BOTT PERIODICITY AND AFFINE GRASSMANNIANS

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ABSTRACT. We give an algebro-geometric proof to the Bott periodicity in the unitary case via the geometry of affine Grassmannians, following the same strategy as Pressley-Segal [Pre86]. This article is a report of partial results of an ongoing project under SURIM (Stanford Undergraduate Research Institute in Mathematics).

1. INTRODUCTION

Recall that $\mathrm{U}(n)$ is the group of n by n complex unitary matrices. Define the *unitary group* U as $\mathrm{U} := \operatorname{colim} \mathrm{U}(n)$ where the transition maps are natural inclusions. Similarly one defines the *orthogonal group* O and *symplectic group* Sp as the colimit of finite orthogonal groups and symplectic groups. The celebrated Bott periodicity theorem states that the homotopy groups of U, O, and Sp are periodic.

Theorem 1.1 ([Bot57, Theorem], [Bot59, Corollary])**.** *The stable homotopy of the classical groups is periodic*

$$\pi_k(\mathrm{U}) = \pi_{k+2}(\mathrm{U}), \pi_k(\mathrm{O}) = \pi_{k+8}(\mathrm{O}), \pi_k(\mathrm{Sp}) = \pi_{k+8}(\mathrm{Sp}).$$

After Bott's original proof by Morse theory, many proofs have been provided, notably Atiyah's proof relating the Bott periodicity to topological K-theory [Ati68]. This article presents an algebro-geometric proof of Bott periodicity for the unitary group via the geometry of affine Grassmannians. In the rest of this article, we mainly restrict our attention to the unitary case.

Let us first fix some notations and reformulate the problem. For a topological space X with a fixed base point, let ΩX be the based loop space of X. Consider the subgroup of polynomial loops $\Omega_{\mathrm{pol}}\mathrm{U}$ consisting of loops $(S^1, 1) \to (\mathrm{U}, I)$ that sends $z \in S^1$ to a matrix with entries being finite Laurent polynomials of z, inheriting the subspace topology of $\Omega\mathrm{U}$. Similarly one may define the polynomial loop subgroup of any symmetric spaces M. A general theorem of Quillen states that the inclusion map $\Omega_{\mathrm{pol}}M \to \Omega M$ is a homotopy equivalence (for a detailed proof, see [Mit88, Theorem 4.1]). For any topological group G, let BG and EG be the classifying space and the total space of G.

Lemma 1.2. *Let G be a topological group, then $\Omega BG \simeq G$ is a weak homotopy equivalence.*

Proof. Consider the fiber bundle $G \to EG \to BG$ and apply [Hat02, Proposition 4.66]. □

Thus $\pi_k(\Omega\mathrm{U}) = \pi_{k+1}(\mathrm{U})$ and $\pi_{k+1}(B\mathrm{U}) = \pi_k(\mathrm{U})$. Hence the unitary case of Theorem 1.1 follows from the following theorem.

Theorem 1.3. *The topological spaces $\mathbb{Z} \times B\mathrm{U}$ and $\Omega_{\mathrm{pol}}\mathrm{U}$ are homotopy equivalent.*

The classical Bott map constructs a homotopy equivalence $\mathbb{Z} \times B\mathrm{U} \to \Omega\mathrm{U}$. The construction relies on a choice of a "generating cycle." This article presents a construction of the "inverse Bott map" that is simple and canonical. The possibility of such a construction was originally announced by Garland-Raghunathan [GR75]. Later Pressley-Segal carries it out with a detailed proof [Pre86, Section 8.8]. However, Pressley-Segal's proof contains a small gap. This article fixes the gap and rewrites the proof in the language of affine Grassmannians.

This article is written as a report on the author's summer research at SURIM (Stanford Undergraduate Research Institute in Mathematics). The project is still ongoing, with the further intention of supplying the proof for the orthogonal and symplectic cases and exploring the relation of the construction with algebraic K-theory and motivic homotopy theory.

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2. Affine Grassmannians

In this section, we briefly introduce affine Grassmannians, the major algebro-geometric input in this article. For a detailed and complete treatment on this topic, see [Zhu16].

