Riemann's Existence Theorem and Moduli of Curves

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Abstract

We study Riemann's Existence Theorem (RET) regarding branched covers of $\mathbb{P}^1_{\mathbb{C}}$ realizing a given monodromy representation. We use this to expand on a known proof of Belyi's theorem, which describes covers over $\mathbb{P}^1_{\mathbb{C}}$ branched over three points as precisely those admitting a model over $\overline{\mathbb{Q}}$. We also describe a possible inductive strategy to algebraically prove RET by gluing together simple branched covers of the sphere at a node, and then resolving the singularity.

Monodromy Representation

Our main object of study are branched covers $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ of Riemann surfaces. Away from some finite $\Delta \in \mathbb{P}^1_{\mathbb{C}}$, we get a topological covering $\varphi\colon X \setminus \varphi^{-1}(\Delta) \to \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$. Since $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, x) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n \rangle$ acts on the fiber $\varphi^{-1}(x) = \{x_1, \ldots, x_d\}$ over x by path lifting, we get a homomorphism

$$\rho\colon \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, x) \to \Sigma_d$$

This is called the *monodromy representation* of $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$.

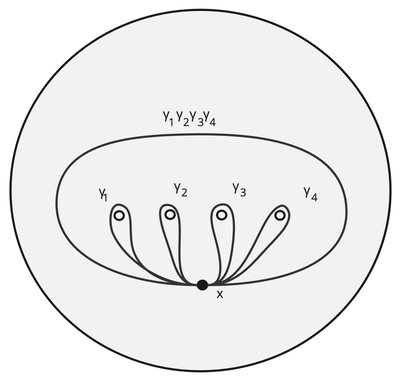

Figure 1. Fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, x) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n \rangle$

Riemann's Existence Theorem

Let $\rho\colon \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, x) \to \Sigma_d$ be a transitive monodromy representation as above. Then the following holds.

Theorem 1. (*Riemann's Existence Theorem*) There is a connected Riemann surface X and branched cover $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ with monodromy representation given by ρ. Further, if ρ, ρ' are conjugate monodromies corresponding to covers φ, φ' on X, X', then there is an isomorphism $p\colon X \to X'$ such that $\varphi = \varphi' \circ p$.

We show this by first constructed a topological covering space $\varphi\colon X_0 \to \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ corresponding to ρ, as for a topological space T we have an equivalence of categories

$$\mathrm{Cov}(T) \xrightarrow{\sim} \mathrm{Sets}^{\pi_1(T,t)}$$

given by $(p\colon U \to T) \mapsto p^{-1}(t)$. Locally, near $x_i \in \Delta$, φ looks like $\sqcup_{j=1}^{d_i} \mathbb{D}^*_j \to \mathbb{D}^*$ given by $z \mapsto z^{n_j}$ where $d_i =$ number of cycles in $\rho(\gamma_i)$ and $n_j =$ length of jth cycle. We can fill in $z = 0$ in each disk to get a map $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$, which induces a Riemann surface structure on X.

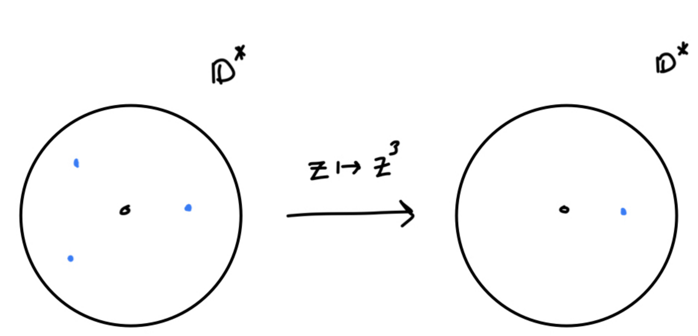

Figure 2. Locally, φ is a map of punctured disks. For example, $z \mapsto z^3$ as above.

Belyi's Theorem

Suppose a cover $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ is branched only over $0, 1, \infty$. What can we say about X?

Theorem 2. (*Belyi*) There is a morphism $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ branched only over $0, 1, \infty$ if and only if X is defined over $\overline{\mathbb{Q}}$. That is, there is a curve X_0 over $\overline{\mathbb{Q}}$ and a morphism $\varphi_0\colon X_0 \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that the following is a pullback diagram.

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\ \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & \mathbb{P}^1_{\overline{\mathbb{Q}}} \end{array}$$

Outline of $\implies$:

