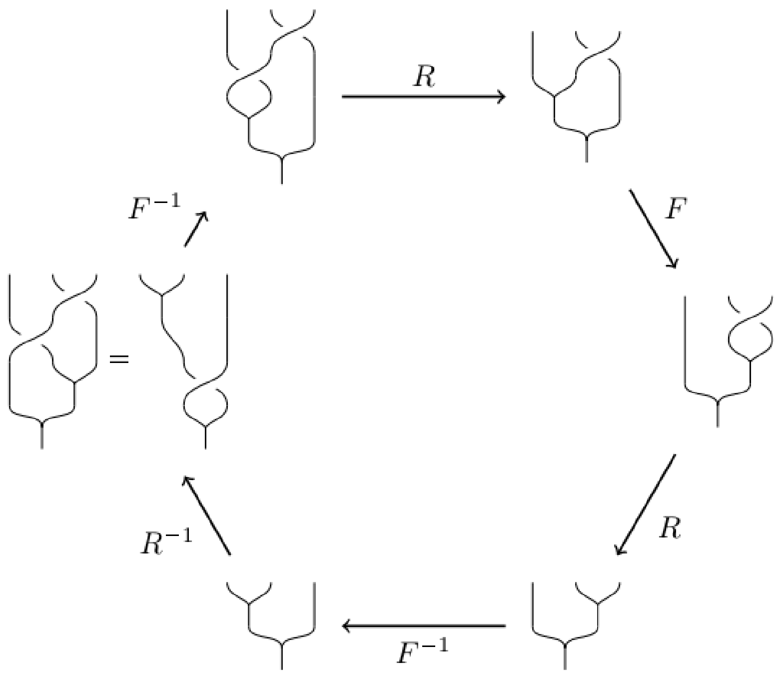

Figure 6: Hexagon axiom

These axioms will be the second tool we use to calculate the F-matrices.

2.1.3 Ribbon Category

A ribbon category $\mathcal{C}$ is a braided monoidal category such that for every object x, there exists an object x^*, such that there are maps $1 \to x \otimes x^*$ and $x \otimes x^* \to 1$, called creation and annihilation. Physically, these are the antiparticles. Furthermore, there are twist maps $a \to a$.

Graphically, the annihilation and creation maps can be represented as in Figure 7, and the twist maps can be represented by Fig. 8. Note that the equation in Fig. 8 is valid under the assumption that $\mathcal{C}$ is also fusion.

Figure 7: Annihilation and creation and are represented by cup and cap diagrams.

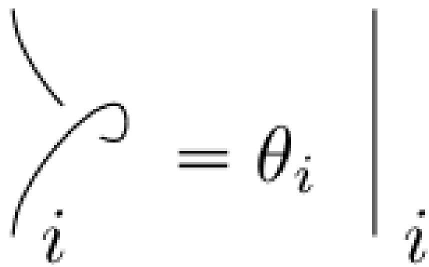

Figure 8: Twist in graphical notation

Furthermore, first the birth and annihilation maps must obey the following identity represented graphically in figure 9. We call these the zig-zag identities.

Figure 9: Straighten a line with creation and annihilation.

With this structure, if we have a map $f : a \to b$, we can define it's dual $f^* : b^* \to a^*$, which we can express graphically in Figure 10. Note that by the zig-zag axioms, $f^{**} = f$.

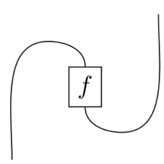

Figure 10: Dual map f^*

The twist must be compatible with the monoidal structure, in the sense that twisting $a \otimes b$ must come from individual twists together with braidings. Graphically, we see this axiom in Fig. 11.

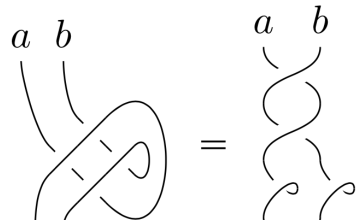

Figure 11: Twist axiom.

Additionally, twisting the vacuum does nothing, and twisting an antiparticle is the dual of twisting the particle.

With this structure, we can define the (left/rightt) trace of a map $f : a \to a$. Graphically, these are represented in Fig. 12

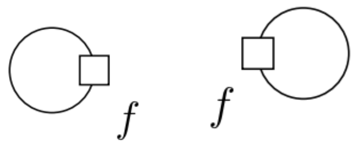

Figure 12: Left and right trace of f

2.1.4 Monoidal Tensor Category

Let $\mathcal{C}$ be a fusion ribbon category. We can construct coefficients $s_{i,j}$ by creating i, i^*, j, j^*, braiding twice i^* and j, then destroying them. Graphically, we have two interlocking rings as in Fig. 13.

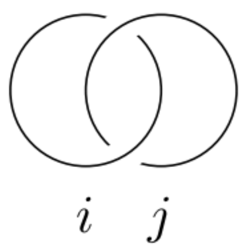

Figure 13: Modular $\tilde{s}_{ij}$ coefficient

We can collect these coefficients into a matrix $\tilde{S}$, indexed by the labels. If $\tilde{S}$ is non-singular, then we call $\mathcal{C}$ a modular tensor category.