2.1. **Basic definitions.** In the rest of this article, let k be a field. Let $k[[t]]$ and $k((t))$ be the rings of formal power series and the field of Laurent series with coefficients in k, respectively. Let Aff_k denote the category of affine k-schemes, the opposite category of k-algebras. Sometimes we mix the notations $\operatorname{Spec} R$ and R for a k-algebra R, when no confusion arises. By a presheaf we mean a covariant functor from Aff_k to the category of sets.

Definition 2.1. Let R be a k-algebra. Define an *R-family of lattices* Λ in $k((t))^n$ to be a finitely generated projective $R[[t]]$-submodule of $R((t))^n$ such that $\Lambda[t^{-1}] = R((t))^n$.

Definition 2.2. The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ (for GL_n) is the presheaf that sends each k-algebra R to the set of R-family of lattices.

The following result forms the foundation for further discussion.

Theorem 2.3. *The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}$ is represented by an ind-projective scheme.*

This is Theorem 1.3 in [Zhu16]. As some aspects of the proof will be useful later, we briefly review the proof.

Proof. In this proof, write $\mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}_n}$ for simplicity. Let $\Lambda \subset k((t))^n$ be an R-family of lattice. Since $\Lambda[t^{-1}] = k((t))^n$, there exists a positive integer N such that $t^N k[[t]]^n \subset \Lambda \subset t^{-N} k[[t]]^n$. Let $\mathrm{Gr}^{(N)}$ denote the subfunctor assigning R to the set of all such Λ, then $\mathrm{Gr} = \operatorname{colim} \mathrm{Gr}^{(N)}$.

Consider the presheaf $\mathrm{Gr}^{(N),f}$ defined by

$$\mathrm{Gr}^{(N),f}(R) = \Big\{ R[[t]]\text{-quotient modules of } t^{-N}R[[t]]^n / t^N R[[t]]^n \text{ that are projective as } R\text{-modules} \Big\}.$$

We have the following key lemma from [Zhu16, Lemma 1.1.5].

Lemma 2.4. *The map*

$$\Lambda \mapsto t^{-N} R[[t]]^n / \Lambda$$

defines an isomorphism between presheaves $\mathrm{Gr}^{(N)} \to \mathrm{Gr}^{(N),f}$.

As an R-module, $t^{-N}R[[t]]^n/t^N R[[t]]^n \simeq R^{2nN}$, with a t-action. Let $\mathrm{Gr}(2nN)$ be the usual Grassmannian variety which assigns R the set of projective R-quotient modules of R^{2nN}, which is a projective variety. Then the natural functor $\mathrm{Gr}^{(N),f} \to \mathrm{Gr}(2nN)$ is a closed embedding as $\mathrm{Gr}(2nN)$ classifies all k-subspaces in k^{2nN} and $\mathrm{Gr}^{(N),f}$ classifies all k-subspaces in k^{2nN} that are fixed by the t-action. Thus $\mathrm{Gr} = \operatorname{colim} \mathrm{Gr}^{(N),f}$ is ind-projective and the natural map

$$\mathrm{Gr} \to \operatorname{colim} \mathrm{Gr}(2nN)$$

is a closed embedding. □

More generally, let $\underline{G}$ be a smooth affine group scheme over $k[[t]]$. Denote $D = \operatorname{Spec} k[[t]]$ and $D^* = \operatorname{Spec} k((t))$ as the *disk* and the *punctured disc* over k. And write $D_R := \operatorname{Spec} R[[t]]$ and $D_R^* := \operatorname{Spec} R((t))$ as a family of discs (resp. punctured discs) parametrized by $\operatorname{Spec} R$. Let $\mathcal{E}^0$ denote the trivial $\underline{G}$-torsor on D_R.

Definition 2.5. The *affine Grassmannian* $\mathrm{Gr}_{\underline{G}}$ *of* $\underline{G}$ is given by

$$\mathrm{Gr}_{\underline{G}}(R) := \Big\{ (\mathcal{E}, \beta) \;\big|\; \mathcal{E} \text{ is a } \underline{G}\text{-torsor on } D_R, \text{ and } \beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ is a trivialization} \Big\}.$$

Theorem 2.6. *The affine Grassmannian* $\mathrm{Gr}_{\underline{G}}$ *is represented by an ind-scheme, ind-of finite type over* k*. If* $\underline{G}$ *is reductive, then* $\mathrm{Gr}_{\underline{G}}$ *is ind-projective.*

Proof. This is [Zhu16, Theorem 1.2.2]. □

2.2. **Affine Grassmannians and loop groups.** An alternative characterization of affine Grassmannians is via jet and loop groups. Now we briefly recall them. In the rest of this article, write $F = k((t))$ and $\mathcal{O} = k[[t]]$.

