CONSTRUCTING SOLUTIONS TO THE ALLEN-CAHN EQUATION

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Abstract. The Allen-Cahn equation $\epsilon^2 \Delta u = u^3 - u = W'(u)$ is an elliptic, semilinear, second-order partial differential equation. Here $W(u) = \frac{1}{4}(1-u^2)^2$ is a double-well potential. Solutions to this equation are exactly the critical points of the energy functional $\int \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)$ in H^1. We investigate general properties of solutions to the Allen-Cahn equation, including their smoothness. We also construct infinite-energy periodic solutions in dimension one, saddle solutions in dimension two, and use gluing and energy methods to construct solutions on S^n.

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1. Introduction

The Allen-Cahn equation is the following partial differential equation (PDE):

$$\epsilon \Delta u = \frac{1}{\epsilon}(u^3 - u). \tag{1.1}$$

Here $\epsilon > 0$ is fixed. More generally, the Allen-Cahn equation can be considered on a complete Riemannian manifold (M, g):

$$\epsilon \Delta_g u = \frac{1}{\epsilon}(u^3 - u) \tag{1.2}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric g.

Solutions to (1.2) are critical points of the Allen-Cahn energy functional:

$$E_\epsilon(u; U) = \int_U \left(\frac{\epsilon}{2} |\nabla_g u|^2 + \frac{1}{\epsilon} W(u) \right) d\mu_g. \tag{1.3}$$

Here $W(\cdot)$ is the "double well potential," which is modelled by the polynomial $W(u) = \frac{1}{4}(1 - u^2)^2$ (more general functions are available), and $d\mu_g$ is the volume form with respect to metric g.

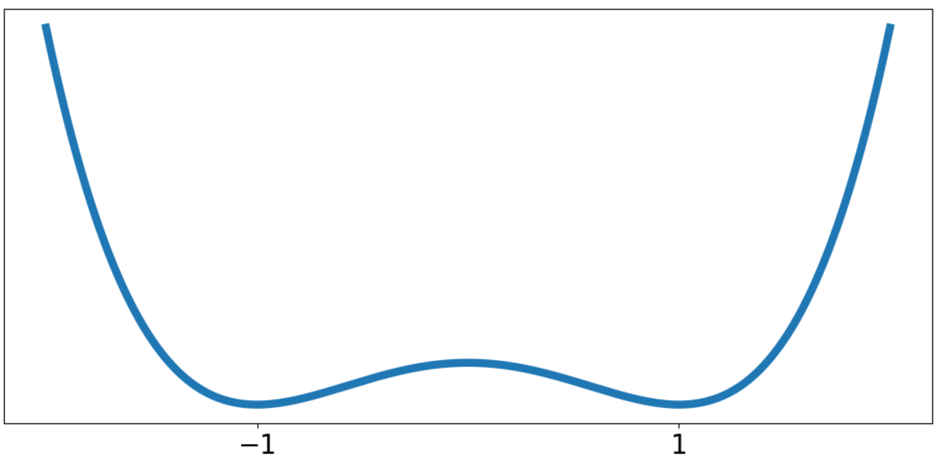

Figure 1. The double well potential $W(u) = \frac{1}{4}(1 - u^2)^2$.

Definition 1.1. Let (M^n, g) be a complete Riemannian manifold. $u : M \to \mathbb{R}$ is a *critical point* of eq. (1.3) if for any $\varphi : M \to \mathbb{R} \in C_c^\infty(M)$ (i.e. φ is smooth with support contained in a precompact open set $\Omega \subset M$), we have $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{d}{dt}\Big|_{t=0} E_\epsilon(u + t\varphi; \Omega) = 0. \tag{1.4}$$

Note that

$$\frac{d}{dt}\Big|_{t=0} E_\epsilon(u + t\varphi; \Omega) = \int_\Omega \left(\frac{\epsilon}{2} g(\nabla_g u, \nabla_g \varphi) + \frac{1}{\epsilon} W'(u)\varphi \right) d\mu_g,$$

where $g(\cdot, \cdot)$ is the inner product with respect to metric g. One can check that if u is a critical point then it weakly solves eq. (1.2).

In section 2, we introduce the regularity theory of second order linear partial differential equations in Sobolev and Hölder spaces in the context of the Allen-Cahn equation. In particular, critical points of the Allen-Cahn energy functional are smooth. After discussing general properties of solutions to Allen-Cahn in section 3, we construct solutions on $\mathbb{R}$, $\mathbb{R}^2$, and S^n. The reader may consult the appendix for the definitions and basic properties of L^p spaces, Sobolev spaces, and Hölder spaces, which we assume familiarity with.

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Our main source for information about the Allen-Cahn equation was a set of notes by Otis Chodosh [1], who also provided insight and hints for some of his exercises. We also learned from notes by Guaraco [2], especially for the constructions in section 4.4. To learn general background about measure theory and L^p spaces, we used Bass's book [3]. We learned about Sobolev spaces, second order linear elliptic PDEs, and calculus of variations from Evans' book [4] and learned about Schauder estimates and other technical results from the book by Gilbarg and Trudinger [5].

2. Elliptic Regularity for Allen-Cahn

Regularity refers to the "niceness" of a solution to a PDE. This often means differentiability, and sometimes integrability. The best one could hope for is a C^∞, or smooth, solution. One also searches for solutions in C^k and the Hölder spaces $C^{k,\alpha}$. It is often easier to find weak solutions (which may be a priori highly discontinuous) in Sobolev spaces $W^{k,p}$, and especially in the Hilbert spaces $H^k = W^{k,2}$.

Elliptic regularity is the phenomenon that weak solutions to certain PDEs have many—even infinitely many—derivatives. The simplest example of this phenomenon is that harmonic functions, namely solutions to $\Delta f = 0$, are smooth. The same miraculous phenomenon applies to the Allen-Cahn equation, whose highest order term is the Laplacian.

2.1. **Allen-Cahn.**

Solutions to Allen-Cahn

$$\Delta u = u^3 - u \tag{2.1}$$

should have at least two derivatives, so that the left side makes sense and the Laplacian can be applied. However, if $u \in C^2$, then so is $\Delta u = u^3 - u$, so we might actually expect u to be C^4. Repeating this reasoning in a so-called "bootstrap argument," we conclude that u should have infinitely many derivatives. This bootstrap phenomenon is part of a much more general picture that begins with the theory of linear elliptic operators.

2.2. H^m **Regularity for Linear Elliptic Operators.**

Suppose u and f are smooth and

$$-\Delta u = f \quad \text{on}\, U \subset \mathbb{R}^n, \text{ with}\, u|_{\partial U} \equiv 0. \tag{2.2}$$

Integrating by parts, we find

$$\begin{aligned}\int_U f^2 = \int_U (\Delta u)^2 &= \int_U \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \\ &= -\int_U \sum_{i,j=1}^n u_{x_i x_i x_j} u_{x_j} = \int_U \sum_{i,j=1}^n u_{x_i x_j} u_{x_i x_j} = \int_U \left|D^2 u\right|^2 .\end{aligned} \tag{2.3}$$

That is, control over the L^2 norm of f gives control of the L^2 norm of the second derivatives of u. More generally, differentiate eq. (2.2) m times and then carry out the calculation in eq. (2.3) to see that the L^2 norm of the order m derivatives of f controls the L^2 norm of the order $m+2$ derivatives of u. This reasoning does not extend directly to weak solutions of $-\Delta u = f$, because we assumed that u had at least three derivatives, but this heuristic calculation gives hope that weak solutions to Poisson's equation $-\Delta u = f$ with $u \in H_0^1$ in fact belong to H^{k+2} whenever $f \in H^k$.

This calculation motivates the study of other second order partial differential operators, namely those whose leading behaviour is "like the Laplacian".

Definition 2.1 (Elliptic operator)**.** A second order linear partial differential operator L on a domain U given by $Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} + cu$, with $a^{ij}, b^i, c : U \to \mathbb{R}$ and $a^{ij} = a^{ji}$, is (uniformly) *elliptic* if there exists $\theta > 0$ with

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \theta \left|\xi\right|^2 \tag{2.4}$$

for almost all $x \in U$ and all $\xi \in \mathbb{R}^n$.

For $a^{ij} = \delta_{ij}$, $b^i \equiv 0$, and $c \equiv 0$, $Lu = -\Delta u$. The condition of uniform ellipticity is to say that for each $x \in U$, the symmetric matrix $\mathbf{A}(x) = ((a^{ij}(x)))$ is positive definite and has smallest eigenvalue at least θ. Elliptic operators resemble the Laplacian in the following way: when a^{ij} are bounded, we have $\theta \mathbf{I} \leq \mathbf{A} \leq \Theta \mathbf{I}$ for some $\Theta \in \mathbb{R}$ (the uniform bound on the largest eigenvalue of $\mathbf{A}$) and for θ the uniform ellipticity constant, where $\mathbf{I}$ is the identity matrix. For the Laplacian, $\theta = \Theta = 1$.

Solutions to elliptic PDEs (namely those given by an elliptic differential operator) enjoy the following regularity.

Theorem 2.2 (Elliptic regularity in Sobolev spaces)**.** *For $m \in \mathbb{N}$, $a^{ij}, b^i, c \in C^{k+1}(U)$, and $f \in H^m(U)$, if $u \in H^1(U)$ is a weak solution of the linear elliptic PDE $Lu = f$ in U, then $u \in H^{k+2}(V)$ for every V compactly contained in U. Moreover,*

$$\|u\|_{H^{k+2}(V)} \leq C(m, U, V, L)(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}). \tag{2.5}$$

If ∂U is C^{k+2} and $u \in H_0^1(U)$, then $u \in H^{k+2}(U)$ and eq. (2.5) holds with C independent of V.

Proof. See Section 6.3 of Lawrence C. Evans *Partial Differential Equations* [4]. □

The upshot is that u has two more derivatives than f. By Sobolev embeddings, sufficient weak regularity of u implies strong regularity of u. In particular, if f and the coefficients of the elliptic operator L are smooth, then so is u.

2.3. **Schauder Estimates.** There are two problems with applying the linear elliptic regularity theory to Allen-Cahn. First, Allen-Cahn, written as $\Delta u = W'(u)$, is a non-linear equation. However, the nonlinearity is present only in the lower order terms, so we can do the following: suppose u solves Allen-Cahn, then set $f := W' \circ u$. Then u solves $\Delta u = f$. Alternatively, we may write $c := u^2 - 1$, so that $\Delta u - cu = 0$. However, there is a more subtle problem in applying the bootstrap argument: regularity of u does not pass to regularity of f. Namely, if $u \in H^k$, it is not necessarily true that $u^3 - u \in H^k$. If u were more regular, say $u \in C^k$, then it would be true that $u^3 - u \in C^k$ as well. It is not true in general that $u \in C^{k+2}$ whenever u solves $\Delta u = f$ with $f \in C^k$. If we ask for a little more regularity, namely for the k-th derivatives of k to be Hölder continuous, then such a statement, due to Schauder, is true.

Theorem 2.3 (Interior Schauder estimates, G-T [5] Theorem 6.2)**.** *For $k \geq 0$, suppose $u \in C^{k+2,\alpha}(U)$ solves the linear elliptic PDE $\Delta u = f$. Then*

$$\|u\|_{C^{k+2,\alpha}(V)} \leq C(\|u\|_{C(U)} + \|f\|_{C^{k,\alpha}(U)}), \tag{2.6}$$

for any V compactly contained in U, where the constant C depends on the dimension n, k, the Holder exponent α, and V.

Proof. For simplicity, we state this here for the Laplacian, but the same is true of general elliptic operators with coefficients bounded uniformly in $C^{k,\alpha}$. Schauder estimates that hold up to the boundary of U (and not just in the interior) also hold. See Chapter 6 of Gilbarg and Trudinger. [5] □

Setting $f := W' \circ u$, notice that $f \in C^{k,\alpha}$ whenever $u \in C^{k,\alpha}$.

The Schauder estimates *assume* that $u \in C^{k+2,\alpha}(U)$. In particular, they do not prove regularity of u, but instead assume it and give bounds on the size of u in $C^{k+2,\alpha}$. Such an estimate is known as an *a priori estimate*, and is common in PDEs. By an approximation argument, one can often prove regularity using a priori estimates. Namely, starting with $u \in C^{k,\alpha}$,

(1) Approximate f by smooth functions f_i with uniformly bounded $C^{k,\alpha}$ norm.
(2) Use existence theory in Sobolev spaces to find the (unique) weak solution v_i to $\Delta v_i = f_i$ with $v_i|_{\partial U} \equiv 0$.
(3) Use elliptic regularity in Sobolev spaces to show that v_i are smooth.
(4) Use Schauder estimates and the maximum principle to bound the v_i in $C^{k+2,\alpha}$ uniformly in i.
(5) Use Arzela-Ascoli to obtain a uniform subsequential limit $v_i \to v$ on compacts, then show that $v \in C^{k+2,\alpha}$ and solves $\Delta v = f$.
(6) Use elliptic regularity in Sobolev spaces to show that because $\Delta(u - v) = 0$ we have $u - v \in C^\infty$, and conclude that $u \in C^{k+2,\alpha}$.

By a deep theorem due to De Giorgi and Nash, the boundedness of f in $\Delta u = f$ ensures that if $u \in H^1$, then $u \in C^\alpha$. That is, we can begin the above bootstrap with $k = 0$, starting with a weak solution.

Theorem 2.4. *Let $U \subset \mathbb{R}^n$ and suppose $u \in H^1(U)$ satisfies $-\Delta u = f$ on U with $f \in L^{\frac{q}{2}}(U)$ for $q > n$. Then for any $V \subset\subset U$, $u \in C^\alpha(\overline{V})$ with*

$$\|u\|_{C^\alpha(\overline{V})} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^{\frac{q}{2}}(U)}), \tag{2.7}$$

where the constant C depends on n, q, U, V, and the Hölder exponent α depends on n, U, V.

Proof. This is stated here for the Laplacian, but again holds for more general elliptic operators. See Gilbarg and Trudinger [5] Chapter 8, and in particular Theorem 8.24. □

In particular, if f is bounded on compact sets, then $f \in L^p(U)$ for all $1 \leq p \leq \infty$ for U bounded.

Theorem 2.5. *Let $U \subset \mathbb{R}^n$ be a (possibly unbounded) domain. If $u \in H^1_{loc}(U) \cap L^\infty_{loc}(U)$ is a bounded weak solution of Allen-Cahn (equivalently a bounded critical point of the energy functional), then u is smooth.*

The above arguments are written for $\mathbb{R}^n$ but carry over to the case of closed manifolds[1] (M, g). In particular, in each chart the Riemannian metric g is smooth and valued in the symmetric positive-definite $n \times n$ matrices. Because M is compact, there is a uniform bound on the smallest eigenvalues of these matrices; that is, g is uniformly elliptic. One can then apply the interior estimates in each chart to obtain analogues of elliptic regularity in Sobolev spaces and Schauder estimates. By compactness we can take the charts to be finite in number and obtain global regularity results.

Corollary 2.6. *Let (M, g) be a complete closed Riemannian manifold. If $u \in H^1(M) \cap L^\infty(M)$ is a weak solution of Allen-Cahn, then u is smooth.*

3. Properties of the Allen-Cahn Equation

In this section, we want to summarize the properties of critical points and solutions to the Allen-Cahn equations. In particular, one can prove that any critical points to the Allen-Cahn equation are smooth and bounded by 1. Next, we proceed to talk about the existence, uniqueness, and stability of solutions.

3.1. General Properties.

Proposition 3.1. *Suppose u is a solution to the Allen-Cahn equation on a closed manifold M. Then $u \in [-1, 1]$.*

Proof. Suppose u solves Allen-Cahn on M (thus it is smooth). Because M is compact, u attains a maximum, say at x. Because M has no boundary, x is an interior point, so $\epsilon^2 \Delta u(x) < 0$. If $u(x) > 1$, then $W'(u(x)) = u^3(x) - u(x) > 0$, which contradicts $\epsilon^2 \Delta u = W'(u)$. □

By the maximum principle, we show that a solution to Allen-Cahn bounded by 1 that achieves ± 1 anywhere is in fact ± 1 everywhere.

Proposition 3.2. *If u is a non-trivial solution to the Allen-Cahn (i.e. $u \not\equiv \pm 1$) and $|u| \leq 1$, then $|u| < 1$.*

Proof. If $u = 1$ somewhere, then $Lu = -\Delta u + 2u$, so that $v := u - 1$ achieves a non-negative interior maximum (because $u \leq 1$) and satisfies $Lv = -\Delta u + 2u - 2 = -u^3 + 3u - 2 \leq 0$ because $u \leq 1$, so we conclude by the maximum principle that $u \equiv 1$. Similarly one shows $-1 < u$. Thus $|u| < 1$ for non-trivial u. □

Remark 3.3. The above arguments also work on bounded open sets U such that u solves Allen-Cahn on U with $|u| \leq 1$ on ∂U.