Note that the coefficients F, R, θ are all dependent on the choice of basis.

2.1.5 Interpretation in TQC

To compute in TQC, we braid in different places. In terms of linear operations, this corresponds to

1. moving things to the correct basis: applying the relevant F matrix
2. using the braiding morphism (applying the relevant R matrix)
3. moving things to the original basis: applying the relevant F^{-1} matrix

This is why we are interested in computing the F (and R) matrices.

2.1.6 R-coefficients

Whenever $N_c^{ab} \neq 1$, R_c^{ab} is a non-zero scalar, since both $\text{Hom}(a \otimes b, c)$ and $\text{Hom}(b \otimes c, a)$ are one-dimensional. By [9], it is possible to choose a gauge [1] such that $(R_c^{ab})^2 = \frac{\theta_c}{\theta_a \theta_b}$ for every a, b, c. From

[1] A gauge is a physics term for a set of choices that fix the degrees of freedom in a system.

this we get, up to a sign, the exact value of R. Furthermore, it is possible to directly compute these twist coefficients [10] and there is an implementation in Sage.

This means we can treat the R coefficients as known.

2.1.7 Ocneanu rigidity

By Ocneanu rigidity [6], there is a finite number of (non-isomorphic) possible F-matrices, for a given label set and fusion rule. In particular, once all the degrees of freedom are fixed, we are guaranteed that a solution to the pentagon and hexagon equations is a valid solution for the coefficients of an MTC. While the graphical notattion is better to understand what is going on, we will need the actual equations to encode this on a computer.

We will restrict ourselves to the case where $N_c^{ab} \in \{0,1\}$. In the case of the pentagon axiom, this states that for all $a,b,c,d,e,f,p,q,m \in \mathcal{L}$:

$$\sum_n F^{bcd}_{q;pn} F^{and}_{f;qe} F^{abc}_{e;nm} = F^{apb}_{f;qm} F^{mcd}_{f;pe}$$

Note how this is a qubic equation in F.

In the case of the hexagon axioms, this states that for all $a,b,c,d,e,m \in \mathcal{L}$

$$(R^{ac}_e) F^{bac}_{d;em} (R^{ab}_m) = \sum_n F^{bca}_{d;en} (R^{an}_d) F^{abc}_{d;nm}$$
$$(R^{ac}_e)^{-1} F^{bac}_{d;em} (R^{ab}_m)^{-1} = \sum_n F^{bca}_{d;en} (R^{an}_d)^{-1} F^{abc}_{d;nm}$$

Note how this is quadratic in F. Since the R coefficients are already known, this greatly simplifies things.

2.2 Theories with $N_a^{bc} \in \{0,1\}$

The concrete MTCs we work with will be the fusion algebras of Wess-Zumino-Witten (WZW) conformal field theories or the Grothendieck groups of tilting modules for quantum groups at roots of unity. [11]. In particular, the product for these fusion algebras is already implemented in SageMat [12] .

While an explanation of the construction of this group is outside the scope of this paper, it suffices to say that they are characterized by Cartan type and level k, with higher levels indicating higher complexity. Among these, the fusion algebras that satisfy $N_a^{bc} \in \{0,1\}$ for all a,b,c are in table 1.

Cartan Type	Constraints
A_1	All k
A_r	$r \geq 1, k \leq 2$
C_2	$k \leq 2$
C_r	$r \geq 2, k = 1$
$B_r, D_r, G_2, F_4, E_6, E_7, E_8$	$k \leq 2$

Table 1: Cartan types and levels we are interested in.

2.3 Groebner basis

Consider a polynomial ring $R = K[x_1, \ldots, x_n]$ and an ideal I of R. We will assume a lexicographic order on the unknowns, so that we can give a total order to the terms. A Groebner basis G is one such that the ideal generated by the leading terms of I is equal to the ideal generated by the leading terms of G. Note that I is potentially infinite and G is ideally finite. Intuitively, this is equivalent to how in Guassian elimination, we try to pivot our way to a triangular matrtix.

Most importantly for us, we have that the variety of G is equal to the variety of I. Thus this is a convenient form to compute the dimension of the variety, or the variety itself if this dimension is zero [13]. Intuitively, a Groebner basis is a computationally convenient structure to compute properties of the variety of an ideal, akin to Gaussian elimination for multivariate non-linear systems.

There are some algorithms to obtain the Groebner basis [14]. Although they are not polynomial time or space, they are fast enough to obtain most of the interesting cases, under certain optimizations.

We will be interacting with the Groebner basis as a semi-black box algorithm. That is, we will be optimizing the input so that we obtain the results we want faster, but we will not be optimizing the algorithm itself.

3 Proposal

Our proposal to calculate the F-matrices is the following.