1. If X is cut out globally by $F_1, \ldots, F_r$, there exists an affine $\overline{\mathbb{Q}}$-variety V such that $\mathfrak{X} = Z(F_1, \ldots, F_r) \subset \mathbb{P}^1_V = \mathbb{P}^1_{\mathbb{C}} \times_{\mathrm{Spec}\overline{\mathbb{Q}}} V$ and we have a fiber diagram
$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & \mathbb{P}^1_V \end{array}$$
2. The singular locus $\mathrm{Sing}(\mathfrak{X}/V)$ is a closed subset of $\mathfrak{X}$, so its image in V is closed. Since the fiber over a point in V is X which is smooth, this is a proper subset so we can remove it. Hence the fibers of Φ are smooth curves.
3. Similarly, the locus $R = \mathrm{Supp}(\Omega_{\mathfrak{X}/\mathbb{P}^1_V}) \cap \Phi^{-1}(\mathbb{P}^1_V \setminus \{0, 1, \infty\} \times V)$ is the closed set of points in $\mathfrak{X}$ not lying over $0, 1, \infty$ that ramify. Again, since X is not in this locus its image in V is a proper closed subset, so we can remove it. Hence the fibers of Φ are covers that ramify only over $0, 1, \infty$.
4. The fibers are smooth covers $\varphi_t\colon X_t \to \mathbb{P}^1_{\mathbb{C}}$ with branch locus $0, 1, \infty$. Any two covers have conjugate monodromies since $\mathfrak{X}$ is connected, hence they are isomorphic by Riemann's existence theorem. One fiber is $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$, and the fiber over a $\overline{\mathbb{Q}}$-point is a cover $\varphi_0\colon X_0 \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ isomorphic to φ.

Dessin's D'Enfants

Consider Belyi morphisms $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ of degree d as above, branched over $0, 1, \infty$. The lift of $[0, 1]$ to X gives a bipartite graph with d edges, where the vertices are the fibers over 0 and 1. This structure is called a *Dessin D'Enfant*, or child's drawing.

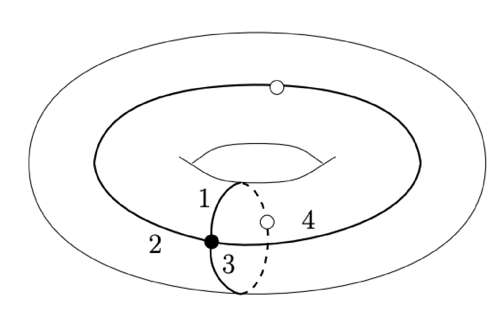

Figure 3. A dessin on a torus, where white vertices lie over 0 and black vertices lie over 1

Conversely, dessins on X give Belyi morphisms. The cycle description of edges incident to a vertex determines a monodromy $\rho\colon \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \Sigma_d$, which by RET determines a Belyi morphism $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$. In Figure 2 above, the image of γ_0 would be the permutation $(1\ 3)(2\ 4)$ and γ_1 would be $(1\ 2\ 3\ 4)$.

Moduli Stack of Curves

Let $\mathcal{M}_g$ be the category of proper, flat morphisms $C \to T$ whose geometric fibers are smooth, connected, genus g curves.

Theorem 3. $\mathcal{M}_g$ is a Deligne-Mumford stack, i.e. there is a scheme U and a surjective étale morphism $U \to \mathcal{M}_g$, and further $\mathcal{M}_g$ is irreducible. [DM69]

Deligne-Mumford proved this using the stack quotient $[H_g/\mathrm{PGL}(5g-6)]$, but there is another proof using Hurwitz schemes.

Isomorphism classes of infinitesimal deformations of C correspond to sections of $H^1(C, T_C)$. For $g \geq 2$, $\deg T_C = 2 - 2g < 0$ so $h^0(C, T_C) = 0$.

$$h^0(C, T_C) - h^1(C, T_C) = 2 - 2g + 1 - g$$

so $\dim \mathcal{M}_g = h^1(C, T_C) = 3g - 3$.

Hurwitz Schemes

Let $\mathcal{H}^{d,b}$ be degree d simple branched covers of $\mathbb{P}^1_{\mathbb{C}}$ with b branch points. The map $\mathcal{H}^{d,b} \to \mathbb{P}^{(b)}_{\mathbb{C}} \setminus \Delta_b$ sending $\varphi \mapsto \mathrm{Branch}(\varphi)$ is unramified, and hence $\mathcal{H}^{d,b}$ is naturally a smooth quasiprojective variety of dimension b.

Theorem 4. $\mathcal{H}^{d,b}$ is connected, and therefore irreducible. [HM98]

We have a natural morphism for $b = 2d + 2g - 2$ sending $(X \to \mathbb{P}^1_{\mathbb{C}}) \mapsto X$.

$$F\colon \mathcal{H}^{d,b} \to \mathcal{M}_g$$

Theorem 5. For $d \geq g + 1$, the map $F\colon \mathcal{H}^{d,b} \to \mathcal{M}_g$ is surjective. [HM98]

Suppose $\varphi\colon C \to \mathbb{P}^1_{\mathbb{C}}$ is degree d, $\mathcal{L} = \varphi^*\mathcal{O}(1)$. Then $h^0(C, \mathcal{L}) \geq 2$ since we can pull back two sections on $\mathbb{P}^1_{\mathbb{C}}$, so by Riemann-Roch

$$2 \leq h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = d + 1 - g$$

Hence, $d \geq g + 1$ is necessary and sufficient for this map to be surjective. Since $\mathcal{H}^{d,b}$ is irreducible, so is $\mathcal{M}_g$.

Acknowledgements

I would like to thank my advisor, Professor Mohan Swaminathan, for his time and invaluable support throughout the course of this project. I would also like to thank Dr. Lernik Asserian for organizing SURIM and helping to navigate the research process.

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