[1] Recall that a closed manifold is a manifold without boundary that is also compact.

Allen-Cahn has three trivial constant solutions, namely $1, -1, 0$. The solutions ± 1 have zero energy, so they are global minimizers of the energy functional introduced in eq. (1.3). On the other hand, 0 is a global maximizer of energy.

Lemma 3.4. *For all $u \in H^1(M)$, we have $E_\epsilon(u) \le E_\epsilon(0)$, with equality iff $u \equiv 0$.*

Proof. Integrate by parts in the Allen-Cahn energy functional:

$$\begin{aligned} \epsilon E_\epsilon(u) &= \int_M \frac{\epsilon^2}{2} |\nabla u|^2 + W(u) = \int_M -\frac{u}{2}\epsilon^2 \Delta u + W(u) \\ &= \int_M -\frac{1}{2}u(u^3 - u) + \frac{1}{4}(1-u^2)^2 = \int_M -\frac{1}{4}u^4 + \frac{1}{4}. \end{aligned} \tag{3.1}$$

If $u \neq 0$ anywhere, then $\epsilon E_\epsilon(u) < \int_M \frac{1}{4} = \int W(0) = \epsilon E_\epsilon(0)$. That is, 0 maximizes E_ϵ. □

Moreover, non-trivial solutions must change sign.

Proposition 3.5. *If u is a non-trivial solution to the Allen-Cahn equation on a closed manifold (M, g), then $\{x \in M \mid u(x) = 0\} \neq \varnothing$.*

Remark 3.6. It is crucial that we are on a closed manifold. Otherwise this property does not hold. A closed manifold is by definition a manifold without boundary and compact.

Proof. Since u is a non-trivial solution, it is a critical point to the Allen-Cahn energy functional and $u \neq 0$. Thus, by definition of u being a critical point we must have

$$\frac{d}{dt} E(u + t\varphi)|_{t=0} = 0$$

for all φ, function that is smooth and compactly supported. Since (M, g) is itself compact and without boundary, $\varphi \equiv 1$ satisfies the requirement. After calculation we get

$$\int_M W'(u) = 0.$$

This cannot happen if $0 < |u| < 1$ (i.e. $\{u = 0\} = \varnothing$) because $W'(u)$ is constant sign and non-zero everywhere. □

Any solution to Allen-Cahn has uniformly bounded derivatives.

Proposition 3.7. *If u is a bounded solution to Allen-Cahn, then $\|u\|_{C^k(\mathbb{R}^n)} \le C(k)$ where $C(k)$ is a constant only depending on k.*

This uses the interior Schauder estimates in theorem 2.3 and the bootstrap process described in the previous section. We first introduce the first Schauder estimates that is useful to get a $C^{1,\alpha}$ bound for u.

Lemma 3.8 ($C^{1,\alpha}$ Schauder estimate)**.** *Let U be a domain in $\mathbb{R}^n$, and let u satisfy $\Delta u = f$ on U, where f is bounded and integrable. Then for any two concentric balls $B_1 = B_R(x_0), B_1 = B_{2R}(x_0) \subset\subset U$[2] we have*

$$\|u\|_{C^{1,\alpha}(B_1)} \le C(n, \alpha, R)(\|u\|_{C(B_2)} + R^2 \|f\|_{C(B_2)}).$$

This is Theorem 4.15 and Equation 4.45 in Gilbarg and Trudinger [5].

[2] We write $A \subset\subset B$ to denote that set A is compactly contained in set B. It means that the closure of A is contained in the interior of B and the closure of A is compact.

Proof of Proposition 3.7. Fix u bounded and solving Allen-Cahn on $\mathbb{R}^n$. Then u satisfies $\Delta u = f$ with $f := W' \circ u$. Fix $x_0 \in \mathbb{R}^n$ and let $B_1 = B(x_0, R)$ and $B_2 = B(x_0, 2R)$. Throughout let C denote a constant depending on n, α, and any extra given parameters. By the estimate in lemma 3.8,

$$\|u\|_{C^1(B_1)} \leq \|u\|_{C^{1,\alpha}(B_1)} \leq C(R)(\|u\|_{C(B_2)} + \|f\|_{C(B_2)}) \leq C(R). \tag{3.2}$$

Now we bound the Holder norm of f by its derivative:

$$\begin{aligned} \|f\|_{C^{0,\alpha}(B_2)} &= \|f\|_{C(B_2)} + [f]_{0,\alpha,B_2} \\ &\leq \|f\|_{C(B_2)} + \|Df\|_{C(B_2)} \\ &\leq \|W'(u)\|_{C(B_2)} + \|W'' \circ u\|_{C(B_2)} \|u\|_{C^1(B_2)} \\ &\leq C(R). \end{aligned} \tag{3.3}$$

Then the interior Schauder estimates (theorem 2.3) say

$$\|u\|_{C^{k+2}(B_1)} \leq \|u\|_{C^{k+2,\alpha}(B_1)} \leq C(k, R)(\|u\|_{C(B_2)} + \|f\|_{C^{k,\alpha}(B_2)}). \tag{3.4}$$

With $k = 0$, this is

$$\|u\|_{C^2(B_1)} \leq C(R) \tag{3.5}$$

by the above. More generally, suppose $\|u\|_{C^j(B_1)} \leq C(k, R)$. for all $j \leq k + 1$. Expanding out $D^j f$ with the product rule, the above calculation gives

$$\|f\|_{C^{k,\alpha}(B_2)} \leq \|f\|_{C^k(B_2)} + \|Df\|_{C^k(B_2)} \leq C(k, R)(\|u\|_{C(B_2)} + \|u\|_{C^{k+1}(B_2)}) \leq C(k, R), \tag{3.6}$$

with the k-dependence in the constant coming from derivatives of W and $\|u\|_{C^j(B_2)}$ for $j \leq k + 1$. Then by induction, the interior Schauder estimate gives

$$\|u\|_{C^{k+2}(B_1)} \leq C(k, R) \tag{3.7}$$

for all k. Now fix R and take a supremum over x_0 to get

$$\|u\|_{C^k(\mathbb{R}^n)} \leq C(k) \tag{3.8}$$

for all k. □

3.2. **Existence and Uniqueness of Positive Dirichlet Solutions.** This section proves theorem 3.9, which comes from exercises in [2]. The following theorem is the backbone of constructing solutions with given zero sets by gluing. Namely, one partitions a manifold into pieces whose boundaries form the prescribed zero set. On each piece, minimize energy to find a solution with zero boundary condition of constant sign. Then, glue these solutions together to get a function on the entire manifold. The resulting function is continuous, because on each boundary portion it is zero, and one hopes to choose the signs in each piece so that the gradient is also continuous. The technical details lie in checking that this function weakly solves Allen-Cahn across the zero set, where the gluing was done.

Broadly speaking, the existence part of this theorem helps us construct a candidate solution, and the uniqueness part lets us pass symmetries of the domain to symmetries of the solution.

Theorem 3.9 (A unique positive solution with Dirichlet boundary data exists if ϵ is small or if the region is large enough)**.** *Let U be a bounded domain and let λ_1 be the first eigenvalue of the Laplacian on U. If $\epsilon^2\lambda_1 < 1$, then $\epsilon^2\Delta u = u^3 - u$ has a unique positive Dirichlet solution on U.*

3.2.1. *Proof of Existence.* First we show existence. It turns out that u minimizes the Allen-Cahn energy over U.

Lemma 3.10. *A minimizer to the Allen-Cahn energy functional over $H_0^1(U)$ exists and is either identically 0 in U or constant sign in the interior of U.*

Proof. Since eq. (1.3) is coercive and convex on U, there exists u minimizing the energy functional over $H_0^1(U)$.[3] That is, there exists $u \in H_0^1(U)$ such that

$$E_\epsilon(u) = \min_{w \in H_0^1(U)} E_\epsilon(w).$$

We claim that this minimizer u is either constant sign in the interior of U (which we can take to be positive) or identically 0. Recall that

$$E_\epsilon(u; U) = \int_U \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right). \tag{3.9}$$

Because $|\nabla |u|| = |\nabla u|$ and $W(|u|) = W(u)$ (W is even), $|u|$ has the same energy as u and is thus also a minimizer. In appendix B, we show that a minimizer of the Allen-Cahn energy is a critical point of the Allen-Cahn energy, equivalently a weak solution of Allen-Cahn. By corollary 2.6, both u and $|u|$ are smooth. By proposition 3.2 and remark 3.3, together with the boundary condition $u = 0$ on ∂U, we conclude $|u| < 1$.

If there exists $x \in U$ such that $|u(x)| = 0$, then by the strong maximum principle, $|u| \equiv 0$ in U, since $\Delta |u| = |u|^3 - |u| \leq 0$ and $|u| \geq 0$ achieves its minimum of 0 on ∂U. Thus either $0 < |u| < 1$. Because u is continuous, we conclude (possibly flipping sign) that either $u \equiv 0$ or $0 < u < 1$. ☐

It remains to show that the minimizer u is not identically 0. We show that a first eigenfunction of the Laplacian has strictly less energy than 0 whenever $\epsilon^2 \lambda_1 < 1$, so that 0 cannot minimize energy.

Let φ be the first eigenfunction of the Laplacian with Dirichlet boundary condition (that is, $-\Delta \varphi = \lambda_1 \varphi$ with $\varphi \equiv 0$ on ∂U). By classical results, λ_1 is real and positive, and so φ is not the zero function. Compute

$$\begin{aligned} E_\epsilon(\varphi) &< E_\epsilon(0) \\ \int \frac{\epsilon}{2} |D\varphi|^2 + \frac{1}{\epsilon} W(\varphi) &< \int \frac{1}{\epsilon} W(0) \\ \int -\frac{\epsilon^2}{2} \varphi \Delta \varphi + \frac{\varphi^4}{4} - \frac{\varphi^2}{2} + \frac{1}{4} &< \int \frac{1}{4} \\ \epsilon^2 \lambda_1 \int \varphi^2 &< \int \varphi^2 - \frac{1}{2} \int \varphi^4 \\ \epsilon^2 \lambda_1 &< 1 - \frac{1}{2} \frac{\int \varphi^4}{\int \varphi^2}. \end{aligned}$$

Because $c\varphi$ satisfies the same conditions as φ for any $c \in \mathbb{R}$, the above calculation holds with $c\varphi$ in place of φ. Then

$$E_\epsilon(c\varphi) < E_\epsilon(0) \iff \epsilon^2 \lambda_1 < 1 - \frac{c^2}{2} \frac{\int \varphi^4}{\int \varphi^2}. \tag{3.10}$$

[3]The argument about the existence of a minimizer can be seen in Evans' Partial Differential Equations [4] Chapter 8 section 2. We specifically use Theorem 2 in that section to prove the existence of the minimizer.

If $\epsilon^2\lambda_1 < 1$, then c can be chosen small enough for the right inequality in to hold, in which case 0 is not a minimizer of the Allen-Cahn energy.

3.2.2. *Proof of Uniqueness.* Next we prove that this solution u is unique. We first show the following lemma.

Lemma 3.11. *Suppose $f, g : \bar{U} \to \mathbb{R}$ are smooth with $f, g = 0$ on ∂U and $f, g \neq 0$ in U. If g has non-vanishing outward normal gradient on the boundary (that is, $\partial_\nu g \neq 0$), then $\frac{f}{g}$ is smooth on $\bar{U}$.*

Proof. Since ∂U is smooth, we can straighten it smoothly. Formally, for any $x \in \partial U$, there exist smooth local coordinates $(y_1, \ldots, y_n)$ and a neighbourhood V of $y(x)$ such that $V \cap \overline{U} = \{z \in V : y_1(x) \geq 0\}$. In these coordinates, the condition $\partial_\nu g = Dg \cdot \nu \neq 0$ on ∂U becomes $\partial_{y_1} g \neq 0$ on ∂U, because $\nu(y) = sy_1$ for some constant s. Because $g = 0$ on ∂U, $g = 0$ in V where $y_1 = 0$, so by the fundamental theorem of calculus,

$$g(y_1, \ldots, y_n) = \int_0^1 \frac{\partial g}{\partial t}(ty_1, \ldots, y_n)\,\mathrm{d}t = y_1 \int_0^1 \frac{\partial g}{\partial y_1}(ty_1, \ldots, y_n)\,\mathrm{d}t. \tag{3.11}$$

Define $h_g : V \cap \overline{U} \to \mathbb{R}$ to be the integral expression on the right. Then $g = y_1 h_g$. Because g is smooth, differentiating under the integral sign shows that h_g is smooth. Define h_f and derive its properties similarly. Moreover,h_g is nonzero on ∂U (because $\partial_{y_1} u_i \neq 0$ on ∂U). Then in V, $\frac{f}{g} = \frac{f}{g} = \frac{h_f}{h_g}$, which is smooth on ∂U. Thus $\frac{f}{g}$ is smooth in a neighbourhood of each point of ∂U, and also smooth in the interior because f, g are non-zero in U. ∎

We now apply this lemma to two positive Dirichlet solutions to Allen-Cahn on U, say u_1 and u_2.

Corollary 3.12. *The quotients $\frac{u_1}{u_2}, \frac{u_2}{u_1}$ are bounded.*

Proof. By assumption, u_1, u_2 are smooth functions nonzero in U and vanishing on U. Because $\Delta u_1, \Delta u_2 \leq 0$ in U, Hopf's lemma implies $\partial_\nu u_1, \partial_\nu u_2 < 0$, so lemma 3.11 implies that $\frac{u_1}{u_2}, \frac{u_2}{u_1}$ bounded because $\bar{U}$ is compact. ∎

For $i = 1, 2$, $-\Delta u_i = -u_i^3 + u_i < u_i$, so $\Delta u_i + u_i > 0$. Write

$$\int \left(\frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2}\right)(u_2^2 - u_1^2) = \int -u_1\Delta u_1 + \Delta u_2 \frac{u_1^2}{u_2} + \Delta u_1 \frac{u_2^2}{u_1} - u_2 \Delta u_2. \tag{3.12}$$

Compute the derivative

$$D\left(\frac{u_1^2}{u_2}\right) = 2\frac{u_1}{u_2} Du_1 - \frac{u_1^2}{u_2^2} Du_2. \tag{3.13}$$

The right side of eq. (3.13) is L^2 due to corollary 3.12, so $\frac{u_1^2}{u_2} \in H_0^1$. Integrating by parts gives

$$\begin{aligned} \int -u_1 \Delta u_1 + \Delta u_2 \frac{u_1^2}{u_2} &= \int |Du|^2 - Du_2 \cdot \left(2\frac{u_1}{u_2} Du_1 - \frac{u_1^2}{u_2^2} Du_2\right) \\ &= \int \left|Du - \frac{u_1}{u_2} Du_2\right|^2 \geq 0. \end{aligned} \tag{3.14}$$

Adding eq. (3.14) to the analogous inequality for u_1, u_2 swapped and substituting into eq. (3.12) gives

$$
\begin{aligned}
0 \geq \int_U \left(\frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2}\right)(u_2^2 - u_1^2) &= \int_U \left(\frac{u_1^3 - u_1}{u_1} - \frac{u_2^3 - u_2}{u_2}\right)(u_2^2 - u_1^2) \\
&= \int_U \left(u_1^2 - u_2^2\right)\left(u_2^2 - u_1^2\right) \\
&= -\int_U \left(u_1^2 - u_2^2\right)^2,
\end{aligned} \tag{3.15}
$$

and thus

$$\int_U (u_1^2 - u_2^2)^2 = 0. \tag{3.16}$$

Because $u_1, u_2 > 0$ in U, we conclude $u_1 = u_2$. This concludes the proof of theorem 3.9.

3.3. Stability of Solutions and the De Giorgi Monotonicity Condition.

On $\mathbb{R}$, the function

$$\mathbb{H}(x) = \tanh(\frac{x}{\sqrt{2}})$$

solves the Allen-Cahn equation. It is called the heteroclinic solution.

Any solution on $\mathbb{R}$ can be made into a so-called *one-dimensional solution* on $\mathbb{R}^n$ by ignoring all but one dimension. Namely, fix $a \in \mathbb{R}^n$ with $|a| = 1$. If u is a solution on $\mathbb{R}$, then $\tilde{u}(x) := u(\langle a, x\rangle)$ is a one-dimensional solution on $\mathbb{R}^n$:

$$\Delta \tilde{u} = \sum_{i=1}^{n} \tilde{u}_{x_i x_i} = \sum_{i=1}^{n} a_i^2 u''(\langle a, x\rangle) = \frac{1}{\epsilon^2} W'(u(\langle a, x\rangle)) = \frac{1}{\epsilon^2} W'(\tilde{u}(x)). \tag{3.17}$$

Notice where we used $|a| = 1$. Similarly, one could translate the origin of this solution or flip its sign: $\tilde{u}(x) := \pm u(\langle a, x - x_0\rangle)$, with a as before and $x_0 \in \mathbb{R}^n$ fixed. In particular, one can extend the one dimensional heteroclinic solution $\mathbb{H}(x)$ to $\mathbb{R}^n$ as $\mathbb{H}(\langle a, x\rangle - b)$.