1. Fix the degrees of freedom (gauges), so that the solution is set. The strategies are the following.
   (a) Enforce the trivial cases, where a, b or c is 1.
   (b) Leverage explicit formulas for R.
   (c) Keep explicit accounting for the rest of the degrees of freedom
2. For a field K, we will consider the polynomial ring $K\left[\left\{X^{abc}_{d;pq} \mid a, b, c, d, p, q \in \mathcal{L}\right\}\right]$. $X^{abc}_{d;pq}$ represents the unknown value $F^{abc}_{d;pq}$
3. Consider the ideals

$$I = \left\langle X^{apb}_{f;qm} X^{mcd}_{f;pe} - \sum_n X^{bcd}_{q;pn} X^{and}_{f;qe} X^{abc}_{e;nm} \right\rangle$$

$$I_\pm = \left\langle (R^{ac}_e)^{\pm 1} X^{bac}_{d;em} (R^{ab}_m)^{\pm 1} - \sum_n X^{bca}_{d;en} (R^{an}_d)^{\pm 1} X^{abc}_{d;nm} \right\rangle$$

   Note that by construction, the variety of I is the set of values of F that satisfy the pentagon axioms, and similarly the variety of $I_\pm$ are the solutions of F that satisfy the hexagon axioms.

4. Calculate the variety $V(I + I_+ + I_-)$. This will be the solutions of F that satisfy both the pentagon and hexagon axioms. By Ocneanu rigidity, this will be the sett of viable soluttions.

There are several things to resolve in the proposal. The first one is to decide over what field we are going to be working with. The most general choice is to pick $\overline{\mathbb{Q}}$, the field of complex algebraic numbers.

However, there is a natural cyclotomic field for a lot of the numbers in the MTC. In particular, the twist coefficients θ_i and the $s_{i,j}$ coefficients both belong to this field. Thus we will work within this field, as it is a more natural environment.

An important thing to note is that while we remain in the cyclotomic field, it might be impossible to find unitary solutions. An example of this is the Fibonaccy anyon theory, with labels $1, x$ and fusion rule $x \otimes x = 1 \oplus x$. This theory does not have unitary solutions within a cyclotomic field.

3.1 Eliminating leftover gauges

We will first explain the simplest procedure. After calculating the Groebner basis, we can compute the dimension of the variety, and if this is zero, the solution set. If this dimension is not zero, we need to fix a gauge and retry. If an equation has more than two variables, this means we have at least one degree of freedom among the variables. Thus, we could fix a variable to a value and recompute that basis under this constraint.

Note that since we can scale linearly the basis, if α is a permissible value for some $F^{abc}_{d;ef}$ with a degree of freedom, so is $\lambda\alpha$, for any λ complex. This means at either 0 or 1 will be permissible for this coefficient. However, note that we are giving up unitarity of the matrices by doing this.

To actually fix the variable $X^{abc}_{d;ef}$ to α, we can add $X^{abc}_{d;ef} - \alpha$ to the ideal considered and recompute the Groebner basis. If at any point in this process we obtain that the solution set is null, we can backtrack and retry. If we find a solution from this procedure, by Ocneanu rigidity we have found a solution for the fusion rule.

3.2 Optimizations

In theory, this procedure eventually returns a solution. However, there are two big problems here. First, the algorithm to compute the Groebner basis is non-polynomial. Second, naively we have $2\,|\mathcal{L}|^6$ hexagon axioms and $|\mathcal{L}|^9$ pentagon axioms. Thus, to be able to solve non-trivial cases we need several optimizations.

3.2.1 Creating the ideals

The first problem we encounter is in actually constructing the set of polynomials implied by the axioms. A naive implementation ends up recomputing the tensor product several times. Here we need to compute and memoize the fusion coefficients for greater efficiency.

Second, not every 9-tuple of labels in the pentagon axiom contains information. In particular, a significant number of equations reduce to $0 = 0$, since the labels are not compatible. Similarly for the hexagon axioms. In one case, this is a reduction from $10,000,000$ pentagon equations to less than $5,000$.

After this, we have a fast way to obtain the ideals we need to start the computation.

3.2.2 Reducing and recomputing the ideal

Our procedure follows a loop of having an ideal, checking if it implies a discrete solution set, and if not fixing a gauge and trying again. In particular, we want to be reducing the ideal whenever we fix a gauge.

After computing the Groebner ideal, we might have some linear terms with just one variable. These clearly fix the value of this variable. Thus we can remove this linear term and substitute this value in the rest of the equations.

Another common polynomial in the Groebner ideal is a linear term with two unknowns. This means that fixing one of the unknowns will fix the other one immediately. If we have the term $\alpha x + \beta y + \gamma$, for unknowns x, y and coefficients α, β, γ, then we can substitute $x = (\beta y + \gamma)\alpha^{-1}$ in every other polynomial, and eliminate this linear term. This will reduce the number of terms. Note that this is particularly viable, since the hexagon axiom has a single linear monomial.

After computing the Groebner basis, if there is an equation with only one indeterminate, then we can solve for it. In particular, if the value is determined, then this equation will be of the form $X - \alpha$. Whenever we encounter a term like this in the ideal, or whenever we want to add a term like this, it turns out to be simpler to substitute α into the variable and erase this variable from the polynomial ring.

4 Future Work

This approach is able to automatically find F-matrices given a fusion rule. However, the main drawback is that it takes a long time to do so, particularly for bigger anyon theories.

With this work, we can solve without intervention a great deal of relevant theories. We expand these theories in the appendix.