Definition 2.7. Let X be a presheaf over $\mathcal{O}$. The *formal jet space* (or the *positive loop space*) of X is the presheaf L^+X given by

$$L^+X(R) := X(R[[t]]).$$

Let Y be a presheaf over F. The *loop space* of Y is the presheaf LX given by

$$LX(R) = X(R((t))).$$

When X is a presheaf over k, we write $L^+(X \otimes_k \mathcal{O})$ by L^+X and $L(X \otimes_k F)$ by LX.

Now consider the natural action

$$L\underline{G} \times \mathrm{Gr}_{\underline{G}} \to \mathrm{Gr}_{\underline{G}}$$

given by

$$(A, (\mathcal{E}, \beta)) \mapsto (\mathcal{E}, A\beta),$$

where $A\beta$ is the map given by β composed with the left multiplication of A. Then one can show that $L\underline{G} \to \mathrm{Gr}_{\underline{G}}$ is a L^+G-torsor. Equivalently we have the following result.

Proposition 2.8. *The affine Grassmannian* $\mathrm{Gr}_{\underline{G}}$ *is isomorphic to the fpqc quotient* $[L\underline{G}/L^+\underline{G}]$.

Proof. This is [Zhu16, Proposition 1.3.6]. In particular, when k is algebraically closed (e.g., $\mathbb{C}$), the k-points of $\mathrm{Gr}_{\underline{G}}$ is the naive group quotient $\underline{G}(F)/\underline{G}(\mathcal{O})$. □

2.3. **Affine Grassmannians over complex number.** In this subsection, we consider the case when $k = \mathbb{C}$. Let G be a complex reductive group and let $\underline{G}$ be the constant group $G \otimes_k \mathcal{O}$. We can endow Gr_G with an analytic topology, regarding it as an infinite dimensional complex analytic space, in particular a CW complex.

Let $K \subset G$ be a maximal compact subgroup. For any $\gamma : S^1 \to K$ in $\Omega_{\mathrm{pol}}K$, the Laurent series corresponding to each matrix entry gives an element in $G(\mathbb{C}((t))) = LG(\mathbb{C})$. Thus we obtain a natural map $\Omega K \to LG(\mathbb{C})$ between topological spaces.

Theorem 2.9. *The natural map* $\Omega K \to LG(\mathbb{C})$ *induces a homeomorphism* $\Omega K \simeq \mathrm{Gr}_G$.

Proof. The case when $G = \mathrm{GL}_n$ is done by Pressley-Segal [Pre86, Section 8.3]. The general case follows from a more general theorem due to Nadler [Nad04, §4]. □

A special case of interest to us is when $G = \mathrm{GL}_n$ and $K = \mathrm{U}$. We have $\Omega\mathrm{U}(n) \simeq \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. The affine Grassmannian over GL_n thus serves as an algebro-geometric model of the loop space. For the proof of Bott periodicity in the orthogonal and symplectic cases, one needs to consider the cases when $G = \mathrm{O}(n; \mathbb{C})$ and $G = \mathrm{Sp}(n; \mathbb{C})$.

2.4. **Sato Grassmannians.** The ind-scheme colim $\mathrm{Gr}(2nN) =$ colim $\mathrm{Gr}(N)$ mentioned above is an instance of a more general notion of Sato Grassmannians, which can be used to construct the determinant line bundle on $\mathrm{Gr}_{\underline{G}}$ via pullback. As this notion and its generalizations are helpful for our proof and even beyond the unitary case, we briefly introduce it here.

Definition 2.10. We say that a topological vector space is *linearly compact* if it is the topological dual of a discrete vector space. A topological vector space is *locally linearly compact* if it admits a basis of neighborhoods of 0 of linearly compact subspaces. A *lattice* in a Tate vector space V is a linearly compact open subspace of V.