This section is centered around the following conjecture of De Giorgi regarding the monotonicity of the heteroclinic solution, which leads to some results in classification of solutions. See [1] for the statements of this section.

Conjecture 3.13. Suppose u is a solution on $\mathbb{R}^n$ and u satisfies $\frac{\partial u}{\partial x_n} > 0$. Is it true that $u(x) = \mathbb{H}(\langle a, x\rangle - b)$?

This conjecture is known to be true for dimensions 2 and 3, and known to be not true for dimension 8 or above. However, it is open for intermediate dimensions. The condition $\frac{\partial u}{\partial x_n}$ is called the De Giorgi monotonicity condition.

The De Giorgi monotonicity condition is closely related to *stability* of solutions:

Definition 3.14. We say a solution u_ϵ to the Allen-Cahn equation $\epsilon^2 \Delta u = u^3 - u$ is *stable* if for compact set V and any $\varphi \in C_c^\infty(V)$, we have

$$\frac{d^2}{dt^2} E_\epsilon(u_\epsilon + t\varphi; V) \geq 0.$$

This relates to the De Giorgi monotonicity condition in the following way:

Theorem 3.15. *Suppose u is a solution on $\mathbb{R}^n$. If u satisfies $\frac{\partial u}{\partial x_n} > 0$, then u is stable.*

Therefore, to solve the De Giorgi Conjecture, it is helpful to classify the stable solutions. In dimension 2 and 3 this is possible, and we present the classification here.

Theorem 3.16 (Ghossob-Gui 1008)**.** *Suppose u is a solution on $\mathbb{R}^2$ and u is stable. Then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$.*

The proof is long and technical. The readers may refer to Ghoussoub-Gui [6]. As an immediate corollary, the De Giorgi conjecture holds in 2 dimensions.

Corollary 3.17. *Suppose u is a solution on $\mathbb{R}^2$ and u satisfies $\frac{\partial u}{\partial x_n} > 0$. Then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$.*

We can utilize the classification given in theorem 3.16 to study solutions on $\mathbb{R}^3$.

Theorem 3.18. *If u is a stable solution on $\mathbb{R}^3$ with $E_1(u; B_R) \leq CR^2$ for some constant C, then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$.*

The above theorem is similar to theorem 3.16 but asks additionally for a quadratic bound on energy growth. We show shat the De Giorgi condition implies this.

Theorem 3.19. *Suppose u is a solution on $\mathbb{R}^3$ and u satisfies $\frac{\partial u}{\partial x_n} > 0$. Then $u(x) = \mathbb{H}(\langle a, x\rangle - b)$.*

Proof. The goal of the proof is to show that $E_1(u; B_R) \leq CR^2$ for a constant C, and conclude that u is the 1-dimensional solution $\mathbb{H}(\langle a, x\rangle - b)$ using theorem 3.18.

Define translation in the last coordinate

$$u^t = u(x_1, x_2, x_3 + t).$$

By monotonicity and boundedness, the limit

$$u^{\pm\infty}(x) = \lim_{t\to\infty} u^t(x)$$

exists, and is a function of x_1, x_2, and is a solution of Allen-Cahn on $\mathbb{R}^2$. Moreover, a compact set in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^3$, and thus $u^{\pm\infty}(x_1, x_2)$ is stable as a solution on $\mathbb{R}^2$. Hence by theorem 3.16

$$u^{\pm\infty}(x_1, x_2) = \mathbb{H}(a_1x_1 + a_2x_2 - b).$$

Since the energy $E_1(\cdot, B_R)$ is radially symmetric, it suffices to let $a_1 = 1$, $a_2 = 0$. We can compute that

$$\begin{aligned} E_1(u^{\pm\infty}, B_R) &= \int_{B_R} \frac{1}{2}|Du^{\pm\infty}|^2 + W(u^{\pm\infty}) \\ &= \frac{1}{2}\int_{B_R} \operatorname{sech}^4(\frac{x_1 - b}{\sqrt{2}}) \leq \frac{1}{2}\int_{[-R,R]^3} \operatorname{sech}^4(\frac{x_1 - b}{\sqrt{2}}) \leq CR^2, \end{aligned}$$

where we bounded the integral over a sphere by the integral over the cube containing it, which can be evaluated explicitly. By dominated convergence theorem,

$$\lim_{t\to\infty} E_1(u^t; B_R) \leq CR^2.$$

Now we would like to obtain information for when $t = 0$. We do this by differentiating $E_1(u^t; B_R)$ with respect to t, and integrate with respect to t, as follows:

$$\begin{aligned}\frac{d}{dt}E_1(u^t;B_R) &= \frac{d}{dt}\int_{B_R}\frac{1}{2}|\nabla u^t|^2 + W(u^t)\\ &= \int_{B_R}\nabla u^t\cdot\nabla\frac{d}{dt}u^t + W'(u^t)\frac{d}{dt}u^t\end{aligned}$$

Integrating by parts in the first term [4], we have

$$\begin{aligned}\frac{d}{dt}E_1(u^t;B_R) &= \int_{B_R}\nabla u^t\cdot\nabla\frac{d}{dt}u^t + W'(u^t)\frac{d}{dt}u^t\\ &= \int_{B_R}\Delta u^t\cdot(-\frac{d}{dt}u^t) + W'(u^t)\frac{d}{dt}u^t + \int_{\partial B_R}\frac{d}{dt}u^t(\nabla u\cdot\nu)\\ &= \int_{\partial B_R}\frac{d}{dt}u^t(\nabla u\cdot\nu)\end{aligned}$$

We know from proposition 3.7 that $|\nabla u| \leq C$ for a constant C. Since $\frac{d}{dt}u^t > 0$ by the De Giorgi condition, we have that

$$\frac{d}{dt}E_1(u^t;B_R) = \int_{\partial B_R}\frac{d}{dt}u^t(\nabla u\cdot\nu) \geq -C\int_{\partial B_R}\frac{d}{dt}u^t.$$

Now, integrating in t, we see that

$$\lim_{t\to\infty}E_1(u^t;B_R) - E_1u;B_R \geq -C\int_{\partial B_R}(u^{+\infty}-u).$$

Since $|\partial B_R| = 4\pi R^2$ and $|u| \leq 1$, we have

$$E_1(u;B_R) \leq \lim_{t\to\infty}E_1(u^t;B_R) + C\int_{\partial B_R}(u^{+\infty}-u) \leq CR^2 + \tilde{C}R^2.$$

This concludes the proof. □

4. Constructing Solutions to Allen-Cahn

One-dimensional constructions using the Jacobi elliptic function have been described in [7]. The construction of the saddle solution is an exercise in [1], and the solutions on S^n are exercises in [2].

4.1. Allen-Cahn in One Dimension.

On $\mathbb{R}$, the Allen-Cahn equation becomes

$$\epsilon^2 u''(t) = W'(u(t)). \tag{4.1}$$

One can derive a solution u_ϵ for general ϵ starting from a solution u at scale $\epsilon = 1$ by setting $u_\epsilon(t) = u(\epsilon^{-1}t)$. For the rest of this section, take $\epsilon = 1$. The reader can check that

$$\mathbb{H}(t) := \tanh\frac{t}{\sqrt{2}}$$

solves Allen-Cahn. It is called the *heteroclinic solution* because of its monotonicity. Indeed, one can find this solution by observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'^2 - 2W(u)) = 2u'u'' - 2u'W'(u) = 0, \tag{4.2}$$

[4] ν here is the outward normal vector field.

so that $u'^2 = 2W(u) - \lambda$ for some $\lambda \in \mathbb{R}$. Setting $\lambda = 0$, supposing that $u' \geq 0$ and separating variables in $u' = \sqrt{2W(u)} = \frac{1}{\sqrt{2}}(1-u^2)$ gives the heteroclinic solution. Moreover, $\mathbb{H}$ has finite energy:

$$E(\mathbb{H}(t), \mathbb{R}) = \int_{-\infty}^{\infty} \frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t))\,\mathrm{d}t = \frac{2}{3}\sqrt{2}. \tag{4.3}$$

It is straightforward to show that the only solutions to Allen-Cahn on $\mathbb{R}$ with finite energy are (up to sign and translation) $u(t) = \mathbb{H}(t)$ and $u(t) \equiv \pm 1$.

Other entire solutions therefore have infinite energy. Finding such solutions might start with varying λ in $u'^2 = 2W(u) - \lambda$. Fixing initial conditions $u(0) = 0$ and $u'(0) \geq 0$, we come to the following setup:

$$\begin{cases} u'' = W'(u) = \frac{1}{4}(1-u^2)^2 \\ u' = \sqrt{2W(u) - \lambda} \\ u(0) = 0 \end{cases} \qquad c \in \mathbb{R} \tag{4.4}$$

The case $\lambda < 0$ results in finite time blowup: from $W(u) \geq 0$ we obtain $u' \geq \sqrt{\lambda}$, and a comparison principle shows that $u(t) \geq \sqrt{c}t$ and so $u(t) > 2$ for t big enough. For $u > 2$, we have $\frac{1}{2}u^2 \leq \sqrt{2W(u)} \leq \sqrt{2W(u) - \lambda}$, so another application of the comparison principle shows that because the solution to $u' = \frac{1}{2}u^2$ blows up in finite time, so must the Allen-Cahn solution $u' = \sqrt{2W(u) - \lambda}$. The case $\lambda > \frac{1}{2}$ is not compatible with the initial condition $u(0) = 0$ (or indeed any initial condition $|u(0)| < 1$), because $u'^2 = 2W(0) - \lambda = \frac{1}{2} - \lambda < 0$.

We may thus restrict our attention to $0 < \lambda \leq \frac{1}{2}$ (knowing that $\lambda = 0$ yields the heteroclinic solution. Indeed, separating variables in eq. (4.4), one finds that λ parametrizes a family of periodic infinite-energy solutions given explicitly by Jacobi elliptic functions.

Theorem 4.1. *For $0 < \lambda \leq \frac{1}{2}$ there exist infinite-energy periodic solutions to Allen-Cahn on $\mathbb{R}$ given by*

$$u_\lambda(t) = \sqrt{1 - \sqrt{2\lambda}}\,\mathrm{sn}\left(\sqrt{\frac{1+\sqrt{2\lambda}}{2}}t, \frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}\right). \tag{4.5}$$

Moreover,

(1) $u_\lambda(t) \to \mathbb{H}(t) = \tanh\frac{t}{\sqrt{2}}$ on a quarter-period $[0, \frac{1}{4}T_\lambda]$,
(2) $T_\lambda = \sqrt{2}\log|\lambda| + O(1)$ as $\lambda \to 0^+$, and
(3) $u_\lambda(t) \to 0$ and $\frac{u_\lambda(t)}{\sqrt{1-\sqrt{2\lambda}}} \to \sin(t)$ as $\lambda \to \frac{1}{2}^-$.

The characterization of solutions to Allen-Cahn on $\mathbb{R}$ using Jacobi elliptic functions is known in the literature, such as in [7]. Also see appendix A.

Here $\mathrm{sn}(x, k)$ is the Jacobi elliptic sine function, the analogue on the ellipse of sine on the circle, defined for $0 \leq k < 1$. One may alternatively parametrize these solutions in terms of their amplitude $A := \sqrt{1 - \sqrt{2\lambda}}$ as

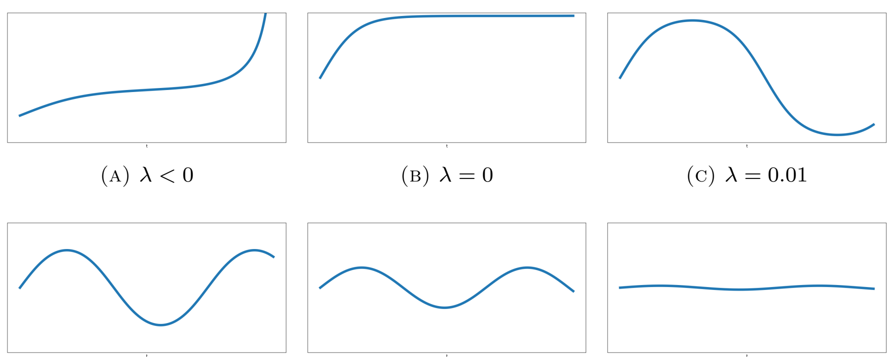
$$u_A(t) = A\,\mathrm{sn}\left(\sqrt{1 - \frac{1}{2}A^2}t, \frac{A^2}{2 - A^2}\right) \qquad 0 \leq A < 1. \tag{4.6}$$

Verifying that the functions in eq. (4.5) solve Allen-Cahn is straightforward once one knows the identity

$$\frac{\partial^2}{\partial x^2}\operatorname{sn}(x,k) = 2k\operatorname{sn}^3(x,k) - (1+k)\operatorname{sn}(x,k) \tag{4.7}$$

for the Jacobi elliptic function. Indeed, noticing the similarity between eq. (4.7) and $u'' = W'(u) = u^3 - u$, one can obtain the solutions as in eq. (4.6) by making the ansatz $u(t) = A\operatorname{sn}(\sqrt{B}t, C)$ and solving for appropriate values of B and C.

When $\lambda = 0$, $u_\lambda(t)$ should be the heteroclinic solution; formally substituting $\lambda = 0$ into eq. (4.5) puts $\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}$ outside the domain of sn, but indeed as $\lambda \to 0$, $u_\lambda(t) \to \mathbb{H}(t)$ on a quarter-period. When $\lambda = \frac{1}{2}$, the amplitude $\sqrt{1-\sqrt{2\lambda}}$ is 0 so $u_{\frac{1}{2}}(t) \equiv 0$ is the trivial infinite-energy solution to Allen-Cahn. Moreover, $\frac{u_\lambda(t)}{\sqrt{1-\sqrt{2\lambda}}} \to \sin(t)$ as $\lambda \to \frac{1}{2}^-$, and one can compute directly from the definition of sn that $\operatorname{sn}(x,0) = \sin x$.



(A) $\lambda < 0$ (B) $\lambda = 0$ (C) $\lambda = 0.01$

(D) $\lambda = 0.2$ (E) $\lambda = 0.4$ (F) $\lambda = 0.499$

FIGURE 2. Solutions to eq. (4.5) for various values of λ. Notice the blowup for $\lambda < 0$ and the heteroclinic solution for $\lambda = 0$.

4.2. **Extending Solutions on $\mathbb{R}$ to $\mathbb{R}^n$.** As noted above, any one dimensional solution can be extended to higher dimensions.

If u is one of the periodic Jacobi elliptic solutions from the previous section with parameter λ, then its one-dimensional extension to $\mathbb{R}^n$ satisfies periodic boundary conditions on $[0, 4T_\lambda]^n$, and so projects to a solution on the flat n-torus $\mathbb{R}^n \big/ 4T_\lambda\mathbb{Z}^n \cong \mathbb{T}^n$.

In general it is difficult to find explicit solutions on $\mathbb{R}^n$ with $n > 1$ apart from these one-dimensional solutions. For the rest of this paper, we construct solutions using non-explicit methods.

4.3. **Saddle Solutions on $\mathbb{R}^2$.**

Theorem 4.2. *There exists a solution to the Allen-Cahn equation (1.2) on $\mathbb{R}^2$ whose nodal set is exactly $\{xy = 0\}$.*

The proof proceeds as follows:

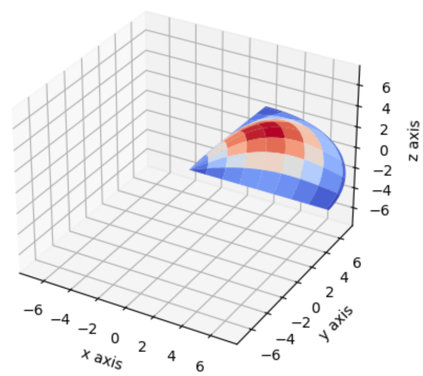


FIGURE 3. An approximation of what u_R will look like.

(1) Use theorem 3.9 to find a positive Dirichlet solution in a quarter disk of radius $\mathbb{R}$.
(2) Extend this solution to a full disk by odd reflection.
(3) Show the solution is smooth across the axes, and use a logarithmic cutoff function to show smoothness at the origin.
(4) By Arzela-Ascoli, obtain a uniform subsequential limit as $R \to \infty$.
(5) Show that the limit function minimizes energy on balls in each quadrant and is thus non-zero away from the axes.