This works utilizes the Groebner basis as a semi-blackbox solver for systems of polynomials. However, the polynomials we are looking at have a lot of structure, and their degrees are limited. Thus there is reason to believe certain optimizations are possible.

However, we should note that the Knapsack problem, for example, can be thought of finding the Groebner basis of an ideal of linear terms and polynomials of degree 2. Thus even this simple structure is not enough to obtain a polynomial solution. At the same time, one should recall the unreasonable effectiveness of boolean SAT solvers. We should be able to solve reasonably sized theories, and in particular theories that are unwieldy to solve by hand.

5 Acknowledgements

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A Some F matrices.

In what follows, we will present the computed tables. For for each Cartan type, we will go through the levels k that we can compute. In each subsection, we will show the fusion rule of the theory and then the computed matrices. For simplicity, we will skip the 1×1 matrices that are equal to 1, and just list them instead. Here ζ_n refers to the n-th root of unity.

A.1 $A1$

A.1.1 $k = 1$

$\otimes$	A0	A1
A0	A0	A1
A1	A1	A0

Trivial matrices: $F_{A0}^{(A0,A0,A0)}$, $F_{A1}^{(A0,A0,A1)}$, $F_{A1}^{(A0,A1,A0)}$, $F_{A0}^{(A0,A1,A1)}$, $F_{A1}^{(A1,A0,A0)}$, $F_{A0}^{(A1,A0,A1)}$, $F_{A0}^{(A1,A1,A0)}$.

$F_{A1}^{(A1,A1,A1)}$	A0
A0	-1

A.1.2 $k = 2$

$\otimes$	A0	A1	A2
A0	A0	A1	A2
A1	A1	A0 + A2	A1
A2	A2	A1	A0

Trivial matrices: $F^{(A0,A0,A0)}_{A0}$, $F^{(A0,A0,A1)}_{A1}$, $F^{(A0,A0,A2)}_{A2}$, $F^{(A0,A1,A0)}_{A1}$, $F^{(A0,A1,A1)}_{A0}$, $F^{(A0,A1,A1)}_{A2}$, $F^{(A0,A1,A2)}_{A1}$, $F^{(A0,A2,A0)}_{A2}$, $F^{(A0,A2,A1)}_{A1}$, $F^{(A0,A2,A2)}_{A0}$, $F^{(A1,A0,A0)}_{A1}$, $F^{(A1,A0,A1)}_{A0}$, $F^{(A1,A0,A1)}_{A2}$, $F^{(A1,A0,A2)}_{A1}$, $F^{(A1,A1,A0)}_{A0}$, $F^{(A1,A1,A0)}_{A2}$, $F^{(A1,A1,A2)}_{A0}$, $F^{(A1,A2,A0)}_{A1}$, $F^{(A1,A2,A1)}_{A2}$, $F^{(A2,A0,A0)}_{A2}$, $F^{(A2,A0,A1)}_{A1}$, $F^{(A2,A0,A2)}_{A0}$, $F^{(A2,A1,A0)}_{A1}$, $F^{(A2,A2,A0)}_{A0}$, $F^{(A2,A2,A2)}_{A2}$.

$F^{(A1,A1,A1)}_{A1}$	A2	A0
A2	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$	$-\frac{1}{2}$
A0	1	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$

$F^{(A1,A1,A2)}_{A2}$	A1
A0	$\zeta_{32}^{12} - \zeta_{32}^{4}$

$F^{(A1,A2,A1)}_{A0}$	A1
A1	-1

$F^{(A1,A2,A2)}_{A1}$	A0
A1	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$

$F^{(A2,A1,A1)}_{A0}$	A2
A1	-1

$F^{(A2,A1,A1)}_{A2}$	A0
A1	$-\frac{1}{2}\zeta_{32}^{12} + \frac{1}{2}\zeta_{32}^{4}$