Remark 2.11. If L_1 and L_2 are lattices in a Tate space V, the quotients are $L_1/(L_1 \cap L_2)$ and $L_2/(L_1 \cap L_2)$ are finite-dimensional, i.e., they are *commensurable.*

Definition 2.12. Let V be a Tate vector space. The *Sato Grassmannian* $\mathrm{Gr}(V)$ is the ind-scheme

$$\mathrm{Gr}(V) := \varinjlim_{L_1 \subset L_2} \mathrm{Gr}(L_2/L_1),$$

where the direct limit is indexed by $L_1 \subset L_2$ that are two lattices in V.

A standard example of a Tate vector space is $V = k((t))^n$ with the usual t-adic topology. It contains $\Lambda_0 = k[[t]]^n$ as a standard lattice, and lattices in $k((t))^n$ are precisely subspaces that are commensurable to Λ_0. The proof of Theorem 2.3 gives a closed embedding

$$\mathrm{Gr}_{\mathrm{GL}_n} \to \mathrm{Gr}(V).$$

3. Schubert Cell Decompositions

Affine Grassmannians have rich geometry. It has various subvarieties of interest to geometric representation theory, including Schubert varieties, opposite Schubert "varieties," and transversal slices. For a detailed exposition, see [Zhu16, Section 2]. In this section, we review Schubert cell decompositions of affine Grassmannians. Such a decomposition amounts to a cell decomposition of the analytic topology of the affine Grassmannian over $\mathbb{C}$.

For the rest of this article, assume that k is algebraically closed and $\underline{G} = G \otimes_k \mathcal{O}$ is a constant reductive group scheme. We write Gr for Gr_G when no confusion arises.

3.1. **A Group Decomposition.** Fix embeddings $T \subset B \subset G$, where T is a maximal torus and B is a Borel subgroup. Let $\mathbb{X}_\bullet(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ denote the coweight lattice and $\mathbb{X}_\bullet(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ be the weight lattice. Denote by $2\rho \in \mathbb{X}^\bullet(T)$ the sum of all positive roots and let $\rho = \frac{1}{2}(2\rho) \in \mathbb{X}^\bullet(T) \otimes \mathbb{Q}$. Let $\mu \in \mathbb{X}_\bullet(T)$ be a coweight; on F-points, it defines a map $\mu : F^\times \to T(F) \subset G(F)$. Denote $t^\mu = \mu(t)$ the image of the uniformizer t in $G(F)$, which is also a k-point of LG. Its image in $\mathrm{Gr}(k) = G(F)/G(\mathcal{O})$ is independent of the uniformizer. Let $\widetilde{W_{\mathrm{aff}}}$ and W_f be the affine Weyl group and finite Weyl group associated with G, respectively.

Let I be the Iwahori group scheme of G. Recall that $I(\mathcal{O})$ is the inverse image in $G(\mathcal{O})$ of $B(k) \subset G(k)$ along the mod t map. We then have the Iwahori decomposition (cf. [Fal03], [PR08])

$$G(F) = \bigsqcup_{w \in \widetilde{W_{\mathrm{aff}}}} I(\mathcal{O}) w I(\mathcal{O}).$$

The double coset $I(\mathcal{O})wI(\mathcal{O})$ is independent of the uniformizer t and the embedding $T \subset G$. As $L^+G = \varprojlim L^nG$ where $L^nG(R) := G(R[t]/t^n)$ is an inverse limit of schemes of finite types and recall that I is the inverse image of the Borel, we obtain a Bruhat decomposition

$$G(\mathcal{O}) = \bigsqcup_{w \in W_f} I(\mathcal{O}) w I(\mathcal{O}).$$

Thus we obtain a group decomposition indexed by $\widetilde{W_{\mathrm{aff}}}/W_f = \mathbb{X}_\bullet(T)$ as follows

$$G(F) = \bigsqcup_{\mu \in \mathbb{X}_\bullet(T)} I(\mathcal{O}) t^\mu G(\mathcal{O}). \tag{3.1}$$