Step 1: Construct a positive solution to Allen-Cahn in a quarter circle. We begin our construction first in the interior of a quarter-circle in the first quadrant of $\mathbb{R}^2$:

$$\Omega_R = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x^2 + y^2 < \mathbb{R}^2\}$$

with the Dirichlet boundary condition. Then by theorem 3.9 we know that there exists $u_R \in C^\infty(\Omega_R)$ and solves the Allen-Cahn. Moreover, we know that either $u_R \equiv 0$ or $u_R \in (0, 1)$ in Ω_R, and u_R is strictly positive for R large enough (because as R increase λ_1 decreases).

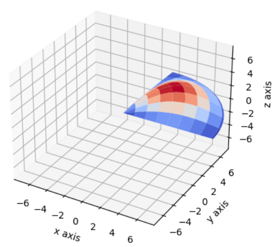
The solution will look something like fig. 3.

Step 2: Use odd reflection to construct a solution on the entire ball, and show the solution is smooth across the axes and at the orign using the log cut-off trick. Thus, we have $u_R \in (0, 1)$ on Ω_R. Next, we use odd reflection to construct $\tilde{u}_R$ that solves the Allen-Cahn eq. (1.2) on the entire ball $B_R(0) \subset \mathbb{R}^2$. One concern when doing odd reflection is the smoothness of $\tilde{u}_R$. We know that $\tilde{u}_R$ restricted to the interior of each quadrant is smooth since u_R is smooth. However, we need to check that $\tilde{u}_R$ is smooth across the axes and at the origin. However, these turn out not to be a problem. We first explicitly define the solution $\tilde{u}_R$ using odd reflection.

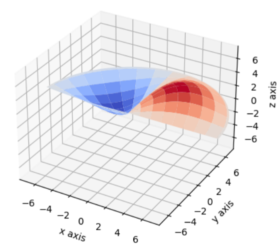
Let $B = B(0, R)$. Let $\tilde{u} = u_R$. Then define $u : B \to \mathbb{R}$ by odd reflections

$$u(x, y) = \begin{cases} \tilde{u}(x, y) & x, y > 0 \\ -\tilde{u}(-x, y) & x < 0 < y \\ \tilde{u}(-x, -y) & x, y < 0 \\ -\tilde{u}(x, -y) & y < 0 < x. \end{cases} \tag{4.8}$$

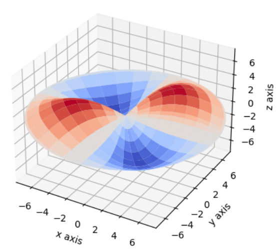
This looks like fig. 4.



(A) No reflection.



(B) Reflected across x-axis



(C) Reflected across two axes; the solution on B

FIGURE 4. Odd reflection of solution from quarter disk in step 1 to the entire disk.

Then $u \in C^1$ at the axes except possibly at 0, so $u \in H_0^1(B - \{0\})$. We next show that u weakly solves Allen-Cahn on $B \setminus \{0\}$. To see this, let R_i be the intersection of $B - \{0\}$ with the i-th quadrant of $\mathbb{R}^2$ and let $v \in C_c^\infty(B - \{0\})$. We can integrate by parts

$$\begin{aligned} \int_{B-0} Du \cdot Dv + W'(u)v \, \mathrm{d}x &= \sum_{i=1}^{4} \int_{R_i} Du \cdot Dv + W'(u)v \\ &= \sum_{i=1}^{4} \int_{R_i} (-\Delta u + W'(u))v \\ &= 0, \end{aligned} \tag{4.9}$$

where the boundary terms vanish because $u = 0$ on the axes and ∂B, and we use the fact that u solves Allen-Cahn strongly on each R_i. An approximation argument lets us replace $v \in C_c^\infty$ to $v \in H_0^1$ above. By elliptic regularity, u is thus smooth on $B - \{0\}$.

Remark 4.3. It is important that we treat the origin and the rest of B separately because the above argument does not work over the whole ball since we don't know $u \in C^1$ at 0, so we can't immediately show u solves Allen-Cahn on the whole ball. If we instead used even reflections to construct u, then u would not be C^1 (jump discontinuity of the derivative at axis), so we couldn't integrate by parts.

To show that u is in fact smooth at the origin we use the "log cut-off" trick. For $0 < r < 1$ define

$$\zeta_r(x) := \begin{cases} 0 & |x| \le r^2 \\ 2 - \frac{\log|x|}{\log r} & r^2 < |x| < r \\ 1 & |x| > r \end{cases} . \tag{4.10}$$

Then $0 \le \zeta_r \le 1$ and ζ_r is supported away from the origin and converges pointwise to 1 on $B \setminus \{0\}$ as $r \to 0$. Then for any $v \in C_c^\infty(B)$, $\zeta_r v \in C_c^\infty(B - \{0\})$, so

$$0 = \int Du \cdot D(\zeta_r v) + W'(u)\zeta_r v = \int \zeta_r Du \cdot Dv + vDu \cdot D\zeta_r + W'(u)\zeta_r v. \tag{4.11}$$

Then

$$|\zeta_r Du \cdot D\zeta_r| \le \frac{1}{2}|Du|^2 + \frac{1}{2}|Dv|^2 < \infty \tag{4.12}$$

by Cauchy's inequality, and because $|u| < 1$,

$$|W'(u)\zeta_r v| \le \frac{1}{2}|W'(u)|^2 + \frac{1}{2}|v|^2 < \infty, \tag{4.13}$$

The right sides in are in L^1 because the domain is finite. On the other hand,

$$\left| \int vDu \cdot D\zeta_r \right| \le \|v\|_{L^\infty} \|Du\|_{L^2} \|D\zeta_r\|_{L^2} \tag{4.14}$$

by Hölder's inequality, and

$$(\zeta_r)_{x_i} = -\frac{x_i}{|x|^2 \log r}, \tag{4.15}$$

so

$$\int_{B\setminus\{0\}} |D\zeta_r|^2 = \int_{r^2<|x|<r} \frac{1}{|x|^2 |\log r|^2} \le \frac{C}{|\log r|^2} \int_{r^2}^{r} \rho^{-1}\, \mathrm{d}\rho \le \frac{C}{|\log r|} \tag{4.16}$$

which goes to 0 as $r \to 0$. Thus we may pass to the limit by the dominated convergence theorem to obtain

$$\int Du \cdot Dv + W'(u)v = 0 \tag{4.17}$$

in the entire ball. Thus u solves Allen-Cahn on the whole ball, so it is smooth.

Step 3: Extend the solution on the ball to an smooth solution to the the entire $\mathbb{R}^2$ *using Arzela-Ascoli, the diagonal argument, and the boundedness of all orders of derivatives.* Now we want to extend the radius of our ball to infinity and obtain a solution on $\mathbb{R}^2$. To do so we use the diagonal argument and Arzela-Ascoli to obtain the subsequence of $\{u_n\}$, where for each $n \in \mathbb{R}$, u_n is a solution for the ball $B_R(0)$ constructed like above, that converges uniformly in $C^\infty_{\text{loc}}(\mathbb{R}^2)$.

The argument goes like this. By proposition 3.7, all derivatives of u_R are bounded uniformly in R. Consider a compact domain in $\mathbb{R}^2$ to apply Arzela-Ascoli. By Arzela-Ascoli, find a sequence $\{n_{k,0}\} \in \mathbb{N}$ such that $u_{n_{k,0}}$ has a uniform limit u. Refine to a subsequence $\{n_{k,1}\}$ such that $Du_{n_{k,1}}$ converges uniformly. In general, if all derivatives up to order m of $u_{n_{k,m}}$ converge uniformly, then refine to a subsequence $\{n_{k,m+1}\}$ so that $D^{m+1}u_{n_{k,m+1}}$ converge uniformly. Then all derivatives of $u_{n_{k,k}}$ converge uniformly, and thus in fact to the corresponding derivatives of u. Thus u is smooth, and passing to a pointwise limit in $\Delta u_R = W'(u_R)$ shows that u is a smooth solution to Allen-Cahn on $\mathbb{R}^n$. For the rest of the problem, we can re-index u_R so that $u_R \to u$ uniformly in C^∞_{loc}.

Step 4: Show that the nodal set is infact two orthogonal axes using the energy method. Now it remains to show that the nodal set is $\{xy = 0\}$. Now we show $\{u = 0\} = \{xy = 0\}$. Because of the symmetry of u, it suffices to show $u \neq 0$ in the interior of the first quadrant. Let $B = B(x_0, r)$ be a ball compactly contained in the first quadrant. We can take r large enough so that by the argument in theorem 3.9 (which applies because u has constant sign in a quadrant), u is non-zero in the interior of the first quadrant if it is a minimizer on such balls. We now show the latter.

For R large enough, Ω_R compactly contains B. Then if $w = u$ on ∂B, the function v that is u on $\Omega_R - B$ and w on B is in $H^1_0(\Omega_R)$, so by Part A, $E(v, \Omega_R) \geq E(u_R, \Omega_R)$, and $v = u_R$ on $\Omega_R - B$, so $E(v, B) \geq E(u_R, B)$. Now we show this property passes to the limit.

Suppose u does not minimize energy on B. Then there exists a minimizer $w \in H^1(B)$ with $w = u$ on ∂B and $E(w, B) \leq E(u, B) - \delta$ for some $\delta > 0$. Moreover $|w| \leq 1$. Define φ_R the log-cutoff function

$$\varphi_R(x) = \begin{cases} 1 & x \in B(x_0, r - \frac{1}{R}) \\ 2 - \frac{\log(r - |x - x_0|)}{\log R} & B(x_0, r - \frac{1}{R^2}) - B(x_0, r - \frac{1}{R}) \\ 0 & x \in B - B(x_0, r - \frac{1}{R^2}) \end{cases}. \tag{4.18}$$

We now claim

$$E((1 - \varphi_R)u_R + \varphi_R w, \Omega_R) = E(\chi_{\Omega_R - B}u_R + \chi_B w, \Omega_R) + o(1) \tag{4.19}$$

as $R \to \infty$. Note that $\chi_{\Omega_R - B}u + \chi_B w \in H^1(\Omega_R)$ because $u = w$ on ∂B. First we estimate the derivatives:

(4.20)
$$\begin{aligned} &\|D((1 - \varphi_R)u_R + \varphi_R w)\|_{L^2(\Omega_R)} - \|D(\chi_{\Omega_R - B}u_R + \chi_B w)\|_{L^2(\Omega_R)} \\ &\leq \|(u_R - w)D\varphi_R\|_{L^2(\Omega_R)} + \|(\chi_B - \varphi_R)Du_R\|_{L^2(\Omega_R)} + \|(\chi_B - \varphi_R)Dw\|_{L^2(\Omega_R)}. \end{aligned}$$

For the second term, the integrand is bounded by $2|Du_R| \leq C$ on B and it is 0 outside of B. The third integrand is bounded by $2|Dw| \in L^2$ on B and 0 outside of B. By the dominated convergence theorem ($\varphi_R \to \chi_B$ a.e.), they both go to 0. For the first term,

$$\int_{\Omega_R} |u_R - w|^2 |D\varphi_R|^2 \le C \int_{B(x_0, r-\frac{1}{R^2}) - B(x_0, r-\frac{1}{R})} \frac{1}{|x-x_0|\,(r-|x-x_0|)\,|\log R|^2}\,\mathrm{d}x \tag{4.21}$$
$$\le \frac{C}{|\log R|^2} \int_{r-\frac{1}{R}}^{r-\frac{1}{R^2}} \frac{\mathrm{d}\rho}{r-\rho}$$
$$= \frac{C}{|\log R|} \to 0.$$

For the potential term,

$$\begin{aligned}&\int_{\Omega_R} |W((1-\varphi_R)u_R + \varphi_R w) - W(\chi_{\Omega_R - B} u_R + \chi_B w)| \\ &= \int_B |W((1-\varphi_R)u_R + \varphi_R w) - W(\chi_{\Omega_R - B} u_R + \chi_B w)|\,,\end{aligned} \tag{4.22}$$

and the integrand is bounded by $2W(0)$ because $|u|, |w| \le 1$. The dominated convergence theorem on the finite domain B and the pointwise convergence of both terms in the integrand to $W(\chi_{\Omega_R - B} u + \chi_B w)$ shows that the difference in potential terms is $o(1)$.

Now we derive a contradiction. Starting from the minimizing property of u_R on Ω_R and applying the above,

$$\begin{aligned} E(u_R, \Omega_R) &\le E((1-\varphi_R)u_R, \varphi_R w, \Omega_R) \\ &= E(\chi_{\Omega_R - B} u_R + \chi_B w, \Omega_R) + o(1) \\ &= E(u_R, \Omega_R - B) + E(w, B) + o(1) \\ &= E(u_R, \Omega_R - B) + E(u, B) - \delta + o(1) \\ &= E(u_R, \Omega_R - B) + E(u_R, B) - \delta + o(1) \\ &= E(u_R, \Omega_R) - \delta + o(1), \end{aligned} \tag{4.23}$$

which gives $\delta \le o(1)$, a contradiction. Notice that we used $E(u_R, B) = E(u, B)$ (because u_R and its derivatives converge uniformly to those of u on B). Thus u vanishes only on $\{xy = 0\}$.

The constructed solution will roughly look like fig. 5.

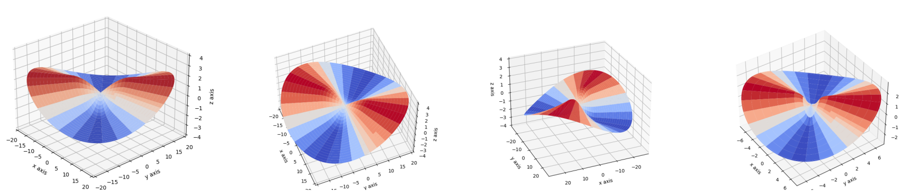


FIGURE 5. Approximation of Solution on $\mathbb{R}^2$

4.4. **Solutions on S^n.** The unit sphere $S^n := \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}$ is naturally a submanifold of $\mathbb{R}^{n+1}$, and its natural metric is the one induced by the ambient Euclidean space. One can parametrize the sphere of radius r by spherical coordinates by γ as

$$\begin{aligned} \gamma : x_1 &\mapsto r\cos\theta_1 \\ x_i &\mapsto r\cos\theta_2 \prod_{j=1}^{i-1} \sin\theta_j \qquad 2 \le i \le n-1 \\ x_{n+1} &\mapsto r \prod_{j=1}^{n} \sin\theta_j \end{aligned} \tag{4.24}$$

for $\theta_j \in [0,\pi]$ for $1 \le j \le n-1$ and $\theta_n \in [0,2\pi)$. Pull back the Euclidean metric g by γ (which in coordinates is $g_{ij} = \delta_{ij}$) to obtain the induced metric on the sphere $\tilde{g} = \gamma^* g$. Use the chain rule to compute $\tilde{g}$ in spherical coordinates $(\theta_1, \ldots, \theta_n)$ as

$$\tilde{g}_{ab} = g_{ij} \frac{\partial x_i}{\partial \theta_a} \frac{\partial x_j}{\partial \theta_b} = \frac{\partial x_i}{\partial \theta_a} \frac{\partial x_i}{\partial \theta_b}. \tag{4.25}$$

Note that we use the Einstein summation convention. The off-diagonal components for $a > b$ and $b > a$ are alternating sums which cancel to give $\tilde{g}_{ab} = 0$. When $a = b$, we obtain the metric of the sphere as a diagonal matrix:

$$\begin{aligned} \tilde{g}_{11} &= r^2 \\ \tilde{g}_{aa} &= r^2 \prod_{j=1}^{a-1} \sin^2\theta_j \qquad 2 \le a \le n. \end{aligned} \tag{4.26}$$

We also recall the expressions for the Laplace-Beltrami operator and the gradient in local coordinates:

$$\begin{aligned} \Delta_g f &= \frac{1}{\sqrt{|\det g|}} \partial_i \left(\sqrt{|\det g|} g^{ij} \partial_j f \right) \\ \nabla_g f &= (\partial_i f) g^{ij} \partial_j, \end{aligned} \tag{4.27}$$

where g^{ij} are the components of the inverse metric tensor.

4.4.1. *Solutions on S^1.* On the circle, the above parametrization reduces the 1×1 matrix $g \equiv [1]$, so the Laplace-Beltrami operator in the θ coordinate system reduces to the ordinary Laplacian. Because $\theta \in [0,2\pi)$, solving Allen-Cahn weakly on the circle S^1 is equivalent to solving Allen-Cahn weakly on $[0,2\pi]$ with periodic boundary conditions. In particular, the gradients at 0 and 2π must match, so that this solution could be extended to a periodic weak solution on $\mathbb{R}$. By elliptic regularity, this is in fact a smooth solution.

That is, the study of solutions of Allen-Cahn on S^1 is exactly the study of periodic solutions of Allen-Cahn on $\mathbb{R}$. In particular, if u_λ is a Jacobi elliptic solution on $\mathbb{R}$, then $\tilde{u}(\theta) := u_\lambda(\frac{T_\lambda}{2\pi}\theta)$ solves Allen-Cahn on S^1 at scale $\epsilon = \frac{2\pi}{T_\lambda}$. Recall that $T_\lambda \in [2\pi, \infty)$ (and $u_\lambda \equiv 0$ when $T_\lambda = 2\pi$), so that Allen-Cahn on the circle has a non-trivial solution for $0 < \epsilon < 1$.