$F^{(A2,A1,A2)}_{A1}$	A1
A1	-1

$F^{(A2,A2,A1)}_{A1}$	A1
A0	$\zeta_{32}^{12} - \zeta_{32}^{4}$

A.1.3 $k = 3$

$\otimes$	A0	A1	A2	A3
A0	A0	A1	A2	A3
A1	A1	A0 + A2	A1 + A3	A2
A2	A2	A1 + A3	A0 + A2	A1
A3	A3	A2	A1	A0

Trivial matrices: $F^{(A0,A0,A0)}_{A0}$, $F^{(A0,A0,A1)}_{A1}$, $F^{(A0,A0,A2)}_{A2}$, $F^{(A0,A0,A3)}_{A3}$, $F^{(A0,A1,A0)}_{A1}$, $F^{(A0,A1,A1)}_{A0}$, $F^{(A0,A1,A1)}_{A2}$, $F^{(A0,A1,A2)}_{A1}$, $F^{(A0,A1,A2)}_{A3}$, $F^{(A0,A1,A3)}_{A2}$, $F^{(A0,A2,A0)}_{A2}$, $F^{(A0,A2,A1)}_{A1}$, $F^{(A0,A2,A1)}_{A3}$, $F^{(A0,A2,A2)}_{A0}$, $F^{(A0,A2,A2)}_{A2}$, $F^{(A0,A2,A3)}_{A1}$, $F^{(A0,A3,A0)}_{A3}$, $F^{(A0,A3,A1)}_{A2}$, $F^{(A0,A3,A2)}_{A1}$, $F^{(A0,A3,A3)}_{A0}$, $F^{(A1,A0,A0)}_{A1}$, $F^{(A1,A0,A1)}_{A0}$, $F^{(A1,A0,A1)}_{A2}$, $F^{(A1,A0,A2)}_{A1}$, $F^{(A1,A0,A2)}_{A3}$, $F^{(A1,A0,A3)}_{A2}$, $F^{(A1,A1,A0)}_{A0}$, $F^{(A1,A1,A0)}_{A2}$, $F^{(A1,A1,A1)}_{A3}$, $F^{(A1,A1,A2)}_{A0}$, $F^{(A1,A1,A3)}_{A1}$, $F^{(A1,A2,A0)}_{A1}$, $F^{(A1,A2,A0)}_{A3}$, $F^{(A1,A2,A3)}_{A0}$, $F^{(A1,A3,A0)}_{A2}$, $F^{(A1,A3,A1)}_{A1}$, $F^{(A1,A3,A1)}_{A3}$, $F^{(A1,A3,A2)}_{A0}$, $F^{(A2,A0,A0)}_{A2}$, $F^{(A2,A0,A1)}_{A1}$, $F^{(A2,A0,A1)}_{A3}$, $F^{(A2,A0,A2)}_{A0}$, $F^{(A2,A0,A2)}_{A2}$, $F^{(A2,A0,A3)}_{A1}$, $F^{(A2,A1,A0)}_{A1}$, $F^{(A2,A1,A0)}_{A3}$, $F^{(A2,A1,A2)}_{A3}$, $F^{(A2,A1,A3)}_{A0}$, $F^{(A2,A2,A0)}_{A0}$, $F^{(A2,A2,A0)}_{A2}$, $F^{(A2,A2,A2)}_{A0}$, $F^{(A2,A3,A0)}_{A1}$, $F^{(A2,A3,A2)}_{A1}$, $F^{(A2,A3,A2)}_{A3}$, $F^{(A3,A0,A0)}_{A3}$, $F^{(A3,A0,A1)}_{A2}$, $F^{(A3,A0,A2)}_{A1}$, $F^{(A3,A0,A3)}_{A0}$, $F^{(A3,A1,A0)}_{A2}$, $F^{(A3,A1,A1)}_{A1}$, $F^{(A3,A1,A3)}_{A1}$, $F^{(A3,A2,A0)}_{A1}$,

$F_{A0}^{(A3,A2,A1)}$, $F_{A0}^{(A3,A3,A0)}$.

$F_{A1}^{(A1,A1,A1)}$	A2	A0
A2	$\zeta_{40}^{12}-\zeta_{40}^{8}$	$\zeta_{40}^{12}-\zeta_{40}^{8}$
A0	1	$\zeta_{40}^{12}-\zeta_{40}^{8}$

$F_{A2}^{(A1,A1,A2)}$	A3	A1
A2	$-\zeta_{40}^{12}+\zeta_{40}^{8}+1$	1
A0	1	-1

$F_{A3}^{(A1,A1,A3)}$	A2
A0	$\zeta_{40}^{12}-\zeta_{40}^{8}-1$

$F_{A0}^{(A1,A2,A1)}$	A1
A1	-1

$F_{A2}^{(A1,A2,A1)}$	A3	A1
A3	$\zeta_{40}^{12}-\zeta_{40}^{8}$	$-\zeta_{40}^{12}+\zeta_{40}^{8}$
A1	1	$-\zeta_{40}^{12}+\zeta_{40}^{8}$

$F_{A1}^{(A1,A2,A2)}$	A2	A0
A3	$\zeta_{40}^{12}-\zeta_{40}^{8}+1$	$\zeta_{40}^{12}-\zeta_{40}^{8}+1$
A1	$\zeta_{40}^{12}-\zeta_{40}^{8}+1$	$\zeta_{40}^{12}-\zeta_{40}^{8}$

$F_{A3}^{(A1,A2,A2)}$	A2
A1	$\zeta_{40}^{12}-\zeta_{40}^{8}$

$F_{A2}^{(A1,A2,A3)}$	A1
A1	-1

$F_{A2}^{(A1,A3,A2)}$	A1
A2	$\zeta_{40}^{12}-\zeta_{40}^{8}-1$

$F_{A1}^{(A1,A3,A3)}$	A0
A2	$\zeta_{40}^{12}-\zeta_{40}^{8}$

$F_{A0}^{(A2,A1,A1)}$	A2
A1	-1

$F_{A2}^{(A2,A1,A1)}$	A2	A0
A3	$-\zeta_{40}^{12}+\zeta_{40}^{8}-1$	$-\zeta_{40}^{12}+\zeta_{40}^{8}-1$
A1	$-\zeta_{40}^{12}+\zeta_{40}^{8}-1$	$-\zeta_{40}^{12}+\zeta_{40}^{8}$