This gives a canonical bijection

$$I(\mathcal{O}) \backslash G(F)/G(\mathcal{O}) \simeq \mathbb{X}_\bullet(T).$$

3.2. **Schubert cells and Schubert varieties.** The Iwahori subgroup I acts on Gr_G by left multiplication. The stabilizer of t^μ for this action of L^+G is $L^+G \cap t^\mu I t^{-\mu}$. This induces a locally closed embedding

$$L^+G/(t^\mu I t^{-\mu}) \to LG/L^+G, \qquad g \mapsto gt^\mu.$$

Denote the image of this embedding Gr_μ as a subfunctor of Gr. This is called a *Schubert cell.* By the decomposition (3.1), the union of Schubert cells is the entire Gr_G, at the level of points over any field. Denote the Zariski closure

$$\mathrm{Gr}_{\leq \mu} := \overline{\mathrm{Gr}_\lambda}.$$

This closed subscheme of Gr_G is called a *Shubert variety.*

Theorem 3.1. *The followings are true.*

1. Gr_μ *forms a single I-orbit and is an affine subscheme with dimension $\ell(\mu)$, i.e.* $\mathrm{Gr}_\mu \simeq \mathbb{A}^{\ell(\mu)}$. *The definition of $\ell(\mu)$ is deferred to the next subsection.*
2. $\mathrm{Gr}_{\leq\mu}$ *is the Zariski closure of* Gr_μ *and so is a projective variety.*

Proof. The proof except for showing that Gr_μ is affine is analogous to [Zhu16, Theorem 2.1.5]. The proof of $\mathrm{Gr}_\mu \simeq \mathbb{A}^{\ell(\mu)}$ can be naturally seen via the connection to Kac-Moody theory, which we briefly introduce in the next subsection. □

3.3. **Affine Kac-Moody algebras.** Affine Grassmannians above are realized as a colimit of closed subschemes of the usual Grassmannian varieties, which are instances of partial flag varieties. The relation between affine Grassmannians and partial flag varieties is deeper, which we shall explain in this subsection.

Consider a connected untwisted affine Dynkin diagram Γ. Such Γ can be obtained from the finite Dynkin diagrams by adding a distinguished node v_0 (see [Kac90, Chap. 4, TABLE Aff1 & Fin]). Let S be a subset of vertices in Γ. To each of these Γ, one may attach two types of groups. Bruhat and Tits attach to Γ a simple and simply connected reductive group G_Γ over F and to S a parabolic group scheme $\underline{G}_S$ over $\mathcal{O}$ (cf. [Tit79, §4.2]). On the other hand, there is a Kac-Moody algebra $\mathfrak{g}_\Gamma$ (see [Kac90, Chap. 4] and [Tit89]) and a corresponding Kac-Moody group G_Γ and to S a parabolic subgroup P_S of G_Γ.

Back to our context, consider $\underline{G}$ a parahoric group scheme in G. This will correspond to a subset S of vertices in the affine Dynkin diagram Γ of G. To (Γ, S) one may attach a parabolic subgroup in an affine Kac-Moody group $P_S \subset G_\Gamma$. The following result relates affine Grassmannians to partial flag varieties explicitly.

Theorem 3.2. *The affine Grassmannian* $\mathrm{Gr}_{\underline{G}}$ *is isomorphic to the partial flag variety G_Γ/P_S as defined in [Kum87] and [Mat88].*

Proof. See [BL94] for the case when $G = \mathrm{SL}_n$ and [Fal03] for the general case. In particular, the Shubert varieties in $\mathrm{Gr}_{\underline{G}}$ are isomorphic to the Schubert varieties arising from the Kac-Moody theory. □

The geometry of Schubert cells in the Kac-Moody theory is well understood. In particular, let $\widetilde{W_{\mathrm{Aff}}}$ be the affine Weyl group associated with G_Γ and let W_f be the finite Weyl group associated with G. Then there is a natural identification

$$\widetilde{W_{\mathrm{Aff}}}/W_f \simeq \mathbb{X}_\bullet(T).$$

For a coweight $\mu \in \mathbb{X}_\bullet(T)$, there is a corresponding coset in $\widetilde{W_{\mathrm{Aff}}}/W_f$. Define the *length* $\ell(\mu)$ of μ to be the length of any element in its preimage in $\widetilde{W_{\mathrm{Aff}}}/W_f$. The general theory of Kac-Moody groups admits a Schubert decomposition into affine spaces index by $\widetilde{W_{\mathrm{Aff}}}/W_f$ with dimension equal to the length of the corresponding element. In particular, this will imply Theorem 3.1 (for details, see [Kum84]).