4.4.2. *Solution Vanishing on an Equator.* By theorem 3.9, there exists u_+ positive minimizing energy with Dirichlet boundary conditions on the half-sphere $S^n_+ := S^n \cap \{x_{n+1} > 0\}$ for ϵ sufficiently small (because the domain is fixed). Define S^n_- and u_- analogously.

Now we show that odd reflecting u_+ and gluing it yields a solution. Define $\tilde{u}_-$ on S^n_- by odd reflection as $\tilde{u}_-(x', x_{n+1}) = -u_+(x', -x_{n+1})$, where $x' = (x_1, \ldots, x_{n-1})$. Then $\tilde{u}_-$ is negative on S^n_- and satisfies Dirichlet boundary conditions. We claim $\tilde{u}_- = u_-$. Indeed, $E(\tilde{u}_-, S^n_-) = E(u_+, S^n_+) = E_0$. We know $E_0 \geq E(u_-, S^n_-)$ because u_- minimizes energy on S^n_-. If the inequality were strict then the odd reflection of u_- to a positive function on S^n_+ with Dirichlet boundary data would have strictly lower energy than u^+, a contradiction. Thus $E_0 = E(\tilde{u}_-, S^n_-) = E(u_-, S^n_-)$, and by uniqueness of u_-, we have $u_- = \tilde{u}_-$.

Because $u_\pm$ as well as spherical coordinates (see eq. (4.24)) are odd with respect to reflection across an equator, we conclude that the gradients of $u_\pm$ agree on the equator $\{x_{n+1} = 0\}$, so the glued solution u which is $u_\pm$ on $S^n_\pm$ and 0 on the equator weakly solves Allen-Cahn on the equator and thus on S^n.

4.4.3. *Solution Vanishing on Orthogonal Equators.* This is a spherical analogue of the saddle solution in $\mathbb{R}^2$, and the argument is much the same.

Let $n \geq 2$. Construct a positive Dirichlet solution u minimizing energy on a quarter-sphere $S^n \cap \{x_1 x_2 = 0\}$ and extend by odd reflection to the entire sphere.

The arguments in step 2 of constructing a saddle solution on $\mathbb{R}^2$ in section 4.3 used to show that the solution is smooth across the axes (in this case across $S^n \cap (\{x_1 x_2 = 0\} \setminus \{x_1 = x_2 = 0\})$) pass to the sphere. However, the log-cutoff function used to show smoothness across the singular point at the origin was defined in Euclidean coordinates, and must be modified accordingly. Recall that the log-cutoff function ζ_r was defined for small r to be zero in a ball of radius r^2 around a point and increase logarithmically to 1 at the ball of radius r so that $\|D\zeta_r\|_{L^2} \to 0$ as $r \to 0$. We construct this cutoff function now, in spherical coordinates.

In our case the singular set is $S^n \cap \{x_1 = x_2 = 0\}$, or $\left\{\theta_1 = \theta_2 = \frac{\pi}{2}\right\}$ for the portion the singular set in the chart of spherical coordinates in eq. (4.24). There are 2^{n-1} such portions, corresponding to $n-1$ choices of sign for the coordinates $x_1, \ldots, x_{n-1}$. By symmetry though, it suffices to compute the log-cutoff function on just of these portions. Define

$$\Theta = \sqrt{(\theta_1 - \frac{\pi}{2})^2 + (\theta_2 - \frac{\pi}{2})^2}$$

$$\zeta_r(\theta_1, \ldots, \theta_n) = \begin{cases} 0 & \Theta \leq r^2 \\ 2 - \frac{\log \Theta}{\log r} & r^2 \leq \Theta \leq r \\ 1 & \Theta \geq r \end{cases} . \tag{4.28}$$

From eq. (4.27), $|Df|^2 = g(Df, Df)$ is $g_{ij}(\partial_i f) g^{ij} (\partial_j f) g^{ji} = g^{ij}(\partial_i f)(\partial_i j)$. In spherical coordinates, this becomes $g^{ii}(\partial_i f)^2$, because the off-diagonal components vanish. We compute

$$\partial_i \zeta_r = \begin{cases} -\frac{\theta_i - \frac{\pi}{2}}{\Theta^2 \log^2 r} & i = 1, 2 \\ 0 & i > 2, \end{cases} \tag{4.29}$$

where ζ_r is non-constant, so that

$$|D\zeta_r|^2 = \frac{(\theta_1 - \frac{\pi}{2})^2}{\Theta^4} + \frac{1}{\sin^2\theta_1}\frac{(\theta_2 - \frac{\pi}{2})}{\Theta^4} \tag{4.30}$$

on $r^2 \le \Theta \le r$. Integrating in local coordinates, we find that

$$\begin{aligned}
\|D\zeta_r\|_{L^2}^2 &= \frac{1}{\log^2 r}\int_{r^2\le\Theta\le r} |D\zeta_r|^2 \, \mathrm{d}\mu_g \\
&= \frac{1}{\log^2 r}\int_{r^2\le\Theta\le r} \sqrt{|\det g|}\left(\frac{(\theta_1 - \frac{\pi}{2})^2}{\Theta^4} + \frac{1}{\sin^2\theta_1}\frac{(\theta_2 - \frac{\pi}{2})^2}{\Theta^4}\right)\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \\
&\le \frac{C}{\log^2 r}\int_{r^2\le\Theta\le r} \frac{(\theta_1 - \frac{\pi}{2})^2}{\Theta^4} + \frac{1}{\sin^2\theta_1}\frac{(\theta_2 - \frac{\pi}{2})^2}{\Theta^4}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2,
\end{aligned}$$

where in the last step we use $\sqrt{|\det g|} = \prod_{j=1}^n \left|\sin^{n-j}\theta_j\right| \le 1$ and integrate along $\theta_3\cdots\theta_n$. Now change coordinates to $\tilde\theta_i = \theta_i - \frac{\pi}{2}$ to obtain

$$\begin{aligned}
\|D\zeta_r\|_{L^2}^2 &\le \frac{C}{\log^2 r}\int_{r^2\le\sqrt{\tilde\theta_1^2+\tilde\theta_2^2}\le r} \frac{\tilde\theta_1^2}{(\tilde\theta_1^2+\tilde\theta_2^2)^2} + \frac{1}{\cos^2\tilde\theta_1}\frac{\tilde\theta_2^2}{(\tilde\theta_1^2+\tilde\theta_2^2)^2}\,\mathrm{d}\tilde\theta_1\,\mathrm{d}\tilde\theta_2 \\
&\le \frac{C}{\log^2 r}\int_{r^2\le\sqrt{\tilde\theta_1^2+\tilde\theta_2^2}\le r} \frac{1}{\tilde\theta_1^2+\tilde\theta_2^2}\,\mathrm{d}\tilde\theta_1\,\mathrm{d}\tilde\theta_2,
\end{aligned} \tag{4.31}$$

because for r small enough, $\tilde\theta_1 \le r$ gives $\frac{1}{\cos^2\tilde\theta_1} \ge 2$. Now change to polar coordinates $\tilde\theta_1 = \rho\cos\varphi, \tilde\theta_2 = \rho\sin\varphi$ and integrate along the φ coordinate:

$$\|D\zeta_r\|_{L^2}^2 \le \frac{C}{\log^2 r}\int_{r^2}^{r}\frac{1}{\rho}\,\mathrm{d}\rho \le -\frac{C}{\log r} \to 0 \,\mathrm{as}\, r \to 0. \tag{4.32}$$

4.4.4. *Solution Projecting to* $\mathbb{RP}^n$. We now construct a solution described by more complicated symmetries than odd reflection. In particular, it is symmetric with respect to the antipodal map $x \mapsto -x$ and thus projects well to a solution on the real projective space $\mathbb{RP}^n$, which can be defined as the quotient ${S^n}/{(x \sim -x)}$, namely the the n-sphere with antipodal points identified. The construction looks like fig. 6. It proceeds roughly as follows:

(1) Cut S^n into three pieces: a band around the equator of width $2t$ and the remaining spherical caps.
(2) Minimize energy by theorem 3.9 to find a non-negative solution on the caps and a non-positive solution on the band.
(3) Use rotational symmetry to show that the normal derivative of the solutions is constant on the boundary and varies continuously in t.
(4) Show that the first eigenvalues of the band and caps are unbounded as $t \to 0$ and $t \to 1$, respectively.
(5) Conclude by theorem 3.9 that the solutions from Step 2 are identically zero on the band and caps for t sufficiently close to 0 or 1, respectively.
(6) By continuity, find some t where the normal derivative of the gradients agree on the boundary. This is a weak solution on all of S^n.

For $0 < t < 1$, let A_t be a band around the equator of width $2t$, namely $A_t := S^n \cap \{|x_1| < t\}$, and let D_t^+ and D_t^- be the remaining two caps of the sphere, so that $S^n \setminus A_t = D_t^+ \cup D_t^-$. We may define the Dirichlet solutions of minimal energy

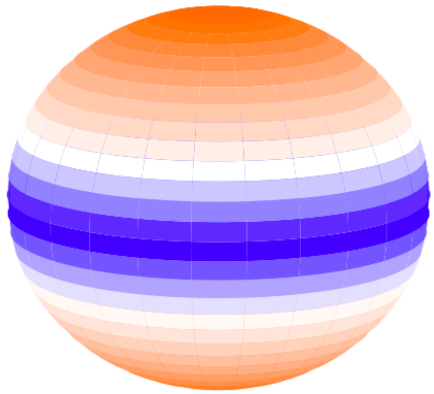


FIGURE 6. Approximation of solution on S^2 projecting to $\mathbb{RP}^2$. Orange, blue, and white, mean $1, -1, 0$, respectively.

on A_t and $D_t^{\pm}$. Note that we do not yet assert that these solutions are non-zero, though this is true for some t_0 and ϵ small enough.

We now show how the symmetries of the domains A_t and $D_t^{\pm}$ pass to symmetries of their energy minimizers. We focus on A_t. Fix a hyperplane through a great circle orthogonal to the equator $\{x_1 = 0\}$ and let v be a non-positive minimizer on A_t (by theorem 3.9, v is either identically zero or negative in the interior). The minimizer v must have the same energy on either side of the hyperplane. If not, then the even reflection of v from the side with less energy would produce a new function $\tilde{v}$ admissible in the minimization problem with strictly less energy than the minimizer. And so $\tilde{v}$ must have the same energy as v. Because v is the unique energy minimizer, $\tilde{v} = v$. The hyperplane was arbitrary, so v is actually rotationally symmetric. The same argument works for (non-negative minimizers on) $D_t^{\pm}$.

Combining the rotational symmetry of the domains (and thus their outward normal vector field) with the rotational symmetry of their energy minimizers, we conclude that the minimizers have constant normal derivatives on the boundary.

Moreover, the above argument shows that the minimizer v on A_t is symmetric with respect to even reflection about the equator $\{x_1 = 0\}$. Because v is smooth, this means that $Dv \cdot \nu = 0$ on the equator.

We now show that the normal derivatives are continuous in t.

Lemma 4.4. *Let u_t be an energy minimizer on the domain A_t or $D_t^{\pm}$ and ν is the outward normal vector field of the domain. Then the normal derivative $Du_t \cdot \nu$ is constant and continuous in t.*

Proof. Fix $\epsilon > 0$. Consider a non-positive Dirichlet minimizer u_t on A_t at scale ϵ; the argument for non-negative solutions on $D_t^{\pm}$ is similar. Let $A_t = A_t^+ \cup A_t^-$, with the sign being that of x_1. Because u_t is symmetric about the equator, we can focus on A_t^+. Let $Du_t \cdot \nu \equiv C_t$ on ∂A_t.

For t_0 fixed, we want to show $C_t \to C_{t_0}$. It suffices to show that this holds on some subsequence of every sequence $t \to t_0$. Because u_t solves Allen-Cahn and $Du_t \cdot \nu \equiv 0$ on the equator, we can integrate by parts to get

$$\int_{A_t^+} W'(u_t) = \int_{A_t^+} \epsilon^2 \Delta u_t = \epsilon^2 \int_{\partial A_t^+} Du_t \cdot \nu = \frac{\epsilon^2}{\left|\partial A_t^+\right|} C_t. \tag{4.33}$$

Evidently $\left|\partial A_t^+\right| \to \left|\partial A_{t_0}^+\right|$ (in $(n-1)$-measure). Extend u_t by 0 (thus continuously) to $S^n \cap \{x_1 \geq 0\}$. By Schauder estimates, as t varies, u_t are uniformly bounded and uniformly equicontinuous, so by Arzela-Ascoli they converge uniformly along a subsequence on $\overline{A_{t_0}^+}$ to u_{t_0}. Then

$$\begin{aligned} \left| \int_{A_{t_0}^+} W'(u_{t_0}) - \int_{A_t^+} W'(u_t) \right| &\leq \int_{A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| \\ &\quad + \int_{A_t^+ - A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| \\ &\leq \int_{A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| + 2\left|A_t^+ - A_{t_0}^+\right|, \end{aligned} \tag{4.34}$$

where the first term goes to 0 by the uniform convergence of $u_t \to u_{t_0}$ and the second term goes to 0 by $|u| < 1$ and the geometry of the domains. In light of eq. (4.33), we conclude that C_t is continuous in t. □

We now characterize the first eigenvalues of the domains A_t and $D_{1-t}^{\pm}$ so that we may apply theorem 3.9. Broadly speaking, small domains correspond to large first eigenvalues. As $t \to 0$, the first eigenvalues of these domains blow up to infinity. First, we show a lower bound on the first eigenvalue of a domain.

Lemma 4.5. *Let M be a Riemannian manifold, let $U \subset M$ be a bounded open set with smooth boundary, and let $f : U \to \mathbb{R}$ be $C^2(U)$ with $f > 0$ in U and $f = 0$ on ∂U. If $-\Delta f \geq \lambda f$, then $\lambda_1(U) \geq \lambda$.*

This type of lemma is attributed to Barta in [8]. We give a proof here.

Proof. If $g \in C_c^2(U) - \{0\}$, then we may write $g = fh$, with $h := \frac{g}{f} \in C_c^2(U)$. Then

$$|Dg|^2 = f^2 |Dh|^2 + h^2 |Df|^2 + 2fhDf \cdot Df. \tag{4.35}$$

Notice that

$$\operatorname{div}(h^2 fDf) = 2fhDh \cdot Df + h^2 |Df|^2 + h^2 f\Delta f, \tag{4.36}$$

so that eq. (4.35) becomes

$$|Dg|^2 = f^2 |Dh|^2 - h^2 f\Delta f + \operatorname{div}(h^2 fDf). \tag{4.37}$$

Because $h \in C_c^2(U)$, by the divergence theorem $\int_U \operatorname{div}(h^2 f Df) = 0$. Using this with $-\Delta f \geq \lambda f$ gives

$$\begin{aligned}\int_U |Dg|^2 &= \int_U f^2 |Dh|^2 - \int_U h^2 f \Delta f \\ &\geq \int_U f^2 |Dh|^2 + \lambda \int_U h^2 f^2 \\ &\geq \lambda \int_U g^2.\end{aligned} \tag{4.38}$$

By an approximation argument, we can take $g \in H_0^1(U)$. Dividing by $\int_U g^2$ and using the Rayleigh quotient for the first eigenvalue, we conclude that

$$\lambda_1(U) = \inf_{g \in H_0^1(U) - \{0\}} \frac{\int_U |Dg|^2}{\int_U g^2} \geq \lambda. \tag{4.39}$$

□

Corollary 4.6. *The first eigenvalues $\lambda_1(A_t), \lambda_1(D_{1-t}^\pm) \to \infty$ monotonically as $t \to 0$.*

The idea is that by lemma 4.5, it suffices to construct positive functions f converging to 0 on A_t and $D_t^\pm$ with $-\Delta f > 0$ bounded below to show that the first eigenvalues of these domains blow up.

Proof. Monotonicity follows from $\lambda_1(U_1) \geq \lambda_1(U_2)$ for any domains $U_1 \subset U_2$.