$F_{A1}^{(A2,A1,A2)}$	A3	A1
A3	$-\zeta_{40}^{12}+\zeta_{40}^{8}$	$\zeta_{40}^{12}-\zeta_{40}^{8}$
A1	-1	$\zeta_{40}^{12}-\zeta_{40}^{8}$

$F_{A2}^{(A2,A1,A3)}$	A2
A1	$-\zeta_{40}^{12}+\zeta_{40}^{8}$

$F_{A1}^{(A2,A2,A1)}$	A3	A1
A2	$\zeta_{40}^{12}-\zeta_{40}^{8}-1$	-1
A0	1	-1

$F_{A3}^{(A2,A2,A1)}$	A1
A2	$-\zeta_{40}^{12}+\zeta_{40}^{8}+1$

$F_{A2}^{(A2,A2,A2)}$	A2	A0
A2	$\zeta_{40}^{12} - \zeta_{40}^{8}$	$\zeta_{40}^{12} - \zeta_{40}^{8} - 1$
A0	$-\zeta_{40}^{12} + \zeta_{40}^{8} - 1$	$-\zeta_{40}^{12} + \zeta_{40}^{8}$

$F_{A1}^{(A2,A2,A3)}$	A1
A2	$-\zeta_{40}^{12} + \zeta_{40}^{8} + 1$

$F_{A3}^{(A2,A2,A3)}$	A1
A0	$-\zeta_{40}^{12} + \zeta_{40}^{8} + 1$

$F_{A0}^{(A2,A3,A1)}$	A2
A1	-1

$F_{A2}^{(A2,A3,A1)}$	A2
A1	$-\zeta_{40}^{12} + \zeta_{40}^{8}$

$F_{A2}^{(A2,A3,A3)}$	A0
A1	$-\zeta_{40}^{12} + \zeta_{40}^{8}$

$F_{A3}^{(A3,A1,A1)}$	A0
A2	$-\zeta_{40}^{12} + \zeta_{40}^{8}$

$F_{A0}^{(A3,A1,A2)}$	A3
A2	-1

$F_{A2}^{(A3,A1,A2)}$	A1
A2	$\zeta_{40}^{12} - \zeta_{40}^{8} - 1$

$F_{A2}^{(A3,A2,A1)}$	A1
A1	-1

$F_{A1}^{(A3,A2,A2)}$	A2
A1	$\zeta_{40}^{12} - \zeta_{40}^{8}$

$F_{A3}^{(A3,A2,A2)}$	A0
A1	$-\zeta_{40}^{12} + \zeta_{40}^{8}$

$F_{A2}^{(A3,A2,A3)}$	A1
A1	-1

$F_{A1}^{(A3,A3,A1)}$	A2
A0	$-\zeta_{40}^{12} + \zeta_{40}^{8} + 1$

$F_{A2}^{(A3,A3,A2)}$	A1
A0	$\zeta_{40}^{12} - \zeta_{40}^{8} - 1$

$F_{A3}^{(A3,A3,A3)}$	A0
A0	-1

A.2 *A2*

$\otimes$	A0	A1	A2
A0	A0	A1	A2
A1	A1	A2	A0
A2	A2	A0	A1

Trivial matrices: $F_{A0}^{(A0,A0,A0)}$, $F_{A1}^{(A0,A0,A1)}$, $F_{A2}^{(A0,A0,A2)}$, $F_{A1}^{(A0,A1,A0)}$, $F_{A2}^{(A0,A1,A1)}$, $F_{A0}^{(A0,A1,A2)}$, $F_{A2}^{(A0,A2,A0)}$, $F_{A0}^{(A0,A2,A1)}$, $F_{A1}^{(A0,A2,A2)}$, $F_{A1}^{(A1,A0,A0)}$, $F_{A2}^{(A1,A0,A1)}$, $F_{A0}^{(A1,A0,A2)}$, $F_{A2}^{(A1,A1,A0)}$, $F_{A0}^{(A1,A1,A1)}$,

$F_{A1}^{(A1,A1,A2)}$, $F_{A0}^{(A1,A2,A0)}$, $F_{A1}^{(A1,A2,A1)}$, $F_{A2}^{(A1,A2,A2)}$, $F_{A2}^{(A2,A0,A0)}$, $F_{A0}^{(A2,A0,A1)}$, $F_{A1}^{(A2,A0,A2)}$, $F_{A0}^{(A2,A1,A0)}$, $F_{A1}^{(A2,A1,A1)}$, $F_{A2}^{(A2,A1,A2)}$, $F_{A1}^{(A2,A2,A0)}$, $F_{A2}^{(A2,A2,A1)}$, $F_{A0}^{(A2,A2,A2)}$. [2]

A.3 $B2$

A.3.1 $k = 1$

⊗	B0	B1	B2
B0	B0	B1	B2
B1	B1	B0	B2
B2	B2	B2	B0 + B1

Trivial matrices: $F_{B0}^{(B0,B0,B0)}$, $F_{B1}^{(B0,B0,B1)}$, $F_{B2}^{(B0,B0,B2)}$, $F_{B1}^{(B0,B1,B0)}$, $F_{B0}^{(B0,B1,B1)}$, $F_{B2}^{(B0,B1,B2)}$, $F_{B2}^{(B0,B2,B0)}$, $F_{B2}^{(B0,B2,B1)}$, $F_{B0}^{(B0,B2,B2)}$, $F_{B1}^{(B0,B2,B2)}$, $F_{B1}^{(B1,B0,B0)}$, $F_{B0}^{(B1,B0,B1)}$, $F_{B2}^{(B1,B0,B2)}$, $F_{B0}^{(B1,B1,B0)}$, $F_{B1}^{(B1,B1,B1)}$, $F_{B2}^{(B1,B1,B2)}$, $F_{B2}^{(B1,B2,B0)}$, $F_{B2}^{(B2,B0,B0)}$, $F_{B2}^{(B2,B0,B1)}$, $F_{B0}^{(B2,B0,B2)}$, $F_{B1}^{(B2,B0,B2)}$, $F_{B2}^{(B2,B1,B0)}$, $F_{B2}^{(B2,B1,B1)}$, $F_{B0}^{(B2,B1,B2)}$, $F_{B0}^{(B2,B2,B0)}$, $F_{B1}^{(B2,B2,B0)}$.

$F_{B2}^{(B1,B2,B1)}$	B2
B2	-1

$F_{B0}^{(B1,B2,B2)}$	B1
B2	$\zeta_{32}^{12} - \zeta_{32}^{4}$

$F_{B1}^{(B1,B2,B2)}$	B0
B2	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$

$F_{B1}^{(B2,B1,B2)}$	B2
B2	-1

$F_{B0}^{(B2,B2,B1)}$	B2
B1	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$

$F_{B1}^{(B2,B2,B1)}$	B2
B0	$\zeta_{32}^{12} - \zeta_{32}^{4}$

$F_{B2}^{(B2,B2,B2)}$	B1	B0
B1	$-\frac{1}{2}\zeta_{32}^{12} + \frac{1}{2}\zeta_{32}^{4}$	$\frac{1}{2}$
B0	1	$\frac{1}{2}\zeta_{32}^{12} - \frac{1}{2}\zeta_{32}^{4}$

A.4 $D3$

A.4.1 $k = 1$

⊕	D0	D1	D2	D3
D0	D0	D1	D2	D3
D1	D1	D0	D3	D2
D2	D2	D3	D1	D0
D3	D3	D2	D0	D1

Trivial matrices: $F_{D0}^{(D0,D0,D0)}$, $F_{D1}^{(D0,D0,D1)}$, $F_{D2}^{(D0,D0,D2)}$, $F_{D3}^{(D0,D0,D3)}$, $F_{D1}^{(D0,D1,D0)}$, $F_{D0}^{(D0,D1,D1)}$, $F_{D3}^{(D0,D1,D2)}$, $F_{D2}^{(D0,D1,D3)}$, $F_{D2}^{(D0,D2,D0)}$, $F_{D3}^{(D0,D2,D1)}$, $F_{D1}^{(D0,D2,D2)}$, $F_{D0}^{(D0,D2,D3)}$, $F_{D3}^{(D0,D3,D0)}$, $F_{D2}^{(D0,D3,D1)}$, $F_{D0}^{(D0,D3,D2)}$, $F_{D1}^{(D0,D3,D3)}$, $F_{D1}^{(D1,D0,D0)}$, $F_{D0}^{(D1,D0,D1)}$, $F_{D3}^{(D1,D0,D2)}$, $F_{D2}^{(D1,D0,D3)}$, $F_{D0}^{(D1,D1,D0)}$, $F_{D1}^{(D1,D1,D1)}$, $F_{D2}^{(D1,D1,D2)}$, $F_{D3}^{(D1,D1,D3)}$, $F_{D3}^{(D1,D2,D0)}$, $F_{D0}^{(D1,D2,D2)}$, $F_{D1}^{(D1,D2,D3)}$, $F_{D2}^{(D1,D3,D0)}$, $F_{D1}^{(D1,D3,D2)}$, $F_{D0}^{(D1,D3,D3)}$,

[2]Note that we can predict this result since the tensor operation is Abelian, so braiding is a trivial operation.

$F_{D2}^{(D2,D0,D0)}$, $F_{D3}^{(D2,D0,D1)}$, $F_{D1}^{(D2,D0,D2)}$, $F_{D0}^{(D2,D0,D3)}$, $F_{D3}^{(D2,D1,D0)}$, $F_{D2}^{(D2,D1,D1)}$, $F_{D1}^{(D2,D1,D3)}$, $F_{D1}^{(D2,D2,D0)}$, $F_{D3}^{(D2,D2,D2)}$, $F_{D2}^{(D2,D2,D3)}$, $F_{D0}^{(D2,D3,D0)}$, $F_{D2}^{(D2,D3,D2)}$, $F_{D3}^{(D3,D0,D0)}$, $F_{D2}^{(D3,D0,D1)}$, $F_{D0}^{(D3,D0,D2)}$, $F_{D1}^{(D3,D0,D3)}$, $F_{D2}^{(D3,D1,D0)}$, $F_{D3}^{(D3,D1,D1)}$, $F_{D1}^{(D3,D1,D2)}$, $F_{D0}^{(D3,D2,D0)}$, $F_{D3}^{(D3,D2,D3)}$, $F_{D1}^{(D3,D3,D0)}$, $F_{D3}^{(D3,D3,D2)}$, $F_{D2}^{(D3,D3,D3)}$.