Any twice-differentiable function $f : S^n \to \mathbb{R}$ can be extended to $\tilde{f} : \mathbb{R}^{n+1} - \{0\}$ by $\tilde{f} : x \mapsto f(x\,|x|^{-1})$. Then $\Delta_{S^n} f = \Delta_{\mathbb{R}^{n+1}} \tilde{f}(x)$. By the chain rule,

$$\begin{aligned}\Delta_{S^n} f(x) &= \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} \left(f_{x_i}(x\,|x|^{-1}) \sum_{j=1}^{n+1} \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \\ &= \sum_{i=1}^{n+1} f_{x_i x_i}(x\,|x|^{-1}) \left(\frac{1}{|x|} - \sum_{j=1}^{n+1} \frac{x_i x_j}{|x|^3} \right) \\ &\quad - \sum_{i=1}^{n+1} f_{x_i}(x\,|x|^{-1}) \left(\frac{x_i}{|x|^3} + \sum_{j=1}^{n+1} \frac{x_j + x_i \delta_{ij}}{|x|^3} - \frac{3x_i^2 x_j}{|x|^5} \right)\end{aligned} \tag{4.40}$$

If f is given by $f : x \mapsto t^2 - x_1^2$, then eq. (4.40) becomes

$$\Delta_{S^n} f(x) = -2(1 - x_1 \sum_{j=1}^{n+1} x_j) + 2x_1 \left(2x_1 + (1 - 3x_1^2) \sum_{j=1}^{n+1} x_j \right) = -2 + Cx_1, \tag{4.41}$$

where $|C| < \infty$. As a result, on A_t we have $-\Delta_{S^n} f(x) \geq 1$ for t small enough, while $f \leq t^2$. By lemma 4.5, $\lambda_1(A_t) \to \infty$ as $t \to 0$. Similarly, taking f to be $f : (1-t)^2 - x_1^2$, we find that $\lambda_1(D_{1-t}^\pm) \to \infty$ as $t \to 0$. □

Recall that theorem 3.9 says that the minimizer is non-zero in the interior when $\epsilon^2 \lambda_1 < 1$ and zero otherwise. Fix $\epsilon > 0$ small enough so that the minimizer is non-zero on both $A_{\frac{1}{2}}$ and $D_{\frac{1}{2}}^\pm$. This is possible because the domains are fixed. Define $u_t \in C(S^n)$ by gluing the minimizers on A_t and $D_t^\pm$. By corollary 4.6 (including the monotonicity statement) and theorem 3.9, the minimizer in A_t is 0 for t small—say for $0 < t \leq t_1$—and the minimizer in $D_t^{\pm}$ is 0 for t large, say for $t_2 \leq t < 1$. Take t_1 as large as possible and t_2 as small as possible. Then the minimizer is non-zero on both A_t and $D_t^{\pm}$ if and only if $t_1 < t < t_2$. By our choice of ϵ, such t exist: $t_1 < \frac{1}{2} < t_2$, so $t_1 < t_2$.

We claim $C_{t_2}(A_{t_2}) > 0$, as otherwise continuity and eq. (4.33) would say $\int_{A_{t_2}} W'(u_{t_2}) \leq 0$, a contradiction with $u_{t_2} < 0$ in A_{t_2}. Similarly $C_{t_1}(D_{t_1}^{\pm}) < 0$. By continuity there is some $t_0 \in (t_1, t_2)$ with

$$Du_{t_0} \cdot \nu_{A_{t_0}}|_{\partial A_{t_0}} = C_{t_0}(A_{t_0}) = -C_{t_0}(D_{t_0}^{\pm}) = -Du_{t_0} \cdot \nu_{D_{t_0}^{\pm}}|_{\partial D_{t_0}^{\pm}}. \tag{4.42}$$

In particular, because $D_{t_0}^{\pm}$ and A_{t_0} share boundary (with opposite orientation), we conclude that the gradients of the minimizers coincide on ∂A_{t_0}. Thus u_{t_0} solves Allen-Cahn weakly on S^n, and by construction its nodal set is exactly $S^n \cap \{x_{n+1} = \pm t_0\}$.

Appendix A. Calculations for Allen-Cahn in One Dimension

This section provides proofs for theorem 4.1.

A.1. **Solving Allen-Cahn in Elliptic Functions.** We analyze u on a quarter-period. Taking $u(0) = 0$ and $u' \geq 0$ so that $u' = \sqrt{2W(t) - \lambda}$ and separating variables gives

$$t = \int_0^{u(t)} \frac{\mathrm{d}u}{\sqrt{2W(u) - \lambda}}. \tag{A.1}$$

Define $C_\lambda := \sqrt{1 - \sqrt{2\lambda}}$ the amplitude of u. Rearranging and substituting $v = C_\lambda^{-1} u$ gives

$$t = \sqrt{2} \int_0^{u(t)} \frac{\mathrm{d}u}{\sqrt{(1-u^2)^2 - 2\lambda}} = \sqrt{2} C_\lambda \int_0^x \frac{\mathrm{d}v}{\sqrt{(1 - C_\lambda^2 v^2) - 2\lambda}}, \tag{A.2}$$

where $x = \frac{u(t)}{C_\lambda}$ is between 0 and 1 (because u increases from 0 to C_λ). Cancel the C_λ out front.

$$\begin{aligned} t &= \sqrt{2} \int_0^x \frac{\mathrm{d}u}{\sqrt{(C_\lambda^{-1} - C_\lambda u^2)^2 - 2\lambda C_\lambda^{-2}}} \\ &= \sqrt{2} \int_0^x \left[\left(\frac{1}{\sqrt{1 - \sqrt{2\lambda}}} - \sqrt{1 - \sqrt{2\lambda}} u^2 \right)^2 - \frac{2\lambda}{1 - \sqrt{2\lambda}} \right]^{-\frac{1}{2}} \mathrm{d}u. \end{aligned} \tag{A.3}$$

Now factor the difference of squares $(1 - \sqrt{2\lambda})(1 + \sqrt{2\lambda}) = 1 - 2\lambda$.

(A.4)

$$\begin{aligned} t &= \sqrt{\frac{2}{1+\sqrt{2\lambda}}} \int_0^x \left[\left(\frac{1}{\sqrt{(1-\sqrt{2\lambda})(1+\sqrt{2\lambda})}} - \sqrt{\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}} u^2 \right)^2 - \frac{2\lambda}{(1-\sqrt{2\lambda})(1+\sqrt{2\lambda})} \right]^{-\frac{1}{2}} \mathrm{d}u \\ &= \sqrt{\frac{2}{1+\sqrt{2\lambda}}} \int_0^x \left[\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}} u^4 - \frac{2}{\sqrt{1-2\lambda}} \sqrt{\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}} u^2 + \frac{1}{1-2\lambda} - \frac{2\lambda}{1-2\lambda} \right]^{-\frac{1}{2}} \mathrm{d}u \end{aligned}$$

Factor the integrand.

$$\begin{aligned} t &= \sqrt{\frac{2}{1+\sqrt{2\lambda}}} \int_0^x \left[\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}} u^4 - \frac{2}{1+\sqrt{2\lambda}} u^2 + 1 \right]^{-\frac{1}{2}} \mathrm{d}u \\ &= \sqrt{\frac{2}{1+\sqrt{2\lambda}}} \int_0^x \left[(u^2 - 1) \left(\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}} u^2 - 1 \right) \right]^{-\frac{1}{2}} \mathrm{d}u, \end{aligned} \tag{A.5}$$

This is now in a well-known form:

$$t = \sqrt{\frac{2}{1+\sqrt{2\lambda}}} F\left(x, \frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}} \right), \tag{A.6}$$

where $F(x,k)$ is the *incomplete elliptic integral of the first kind*[5]

$$F(x,k) = \int_0^x \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-kt^2)}} \qquad 0 \le x \le 1, 0 \le k < 1. \tag{A.7}$$

The inverse to this function is known as the Jacobi elliptic function $\operatorname{sn}(x,k)$; namely, $\operatorname{sn}(F(x,k),k) = x$. Recalling $x = \frac{u(t)}{C_\lambda}$, we have

$$\begin{aligned} \sqrt{\frac{1+\sqrt{2\lambda}}{2}}t &= F\left(\frac{u(t)}{C_\lambda}, \frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}\right) \\ u(t) &= C_\lambda \operatorname{sn}\left(\sqrt{\frac{1+\sqrt{2\lambda}}{2}}t, \frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}\right). \end{aligned} \tag{A.8}$$

A.2. **Asymptotic on Period.** The goal of this section is to derive an asymptotic on T_λ as $\lambda \to 0$. Similar calculations to those above show that $\operatorname{sn}(x,k)$ has period $4K(k)$, where $K(k)$ is the *complete elliptic integral of the first kind*

$$K(k) = F(1,k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-kt^2)}}. \tag{A.9}$$

In our particular case, set $x = 1$ in eqs. (A.6) and (A.7) to find that the period of $u(t)$ is

$$4T_\lambda = 4\sqrt{\frac{2}{1+\sqrt{2\lambda}}}K\left(\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}}\right). \tag{A.10}$$

The main work is understanding $K(1-\epsilon)$ as $\epsilon \to 0$. In eq. (A.9), substitute $u = 1-(1-\epsilon)t^2, \mathrm{d}u = -2t(1-\epsilon)\,\mathrm{d}t = -2\sqrt{(1-\epsilon)(1-u)}$:

$$\begin{aligned} K(1-\epsilon) &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-(1-\epsilon)t^2)}} \\ &= \frac{1}{2}\int_\epsilon^1 \frac{1}{\sqrt{(1-u)(1-\epsilon)}} \frac{1}{\sqrt{u\frac{u-\epsilon}{1-\epsilon}}}\,\mathrm{d}u \\ &= O(1) + \frac{1}{2}\int_\epsilon^{\frac{1}{2}} \frac{\mathrm{d}u}{\sqrt{u(1-u)(u-\epsilon)}}. \end{aligned} \tag{A.11}$$

Recall the Taylor series expansion $(1+x)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} x^k$ (for $\alpha \in \mathbb{R}$ and $|x| < 1$), where $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ is a generalized binomial coefficient. Moreover,

[5]One place this integral comes up is in physics when analyzing the motion of a pendulum without using the small angle approximation $\sin\theta \approx \theta$. This application is apparent after making the substitution $\theta = \sin t$ to get $\int_0^{\arcsin x} \frac{\mathrm{d}\theta}{\sqrt{1-k\sin^2\theta}}$.

$\left|\binom{\alpha}{k}\right| \leq \frac{C}{k^{1+\alpha}}$ for $\alpha \notin \mathbb{Z}_{\geq 0}$.[6] Then

$$\begin{aligned}\int_{\epsilon}^{\frac{1}{2}} \frac{\mathrm{d}u}{\sqrt{u(1-u)(u-\epsilon)}} &= \int_{\epsilon}^{\frac{1}{2}} \frac{1}{u\sqrt{(1-u)}} \frac{1}{\sqrt{1-\frac{\epsilon}{u}}}\,\mathrm{d}u \\ &= \int_{\epsilon}^{\frac{1}{2}} \frac{1}{u\sqrt{(1-u)}} \left(\sum_{k=0}^{\infty}(-1)^k \binom{-\frac{1}{2}}{k}\epsilon^k u^{-k}\right)\mathrm{d}u \\ &= \int_{\epsilon}^{\frac{1}{2}} \frac{\mathrm{d}u}{u\sqrt{(1-u)}} + \sum_{k=1}^{\infty}(-1)^k \binom{-\frac{1}{2}}{k}\epsilon^k \int_{\epsilon}^{\frac{1}{2}} \frac{u^{-k-1}}{\sqrt{(1-u)}}\,\mathrm{d}u.\end{aligned} \tag{A.12}$$

We can bound the higher order terms as

$$\begin{aligned}\left|\sum_{k=1}^{\infty}(-1)^k \binom{-\frac{1}{2}}{k}\epsilon^k \int_{\epsilon}^{\frac{1}{2}} \frac{u^{-k-1}}{\sqrt{(1-u)}}\,\mathrm{d}u\right| &\leq \sum_{k=1}^{\infty} \frac{C}{\sqrt{k}}\epsilon^k 2 \int_{\epsilon}^{\frac{1}{2}} u^{-k-1}\,\mathrm{d}u \\ &= \sum_{k=1}^{\infty} \frac{C}{k\sqrt{k}}\epsilon^k \left[\epsilon^{-k} - 2^k\right] \\ &= O(1)\end{aligned} \tag{A.13}$$

as $\epsilon \to 0$. On the other hand, we can explicitly evaluate the main term in eq. (A.12) (which is unbounded),[7] so

$$K(1-\epsilon) = \frac{1}{2}\int_{\epsilon}^{\frac{1}{2}} \frac{\mathrm{d}u}{u\sqrt{1-u}} + O(1) = -\frac{1}{2}\log(1-\sqrt{1-\epsilon}) + O(1). \tag{A.14}$$

One can check that $\frac{\log(1-\sqrt{1-\epsilon})}{\log\epsilon} \to 1$ as $\epsilon \to 0$, and moreover that $\log(1-\sqrt{1-\epsilon}) - \log\epsilon \leq \frac{1}{2}$ near $\epsilon = 0$, so we conclude that

$$K(1-\epsilon) = -\frac{1}{2}\log\epsilon + O(1). \tag{A.15}$$

The main work is done now. To relate this to λ, recall eq. (A.10) and observe that $\frac{1-\sqrt{2\lambda}}{1+\sqrt{2\lambda}} = 1 - \frac{2\sqrt{2\lambda}}{1+\sqrt{2\lambda}}$. It is readily verified that $\frac{2\sqrt{2\lambda}}{1+\sqrt{2\lambda}} = 2\sqrt{2\lambda} + O(\lambda)$, and that $\sqrt{\frac{2}{1+\sqrt{2\lambda}}} = \sqrt{2} + O(\sqrt{\lambda})$ (as $\lambda \to 0$). Thus

$$\begin{aligned}4T_\lambda &= 4(\sqrt{2} + O(\sqrt{\lambda}))\Big(-\frac{1}{2}\log(2\sqrt{2\lambda} + O(\lambda)) + O(1)\Big) \\ &= -2\sqrt{2}\log\sqrt{\lambda} + O(\sqrt{\lambda}\log\sqrt{\lambda}) + O(1) \\ &= \sqrt{2}\,|\log\lambda| + O(1).\end{aligned} \tag{A.16}$$

[6]This is a consequence of Gauss's limit formula for the gamma function $\Gamma(\alpha) = \lim_{k\to\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k)}{k!k^\alpha}$, which holds where the gamma function does not have poles, namely for $\alpha \notin \mathbb{Z}_{\leq 0}$. Rearranging gives $\lim_{k\to\infty}\left|\binom{\alpha}{k}\Gamma(-\alpha)k^{1+\alpha}\right| = 1$, which implies $\left|\binom{\alpha}{k}\right| \leq \frac{C}{k^{1+\alpha}}$ for $\alpha \notin \mathbb{Z}_{\geq 0}$.

[7]The integrand has anti-derivative $\log\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}$.

Appendix B. Minimizers of Allen-Cahn Energy are Critical Points

We show that a minimizer u of the Allen-Cahn energy functional over $H_0^1(U)$ is a weak solution to Allen-Cahn, as needed in the proof of theorem 3.9. Fix $v \in C_c^\infty(U)$. Define

$$i[\tau] = E_\epsilon(u + \tau v) \quad (\tau \in \mathbb{R}).$$

$i[\tau]$ is finite for all $\tau \in \mathbb{R}$ because

$$\begin{aligned} i[\tau] &= E_\epsilon(u+\tau v) \\ &= \int_U \frac{\epsilon}{2}|Du + \tau Dv|^2 + \frac{1}{\epsilon}W(u+\tau v)\, d\mu_g \\ &\le C\int_U \left(|Du|^2 + |Dv|^2 + |Du|\,|Dv| + |u|^4 + |v|^4 + |u|^2|v|^2 + 1\right) d\mu_g \\ &\le C\int_U \left(|Du|^2 + |Dv|^2 + |u|^4 + |v|^4 + 1\right) d\mu_g < \infty \end{aligned}$$

because $v \in C_C^\infty(U)$, $u \in H_0^1(U)$, and

$$\int_U |u|^4 \le \int_U |W(u)| \le CE_\epsilon(u) < CE_\epsilon(0) < C|U| < \infty \tag{B.1}$$

as U is a finite domain. We went from the third inequality to the last inequality using Cauchy's inequality[8]. Now fix $\tau \neq 0$ and write the difference quotient

$$\begin{aligned} \frac{i[\tau]-i[0]}{\tau} &= \frac{E_\epsilon(u+\tau v) - E_\epsilon(u)}{\tau} \\ &= \int_U \frac{L(Du+\tau Dv, u+\tau v) - L(Du,u)}{\tau}\, d\mu_g \\ &= \int_U L^\tau(u,v)\, d\mu_g \end{aligned}$$

where

$$L(u, Du) = \frac{\epsilon}{2}|Du|^2 + \frac{1}{\epsilon}W(u)$$

and

$$L^\tau(u,v) = \frac{L(Du+\tau Dv, u+\tau v) - L(Du,u)}{\tau}.$$

Clearly, we have

$$\lim_{\tau\to 0} L^\tau(u,v) = g\left(DL(Du,u),(Dv,v)\right) = \epsilon g(Du,Dv) + \frac{1}{\epsilon}W'(u)v.$$

On the other hand,

$$\begin{aligned} L^\tau(u,v) &= \frac{\epsilon}{2}|Du+\tau Dv|^2 - |Du|^2 + \frac{1}{\epsilon}W(u+\tau v) - \frac{1}{\epsilon}W(u) \\ &\le C\left(|Du|^2 + |Dv|^2 + |u|^4 + |v|^4 + 1\right) < \infty \end{aligned}$$

by inequality B.1, $u \in H_0^1(U)$, $v \in C_c^\infty(U)$, Cauchy's inequality, and the finiteness of $|U|$. Thus, we apply Dominated convergence theorem and get that

$$\lim_{\tau\to 0}\int_U L^\tau(u,v)\, d\mu_g = \int_U \epsilon g(Du,Dv) + \frac{1}{\epsilon}W'(u)v\, d\mu_g = i'[0] = 0$$

[8] $a^2 + b^2 \ge 2ab$ for $a, b \in \mathbb{R}$.

since u is a minimizer of $E_\epsilon(\cdot)$, $E_\epsilon(u) = i[0]$ is a critical point of $i[\cdot]$. This holds for any $v \in C_c^\infty(U)$. Thus, by definition u is a weak solution to the Allen-Cahn (1.2). Once we know that u weakly solves the Allen-Cahn, we know in particular u satisfies corollary 2.6 and proposition 3.1.

Appendix C. Measure and Integration

C.1. **Measures and Measurable Functions.** In this section, we want to build some of the fundamental definitions and results about measures and measurable functions in order to lead our discussion to Lebesgue Integrals and eventually L^p and $W^{k,p}$ spaces. Although the topics in this section are not directly related to the main goal of this paper about constructing solutions to the Allen-Cahn equations, they create a good enough environment about the sets and functions for us to work with while retaining a lot of freedom for exploration.

C.1.1. *Algebras, σ-algebras.* As it turns out, we cannot define measure on any arbitrary sets. Thus, we need to restrict our family of sets to a "nice" enough one. This leads to the notion of algebras and σ-algebras.

Definition C.1. Let X be a set. An *algebra* is a collection $\mathcal{A}$ of subsets of X such that

(1) $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$.
(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
(3) If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then $\cup_{i=1}^n A_i \in \mathcal{A}$ and $\cap_{i=1}^n A_i \in \mathcal{A}$.

Additionally, $\mathcal{A}$ is a *σ-algebra* if the above three together with inclusion of countable unions and intersections:

(4) If $A_1, A_2, \dots \in \mathcal{A}$, then $\cup_{i=1}^\infty A_i \in \mathcal{A}$ and $\cap_{i=1}^\infty A_i \in \mathcal{A}$

are satisfied.

We call the ordered pair $(X, \mathcal{A})$ **measurable space**.

Example C.2. Let $X = \mathbb{R}$. Let $\mathcal{A}$ be the collection of all subsets of X, then $\mathcal{A}$ is a σ-algebra.

Example C.3. Let $X = [0, 1]$. Let $\mathcal{A} = \{[0, 1], \varnothing, [0, \frac{1}{2}], (\frac{1}{2}, 1]\}$. Then $\mathcal{A}$ is a σ-algebra.

Example C.4. If $\mathcal{A}_\alpha$ is a σ-algebra for each α element in the index set $\mathcal{I}$, then the set $\cap_{\alpha\in\mathcal{I}}\mathcal{A}_\alpha$ is also a σ-algebra.

With the above example C.4, let use define

$$\sigma(\mathcal{C}) = \cap\{\mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is } \sigma\text{-algebra}, \mathcal{C} \subset \mathcal{A}_\alpha\}$$

given $\mathcal{C}$ a collection of subsets of the set X. The intersection is taken over all family of subsets of X that is itself a σ-algebra and contains $\mathcal{C}$. In view of C.4, $\sigma(\mathcal{C})$ is a intersection of σ-algebras, thus itself is a σ-algebra. Notice that there exists at least one collections of subsets (the collection that contains all subsets of X) that satisfies the condition. Thus, we are not taking an intersection of empty sets. Moreover, we call $\sigma(\mathcal{C})$ the σ-algebra *generated by collection $\mathcal{C}$*. $\mathcal{C}$ is called the *generating collection.*

With these notaions, we define one of the most commonly used σ-algebra: Borel σ-algebra.

Proposition C.5. *Let $X = \mathbb{R}$, then the Borel σ-algebra $\mathcal{B}$ is generated by each of the following four generating collections:*

(1) $\mathcal{C} = \{(a, b) : (a, b) \subset \mathbb{R}\}$
(2) $\mathcal{C} = \{[a, b] : [a, b] \subset \mathbb{R}\}$
(3) $\mathcal{C} = \{(a, b] : (a, b] \subset \mathbb{R}\}$
(4) $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$.

C.1.2. *Measure.* Informally speaking, measure is a generalization of the notion of length in one dimension, area in two dimensions, and volume in three dimensions. As the readers can see in the definition of a measure, it includes some of the basic properties that appears in these analogies.

Definition C.6. Let X be a set and $\mathcal{A}$ a σ-algebra consists of subsets of X. Then a *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(1) $\mu(\varnothing) = 0$.
(2) *Countable additivity:* If $A_i \in \mathcal{A}, i = 1, 2, \ldots$ are pairwise (i.e. $A_k \cap A_k = \varnothing$ if $k \neq l$) disjoint, then
$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The ordered triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.

Example C.7. Let X be a set. $\mathcal{A}$ be the collection of all subsets of X, and $\mu(A)$ for $A \in \mathcal{A}$ counts the number of elements in the set A. Such μ is called the *counting measure.*

Example C.8. Let $\delta_x(A) = 1$ if $x \in A$ and 0 is not, then such a δ_x is called the *point mass* at x.

C.1.3. *Measurable Functions.* We next introduce measurable functions. Not surprisingly, since we restrict our domain to only the σ-algebras of sets, we need some criteria for our functions to make sure that their inverse image is not on some sets that is not inside the σ-algebras.

Definition C.9. Let $(X, \mathcal{A})$ be a measurable space. Then a function $f : X \to \mathbb{R}$ is *measurable or $\mathcal{A}$-measurable* if $\{x : f(x) > a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$.

Proposition C.10. *If X is a metric space, and $\mathcal{A}$ is the Borel σ-algebra of X. Then any continuous function $f : X \to \mathbb{R}$ is measurable*

Proof. Since f is continuous, for each $a \in \mathbb{R}$, $\{x : f(x) > a\} = f^{-1}((a, \infty))$, which is open, thus is contained in $\mathcal{A}$. □

Proposition C.11. *Let $c \in \mathbb{R}$. If f, g are measurable functions, then $-f, f + g, cf, \max(f, g), \min(f, g)$ are all measurable.*

All non-negative measurable functions can be well approximated by an increasing sequence of simple functions. This property is crucial and it is directly related to the definition of Lebesgue integrals.

Definition C.12. Let $(X, \mathcal{A})$ be a measurable space. If $E \in \mathcal{A}$, define the *indicator function of set E* to be $\chi_E : E \to \{0, 1\}$ with
$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

A *simple function* s is of the form

$$s_E(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

with each $a_i \in \mathbb{R}$ and measurable sets $E_i \in \mathcal{A}$.

Proposition C.13. *Suppose f is a measurable, non-negative function. Then there exists a sequence of non-negative simple functions increasing to f.*

Proof. Define

$$A_{in} = \left\{ x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}$$

for $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, n2^n$. Moreover, let

$$B_n = \{x : f(x) \geq n\}.$$

Then let the simple functions be

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{A_{in}} + \chi_{B_n}.$$

One can check that $\{s_n\}_{n=1}^{\infty}$ approaches f. □

C.2. **Lebesgue Integrals.** In this section we give the definition of Lebesgue integral and some of its properties. Since we are working with partial differential equations, we are going to frequently use integration. Thus, it is important to give the right definition here once. The readers can see through the definition of Lebesgue integral that unlike Riemann integral which is not defined for some functions that are quite useful (e.g. $\chi_{\mathbb{Q}}$). But as we will show later that the space of Lebesgue integrable functions form a complete space.

C.2.1. *Definition and Some Properties.* We first define Lebesgue integral for simple functions, which the readers can see that the definition is natural. Then we use fact C.13 to define the integral for all measurable functions.

Definition C.14. Let $(X, \mathcal{A}, \mu)$ be a measure space. If

$$s(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

is a non-negative measurable simple function, define its Lebesgue integral to be

$$\int s(x)\, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Here, if $a_i = 0$ and $\mu(A_i) = \infty$, then we use the convention that $a_i \mu(A_i) = 0$. If $f \geq 0$ is a measurable function, then define its Lebesgue integral to be

$$\int f\, d\mu = \sup \left\{ \int s\, d\mu : 0 \leq s \leq f, s \text{ simple} \right\}.$$

Let f be a measurable function. Define $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$. Then define

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu.$$

Definition C.15. If f is measurable and $\int |f|\, d\mu < \infty$, we say that f is *integrable*.

We next introduce some corollary properties of Lebesgue integrals that follows from the definition.

Proposition C.16. *(1) If f is a real-valued, measurable function with $0 \leq a \leq f \leq b$ for some constants $a, b \in \mathbb{R}$, then $a\mu(X) \leq \int f\,d\mu \leq b\mu(X)$.*

(2) If f, g integrable, real-valued, measurable, and $0 \leq g \leq f$, then $\int g\,d\mu \leq \int f\,d\mu$.

(3) If f is real-valued, integrable, non-negative, and $c \in \mathbb{R}$ is a non-negative constant, then $\int cf d\mu = c \int f\,d\mu$.

(4) If $\mu(A) = 0$ and f is non-negative and measurable, then $\int f\chi_A\,d\mu = 0$.

We also write $\int f\chi_A\,d\mu = \int_A f\,d\mu$. And when the measure used is obvious through the context, we will omit the $d\mu$ to write the integral as $\int f$. Moreover, $\int_a^b f\,d\mu = \int_{[a,b]} f\,d\mu$

C.2.2. *Two Theorems about Taking Limits.* We next introduce two theorems that take care of integration with limits involved. They are the *monotone convergence theorem* and the *dominated convergence theorem*. The first theorem can be used to prove linearity of Lebesgue integrals and the second theorem is used often to exchange integration with limits as the readers will see in later sections. We will not present to proofs of them. For the proofs please refer to Bass's *Real Analysis* [3] Chapter 7.

Theorem C.17 (Monotone Convergence Theorem)**.** *Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $0 \leq f_1 \leq f_2 \leq \cdots$ and with*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

for all x. Then $\int f_n\,d\mu \to \int f\,d\mu$.

Once we have monotone convergence theorem, we can prove that Lebesgue integrals are linear.

Proposition C.18. *If f, g are non-negative and measurable, or if f or g are integrable, then*

$$\int f + g\,d\mu = \int f\,d\mu + \int g\,d\mu.$$

Remark C.19. With linearity of Lebesgue integrals, we can extend proposition C.16 to all measurable functions by writing $f = f^+ - f^-$.

We also have this important inequality when coming to estimating integrals.

Proposition C.20. *If f is integrable, then*

$$\left|\int f\,d\mu\right| \leq \int |f|\,d\mu.$$

We next state the dominated convergence theorem, which is used more often in our setting.

Theorem C.21 (Dominated Convergence Theorem)**.** *Suppose f is measurable and real-valued and $f_n(x) \to f(x)$ for each x. Suppose there exists g, a non-negative integrable function such that $|f_n(x)| \leq g_n(x)$ for each x. Then*

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu.$$

Remark C.22. For theorems C.17 and C.21, the conclusions still hold if we change the point wise convergence condition of $f_n(x) \to f$ to an almost everywhere convergence (i.e. convergent to f except on a measure zero set). This is very important because when talking about L^p and $W^{k,p}$ spaces, everything is identified almost everywhere and almost no point wise information can be drawn.

Appendix D. Banach and Hilbert Spaces

Before diving into stating the definition of L^p spaces, we need to define what is a norm, inner product and how do these relates to the definition of Banach and Hilbert spaces.

D.1. **Norm and Banach Space.** In words, Banach spaces are normed linear spaces that are complete. Linear spaces are equivalent to vector spaces. We will omit the definition of that in this paper. We will start with defining what is a norm.

Definition D.1. Let X be a linear space over $\mathbb{R}$. Then X is a *normed linear space* if there exists a map $x \mapsto \|x\|$ such that

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|cx\| = |c| \|x\|$ for all $x \in X$ and $c \in \mathbb{R}$.
(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Remark D.2. Given a normed linear space X, X is a metric space with the metric $d(x, y) = \|x - y\|$ for all $x, y \in X$.

Example D.3. In $\mathbb{R}^n$, we have the sup norm which is defined by given $a = (a_1, \ldots, a_n)$, then $\|a\|_{sup} = \sup_{1\leq i\leq n} |a_i|$. We also have the L^2 norm which is defined by $\|a\|_2 = \sqrt{\sum_{i=1}^{n} a_i^2}$. The metric induced by the L^2 norm is our standard Euclidean distance function.

Definition D.4. A *Banach space* is a normed linear space such that, with respect to the induced metric $\|x - y\|$, it is complete, meaning that every Cauchy sequence converges.

Example D.5. The normed linear space $\mathbb{R}^n$ with respect to either of the two norms in D.3 are complete. Thus, both of them are Banach spaces.

Example D.6. Let $C([0, 1])$ be space of continuous function with domain being $[0, 1]$. One can check that it is a linear space over $\mathbb{R}$. Furthermore, it has the L^2 norm of $\|f\|_2 = (\int_{[0,1]} |f|^2 \, dx)^{\frac{1}{2}}$. However, this normed linear space $C([0, 1])$ is not complete with respect to the induced metric, thus not a Banach space. On the other hand, we can also have the sup norm of $\|f\|_{sup} = \sup_{x\in[0,1]} |f(x)|$ on $C([0, 1])$. The normed linear space under this norm is complete, thus a Banach space.

D.2. **Inner Product and Hilbert Space.** Another useful space is Hilbert space. A Hilbert space is a complete normed linear space that have an inner product. The additional structure on this space allows the existence of many useful theorems. In particular, as the readers will see in later section, the Riesz Representation theorem is very important when talking about the existence of certain PDEs.

We first define inner product over a linear space.

Definition D.7. Let H be a vector space over $\mathbb{R}$. H is a *inner product space* if there exists a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ such that

(1) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in H$.
(2) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in H$.
(3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in H$ and $\alpha \in \mathbb{R}$.
(4) $\langle x, x\rangle \geq 0$ for all $x \in H$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

With such an inner product, we can use it and induce a norm defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in H$.

We have the following important inequality, called the *Cauchy-Schwarz Inequality*.

Proposition D.8. *For all $x, y \in H$, we have*

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| .$$

With the Cauchy-Schwarz inequality, we can prove that the induced norm actually satisfies the definition of norm. In particular, the readers need to check that $\|x + y\| \leq \|x\| + \|y\|$, which follows from the Cauchy-Schwarz.

We give the definition of Hilbert spaces

Definition D.9. A *Hilbert space* H is an inner product space that is complete with respect to the induced metric $d(x, y) = \|x - y\|$.

Example D.10. $\mathbb{R}^n$ with inner product being the dot product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ given $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is a Hilbert space.

Appendix E. Function Spaces: L^p, Sobolev, and Hölder Spaces

E.1. L^p **Spaces.** In this section we introduce the space of Lebesgue integrable functions L^p spaces. These L^p spaces will be the place we construct solutions to the Allen-Cahn equation. The reason we like to work in these spaces is because that they satisfy a lot of nice properties as we will introduce in this section. In particular, L^p spaces are Banach spaces and L^2 is additionally a Hilbert space. Moreover, it can be shown that all functions in these spaces can be approximated by smooth functions. These properties allow us to argue for the existence, and smoothness of solutions to not only Allen-Cahn, but also solutions to various kinds of PDEs.

E.1.1. *Two Inequalities and L^p Completeness.* In this section we give the definition of L^p spaces and its associated norms. We then state Hölder's inequality and Minkowski's inequality that are necessary for later results.

Definition E.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. For $1 \leq p < \infty$. The space $L^p(X)$ consists of equivalence classes of measurable functions $f : X \to \mathbb{R}$ with

$$\int |f|^p \, d\mu < \infty,$$

where two measurable functions are equivalent if they are equivalent almost everywhere. The L^p norm is defined by

$$\|f\|_{L^p(X)} = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} .$$

Moreover, for $p = \infty$, $L^\infty(X)$ consists of all the functions such that its essential supremum:

$$\operatorname{ess\,sup}_X f = \inf\{M \geq 0 : \mu(\{x : |f(x)| \geq M\}) = 0\}$$

is finite. The essential supremum is also the norm for L^∞ spaces.