$F_{D2}^{(D1,D2,D1)}$	D3
D3	-1

$F_{D3}^{(D1,D3,D1)}$	D2
D2	-1

$F_{D0}^{(D2,D1,D2)}$	D3
D3	-1

$F_{D0}^{(D2,D2,D1)}$	D3
D1	-1

$F_{D1}^{(D2,D3,D1)}$	D2
D0	-1

$F_{D3}^{(D2,D3,D3)}$	D1
D0	-1

$F_{D0}^{(D3,D1,D3)}$	D2
D2	-1

$F_{D1}^{(D3,D2,D1)}$	D3
D0	-1

$F_{D2}^{(D3,D2,D2)}$	D1
D0	-1

$F_{D0}^{(D3,D3,D1)}$	D2
D1	-1

A.5 $G2$

A.5.1 $k = 1$

$\oplus$	G0	G1
G0	G0	G1
G1	G1	G0 + G1

Trivial matrices: $F_{G0}^{(G0,G0,G0)}$, $F_{G1}^{(G0,G0,G1)}$, $F_{G1}^{(G0,G1,G0)}$, $F_{G0}^{(G0,G1,G1)}$, $F_{G1}^{(G0,G1,G1)}$, $F_{G1}^{(G1,G0,G0)}$, $F_{G0}^{(G1,G0,G1)}$, $F_{G1}^{(G1,G0,G1)}$, $F_{G0}^{(G1,G1,G0)}$, $F_{G1}^{(G1,G1,G0)}$, $F_{G0}^{(G1,G1,G1)}$.

$F_{G1}^{(G1,G1,G1)}$	G1	G0
G1	$\zeta_{60}^{14} - \zeta_{60}^{6} - \zeta_{60}^{4} + 1$	$-\zeta_{60}^{14} + \zeta_{60}^{6} + \zeta_{60}^{4} - 1$
G0	1	$-\zeta_{60}^{14} + \zeta_{60}^{6} + \zeta_{60}^{4} - 1$

A.6 $E8$

A.6.1 $k = 2$

$\oplus$	E0	E1	E2
E0	E0	E1	E2
E1	E1	E0	E2
E2	E2	E2	E0 + E1

Trivial matrices: $F_{E0}^{(E0,E0,E0)}$, $F_{E1}^{(E0,E0,E1)}$, $F_{E2}^{(E0,E0,E2)}$, $F_{E1}^{(E0,E1,E0)}$, $F_{E0}^{(E0,E1,E1)}$, $F_{E2}^{(E0,E1,E2)}$, $F_{E2}^{(E0,E2,E0)}$, $F_{E2}^{(E0,E2,E1)}$, $F_{E0}^{(E0,E2,E2)}$, $F_{E1}^{(E0,E2,E2)}$, $F_{E1}^{(E1,E0,E0)}$, $F_{E0}^{(E1,E0,E1)}$, $F_{E2}^{(E1,E0,E2)}$, $F_{E0}^{(E1,E1,E0)}$, $F_{E1}^{(E1,E1,E1)}$, $F_{E2}^{(E1,E1,E2)}$, $F_{E2}^{(E1,E2,E0)}$, $F_{E2}^{(E2,E0,E0)}$, $F_{E2}^{(E2,E0,E1)}$, $F_{E0}^{(E2,E0,E2)}$, $F_{E1}^{(E2,E0,E2)}$, $F_{E2}^{(E2,E1,E0)}$, $F_{E2}^{(E2,E1,E1)}$, $F_{E0}^{(E2,E1,E2)}$, $F_{E0}^{(E2,E2,E0)}$, $F_{E1}^{(E2,E2,E0)}$.

$F_{E2}^{(E1,E2,E1)}$	E2
E2	-1

$F_{E0}^{(E1,E2,E2)}$	E1
E2	$-\zeta_{128}^{48}+\zeta_{128}^{16}$

$F_{E1}^{(E1,E2,E2)}$	E0
E2	$-\frac{1}{2}\zeta_{128}^{48}+\frac{1}{2}\zeta_{128}^{16}$

$F_{E1}^{(E2,E1,E2)}$	E2
E2	-1

$F_{E0}^{(E2,E2,E1)}$	E2
E1	$-\frac{1}{2}\zeta_{128}^{48}+\frac{1}{2}\zeta_{128}^{16}$

$F_{E1}^{(E2,E2,E1)}$	E2
E0	$-\zeta_{128}^{48}+\zeta_{128}^{16}$

$F_{E2}^{(E2,E2,E2)}$	E1	E0
E1	$\frac{1}{2}\zeta_{128}^{48}-\frac{1}{2}\zeta_{128}^{16}$	$\frac{1}{2}$
E0	1	$-\frac{1}{2}\zeta_{128}^{48}+\frac{1}{2}\zeta_{128}^{16}$