Remark E.2. We will write from now on that $\operatorname{ess\,sup} f$ as $\sup f$. This should not give rise to any confusion because we rarely use $\sup f$ when talking about functions. We say that $f_n \to f$ in $L^p(X)$ to mean that $\|f_n - f\|_{L^p(X)} \to 0$. The reason that we identify functions within equivalence classes is to make the L^p norm actually a norm (i.e. it satisfies the conditions in D.1). For example, we have $\|\chi_{\mathbb{Q}}\|_{L^p(\mathbb{R})} = 0$ and $\chi_{\mathbb{Q}} \neq 0$, but $\chi_{\mathbb{Q}} = 0$ almost everywhere. Therefore we must identify functions within the same equivalence class.

We next introduce Hölder's inequality, which is of extreme importance when working with L^p spaces.

Proposition E.3 (Hölder's Inequality)**.** *If $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, and if f, g are measurable, then*

$$\int |fg| \, d\mu \leq \|f\|_{L^p} \|g\|_{L^p}.$$

This holds for $p = 1$, $q = \infty$ or vice versa.

Proof. If $p = 1$ and $q = \infty$, then we have $|g(x)| \leq \|g\|_{L^\infty(X)} = M$ for all $x \in X$. Thus

$$\int |fg| \, d\mu \leq M \int |f| \, d\mu = M \|f\|_{L^1(X)}$$

as desired. Thus, we assume $1 < p, q < \infty$. If either $\|f\|_{L^p}$ or $\|g\|_{L^q}$ equal to zero, then we have f or g equal to zero a.e. Thus, $\int |fg| \, d\mu = 0$, which satisfies the inequality. So assume f, g both non zero. On the other hand, if one of f or g has infinite norm, then the inequality automatically holds. Thus, it suffices to consider only the case where both of their norm are non zero and finite.

Let $F(x) = \frac{|f(x)|}{\|f\|_{L^p}}$ and $G(x) = \frac{|g(x)|}{\|g\|_{L^q}}$. Note that $\|F\|_{L^p}, \|G\|_{L^q} = 1$, and it suffices to show that $\int FG \, d\mu \leq 1$. By Young's inequality for product[9], we obtain that

$$FG \leq \frac{F^p}{p} + \frac{G^q}{q}.$$

Integrating both sides and we get that

$$\int FG \, d\mu \leq \frac{\|F\|_{L^p}^p}{p} + \frac{\|G\|_{L^q}^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

as desired. ☐

Next we present the *Minkowski's inequality*, which acts as the triangular inequality in the L^p spaces.

Proposition E.4. *If $1 \leq p < \infty$, and f, g are measurable functions, then*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^q}.$$

With the L^p norms for $1 \leq p \leq \infty$, the L^p space is in fact complete.

Theorem E.5. *If $1 \leq p \leq \infty$, the L^p spaces are Banach spaces.*

Theorem E.6. *In addition, L^2 with the inner product*

$$\langle f, g \rangle = \int fg \, d\mu$$

is a Hilbert space if the domain is any subsets of $\mathbb{R}$.

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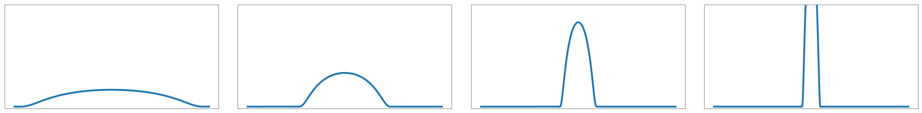
[9]If $a, b \geq 0, p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{a^p}{p} + \frac{a^q}{q} \geq ab$.



(A) $\epsilon = 1$ (B) $\epsilon = 0.5$ (C) $\epsilon = 0.2$ (D) $\epsilon = 0.1$

FIGURE 7. Standard mollifier φ_ϵ on $[-1, 1]$.

E.1.2. *Convolution and Mollification.* Convolution is a way to average a given function against another function. It is useful in a process called mollification, which can be used to prove that the set of smooth functions with compact support is dense in L^p, for $1 \leq p \leq \infty$.

Definition E.7. The *convolution* of two measurable functions f and g is defined by

$$f * g(x) = \int f(x-y)g(y)\,dy$$

provided the integral exists.

Remark E.8. The convolutions of two measurable functions are always measurable. Moreover, by change of variables, $f * g = g * f$.

One application of convolution is to approximate functions in L^p spaces with smooth functions with compact support (i.e. the function is equal to zero outside of a compact set). This process is called *mollification.*

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support in $\mathbb{R}^n$, non-negative, and $\int_{\mathbb{R}^n} \varphi\, dx = 1$. One example of such function is

$$\varphi(x) = \begin{cases} \frac{\exp\left(\frac{-1}{1-\|x\|^2}\right)}{I_n} & \|x\| < 1 \\ 0 & \|x\| \geq 1. \end{cases}$$

where I_n is $\int_{\{\|x\|<1\}} \exp\left(\frac{-1}{1-\|x\|^2}\right) dx$ to ensure φ integrates to one.

Theorem E.9. *Let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. Suppose $1 \leq p \leq \infty$ and $f \in L^p$. Define $f_\epsilon = f * \varphi_\epsilon$. Then*

(1) For each $\epsilon > 0$, $f * \varphi \in C^\infty$ (i.e.infinitely differentiable) and for each multiindex [10]*. $\alpha = (\alpha_1, \ldots, \alpha_n)$,*

$$D^\alpha f_\epsilon = f * D^\alpha \varphi_\epsilon.$$

We use the convention that 0-th derivative is the function f_ϵ itself.

(2) $f_\epsilon \to f$ almost everywhere as $\epsilon \to 0$.

(3) If f is continuous, then $f_\epsilon \to f$ uniformly on compact sets as $\epsilon \to 0$.

(4) If $1 \leq p < \infty$ and $f \in L^p$, we have $f_\epsilon \to f$ in L^p.

Proof. Please see Bass's Real Analysis Theorem 15.8 (linked above). □

[10] A multiindex α is a vector $\alpha = (a_1, a_2, \cdots, a_n)$ with $a_i \in \mathbb{N}$ and $D^\alpha f$ means $\frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_1^{a_n}} f$, i.e. taking partial derivative in the x_i direction a_i times. $|\alpha|$ means the order of the derivative, i.e. $|\alpha| = a_1 + \cdots + a_n$

E.2. **Sobolev Spaces.** In order to study PDEs, we hope to work with functions whose partial derivatives exist. However, it is often times too constraining to consider only differentiable functions. The introduction of Sobolev spaces and weak derivatives reconcile this problem.

E.2.1. *Weak Derivatives.* The notion of weak derivatives, while not requiring any smoothness property, captures an important feature of classical derivatives, which is integration by parts.

Definition E.10. Let α be a multiindex. Let f be a locally integrable function defined on an open domain $U \subseteq \mathbb{R}^n$. We say g is the α-th order weak derivative of f in the x_i direction if for any smooth function φ with compact support in U,

$$\int_U f D^\alpha \varphi = (-1)^{|\alpha|} \int_U g\varphi.$$

We write

$$g = D^\alpha f.$$

In other words, the weak derivative g is the function which acts like a classical derivative when integrating by parts. There is no boundary term since the functions φ have compact support. It follows from the definition that the weak derivative is unique up to a set of measure zero.

Example E.11. Let $f(x) = |x|$ be defined on $(-1, 1)$. We show that

$$g(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 \leq x < 1 \end{cases}$$

is the first order derivative of f. Indeed, for any compactly support smooth function φ,

$$\begin{aligned} \int_{-1}^{1} f(x)\varphi'(x)dx &= \int_{-1}^{0} -x\varphi'(x)dx + \int_{0}^{1} x\varphi'(x)dx \\ &= \int_{-1}^{0} \varphi(x)dx + \int_{0}^{1} -\varphi(x)dx = -\int_{-1}^{1} g(x)\varphi(x). \end{aligned}$$

In fact, the value of g at zero is irrelevant.

E.2.2. $W^{k,p}$ *Spaces.*

Definition E.12. Suppose $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(U)$ is the space of functions whose α-th weak derivatives exist and are in $L^p(U)$ for all $|\alpha| \leq k$.

$W^{k,p}$ spaces are Banach spaces under the following norm:

$$\|f\|_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(U)}$$

This is indeed a norm if we identify functions that agree almost everywhere. To see this,

$$\begin{aligned} \|f + g\|_{W^{k,p}(U)} &= \sum_{|\alpha| \leq k} \|D^\alpha f + D^\alpha g\|_{L^p(U)} \\ &\leq \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(U)} + \|D^\alpha g\|_{L^p(U)} \\ &= \|f\|_{W^{k,p}(U)} + \|g\|_{W^{k,p}(U)} \end{aligned}$$

The inequality is true since the L^p norm is a norm.

To see $W^{k,p}(U)$ is complete, let $f_m \in W^{k,p}(U)$ be a Cauchy sequence. Since for each $|\alpha| \leq k$,

$$\|D^\alpha f_m - D^\alpha f_n\|_{L^p(U)} \leq \|D^\alpha f_m - D^\alpha f_n\|_{W^{k,p}(U)},$$

it follows that $D^\alpha f_m$ is a Cauchy sequence in $L^p(U)$. We have seen that $L^p(U)$ is complete, so let f_α be the limit of $D^\alpha f_m$ in $L^p(U)$, and in particular let f be the limit of f_m in $L^p(U)$. It now suffices to show that $f_\alpha = D^\alpha f$ for every $|\alpha| \leq k$. That is, for any smooth φ with compact support in U, we want to show that

$$\int_U f_\alpha \varphi = \int_U f D^\alpha \varphi.$$

First observe that

$$\Big| \int_U D^\alpha f_m \varphi - \int_U f_\alpha \varphi \Big| \leq \int_U |D^\alpha f_m - f_\alpha||\varphi| \leq \|D^\alpha f_m - f_\alpha\|_{L^p(U)} \|\varphi\|_{L^q(U)}$$

by Hölder's inequality, for $\frac{1}{p} + \frac{1}{q} = 1$. Since φ has compact support, its L^q norm is finite. Therefore

$$\lim_{m\to\infty} \int_U D^\alpha f_m \varphi = \int_U f_\alpha \varphi.$$

Hence

$$\begin{aligned}\int_U f_\alpha \varphi &= \lim_{m\to\infty} \int_U D^\alpha f_m \varphi \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U f_m D^\alpha \varphi \\ &= (-1)^{|\alpha|} \int_U f D^\alpha \varphi,\end{aligned}$$

where we used the above observation in the first and last equality, and the definition of weak derivative in the second equality. We conclude that $W^{k,p}(U)$ is complete under this norm.

Similar to L^2, $W^{k,2}$ is a Hilbert space, and we will denote $W^{k,2}$ by H^k to emphasize this fact.

Example E.13. Although $W^{k,p}$ is a Banach space, functions in it can still behave badly: discontinuous, unbounded, etc. Consider the unit ball $B_1(0) \subset \mathbb{R}^n$. Let $\{r_k\}$ be a sequence of countable points that is dense in $B_1(0)$, for example $\mathbb{Q}^n \cap B_1(0)$. Then the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^a$$

is in $W^{1,p}(B_1(0))$ for $a < \frac{n-p}{p}$ (in particular, a can be less than 1). It is discontinuous on the set $\{r_k\}$ and is unbounded in any open set within $B_1(0)$. (On the points r_k, the formula is undefined, but this is allowed since we consider functions up to redefining on a set of measure zero, so we could assign arbitrary values on the points r_k.) For details of why it is in $W^{1,p}(B_1(0))$, see page 260 in Lawrence C. Evans's *Partial Differential Equations*.

E.2.3. *Trace.* In studying PDEs, boundary conditions play an important role. However, the boundary of a region in $\mathbb{R}^n$ is of measure zero, and as illustrated in the example above, functions in $W^{1,p}$ spaces can be discontinuous, and their values on a set of measure zero can be defined arbitrarily. It is therefore unclear what boundary value means. The notion of a *trace operator* is introduced for this issue. For proofs of the following theorems, see Lawrence C. Evans's *Partial Differential Equations*.

Theorem E.14. *Let U be a bounded open domain with C^1 boundary. There exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\bar{U})$ such that*

$$\|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}}$$

for a constant C depending only on p and U. Moreover, $Tu = u|_{\partial U}$ for $u \in W^{1,p} \cap C(\bar{U})$.

In other words, the trace operator does nothing to functions whose boundary values are already defined.

Theorem E.15. *Let U be the same as above. Suppose $u \in W^{1,p}(U)$. Then $Tu = 0$ if and only if $u \in \overline{C_c^\infty(U)}$* [11].

This is saying that a function $u \in W^{1,p}$ is 0 on the boundary if and only if it is a limit of compactly supported smooth functions. This is particularly useful since in circumstances when the boundary condition of a PDE is zero. This is called the *Dirichlet boundary condition of* 0.

E.3. **Hölder Spaces.** We introduce yet another family of spaces of functions, which will be useful when proving solutions to PDEs are smooth/regular.

Definition E.16. Let $U \in \mathbb{R}^n$ be an open domain. Let $\alpha \in (0, 1]$, and let f be a bounded continuous function. The *Hölder seminorm with exponent* α is

$$[f]_{C^{0,\alpha}(\bar{U})} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

The *Hölder norm* is

$$\|f\|_{C^{0,\alpha}(\bar{U})} = \|f\|_{C(\bar{U})} + [f]_{C^{0,\alpha}(\bar{U})}.$$

[12] Now the *Hölder space* $C^{0,\alpha}(\bar{U})$ is the set of functions with finite Hölder norm.

Definition E.17. Let U and α be same as above. Let $f \in C^k(\bar{U})$. The $C^{k,\alpha}$ norm of f is

$$\|f\|_{C^{k,\alpha}(\bar{U})} = \sum_{|j| \leq k} \left\|D^j f\right\|_{C(\bar{U})} + \sum_{|j|=k} [D^j f]_{C^{0,\alpha}(\bar{U})}.$$

The *Hölder space* $C^{k,\alpha}(\bar{U})$ is the set of functions with finite $C^{k,\alpha}$ norm.

The fact that these Hölder norms are indeed norms follow from simple applications of triangle inequalities. The Hölder spaces are also Banach spaces. This follows from the fact that a Cauchy sequence in $C^{0,\alpha}(\bar{U})$ is also a Cauchy sequence in $C(U)$, and thus converges uniformly. One can check that the limit is indeed in $C^{0,\alpha}$.

[11] The notation $C_c^\infty(U)$ denotes the set of smooth functions defined on U with compact support. C^∞ means infinitely differentiable, and the subscript c means compact support.

[12] The $\|\cdot\|_{C(\bar{U})}$ norm is the usual sup norm.

Proposition E.18. *The Hölder spaces $C^{k,\alpha}(\bar{U})$ are closed under multiplication.*

Proof. First let $k=0$. Let $f,g \in C^{0,\alpha}(\bar{U})$. We compute

$$\begin{aligned}
[fg]_{C^{0,\alpha}(\bar{U})} &= \sup_{x\neq y} \frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^\alpha} \\
&= \sup_{x\neq y} \frac{|f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)|}{|x-y|^\alpha} \\
&= \sup_{x\neq y} \frac{|f(x)(g(x)-g(y))+(f(x)-f(y))g(y)|}{|x-y|^\alpha} \\
&\leq \|f\|_{C(\bar{U})}\,[g]_{C^{0,\alpha}(\bar{U})} + \|g\|_{C(\bar{U})}\,[f]_{C^{0,\alpha}(\bar{U})} \\
&\leq \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|fg\|_{C^{0,\alpha}(\bar{U})} &= \|fg\|_{C(\bar{U})} + [fg]_{C^{0,\alpha}(\bar{U})} \\
&\leq \|f\|_{C(\bar{U})}\,\|g\|_{C(\bar{U})} + [fg]_{C^{0,\alpha}(\bar{U})} \leq \infty.
\end{aligned}$$

Now use induction and product rule of derivatives to conclude that $C^{k,\alpha}(\bar{U})$ is also closed under multiplication. ☐

References

[1] O. Chodosh, "Lecture notes on geometric features of the allen–cahn equation (princeton, 2019).".

[2] M. A. M. Guaraco, "Min-max for the allen-cahn equation and other topics (princeton, 2019).".

[3] R. Bass, *Real Analysis for Graduate Students*. Createspace Ind Pub, 2013.

[4] L. Evans, *Partial Differential Equations*. Graduate studies in mathematics, American Mathematical Society, 1998.

[5] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2013.

[6] N. Ghoussoub and C. Gui, "On a conjecture of de giorgi and some related problems," *Mathematische Annalen*, vol. 311, no. 3, pp. 481–491, 1998.

[7] S. Kosugi, Y. Morita, and S. Yotsutani, "Stationary solutions to the one-dimensional cahn-hilliard equation: Proof by the complete elliptic integrals," *Discrete and Continuous Dynamical Systems*, vol. 19, pp. 609–629, 2007.

[8] A. Kasue, "On a lower bound for the first eigenvalue of the laplace operator on a riemannian manifold," *Annales scientifiques de l'École Normale Supérieure*, vol. 17, no. 1, pp. 31–44, 1984.