4. Proof of Theorem 1.3

Recall that the classifying space of $\mathrm{U}(n)$ is $B\mathrm{U}(n) = \varinjlim \mathrm{Gr}(n, N)$. Hence the classifying space of U is $B\mathrm{U} = \varinjlim \mathrm{Gr}(n, N)$, where the transition maps are taken to be closed embeddings $\mathrm{Gr}(n, N) \to \mathrm{Gr}(n, N+1)$ and $\mathrm{Gr}(n, N) \to \mathrm{Gr}(n+1, N+1)$. It's then not hard to see that $\mathrm{Gr}(V)(\mathbb{C})$ can be identified with $B\mathrm{U} \times \mathbb{Z}$ as topological spaces. Recall by Theorem 2.9, $\Omega\mathrm{U}(n) = \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. Taking direct limit we obtain $\Omega\mathrm{U} = \mathrm{colim}\ \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. Write $\mathrm{Gr}_{\mathrm{GL}} := \mathrm{colim}\ \mathrm{Gr}_{\mathrm{GL}_n}$ as an ind-scheme. We have a sequence of closed embeddings

$$\begin{array}{ccccccccc}
\mathrm{Gr}_{\mathrm{GL}_1} & \hookrightarrow & \mathrm{Gr}_{\mathrm{GL}_2} & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Gr}_{\mathrm{GL}_n} & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathrm{Gr}(\mathbb{C}((t))) & \hookrightarrow & \mathrm{Gr}(\mathbb{C}((t))^2) & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Gr}(\mathbb{C}((t))^n) & \hookrightarrow & \cdots,
\end{array}$$

which will induce a closed embedding of their direct limits

$$\mathrm{Gr}_{\mathrm{GL}} \to \mathrm{Gr}(\mathbb{C}((t))^{\infty}) \simeq \mathrm{Gr}(V). \tag{4.1}$$

Here by the standard diagonal argument, one sees that there is an identification $\mathrm{Gr}(\mathbb{C}((t))^{\infty}) \simeq \mathrm{Gr}(V)$ for $V = \mathbb{C}((t))$. The closed embedding (4.1) is the desired *inverse Bott map*. Now the proof of Theorem 1.3 amounts to showing that the (4.1) induces a homotopy equivalence on the level of $\mathbb{C}$-points. The following lemma will then suffice.

Lemma 4.1. *The closed embedding*

$$\mathrm{Gr}_{\mathrm{GL}_n} \to \mathrm{Gr}(V^n)$$

induces isomorphisms on homotopy groups on the level of $\mathbb{C}$ *up to dimension* $2n-2$ *(with the analytic topology).*

The proof relies on the Schubert cell decomposition given in Section 3. We first need a combinatorial description of the dimension of Schubert cells.

Let $\mu = (a_1, \dots, a_n) \in \mathbb{X}_\bullet(T) \simeq \mathbb{Z}^n$ be a coweight. Consider the strictly increasing integer sequence $\{s_i\}_{i \in \mathbb{Z}_+}$ formed by elements of

$$S := \left\{ mn + i \;\middle|\; i = 1, 2, \dots, n, m \geq a_i \right\},$$

then one may shift the indices of $\{s_i\}$ by an integer such that $s_k = k$ when k is sufficiently large. Denote

$$\ell^0(\{s_i\}) := \sum_{k \geq 0} (k - s_k), \qquad \ell(\{s_i\}) := \sum_{k \in (S+n)-S} (k - s_k)$$

which are positive integers. Since the analytic space of complex points of the sato, Grassmannian is the filtered colimit of all finite Grassmannians, it is naturally the filtered colimit of all finite Schubert cells in the classical sense, which are easily seen to be indexed by S and have dimensions $\ell^0(S)$.

Lemma 4.2. *We have that* $\ell(\mu) = \ell(\{s_i\})$.

Proof. The proof immediately follows from the definition of length in affine Weyl groups. □

Proof of Lemma 4.1. It's sufficient to consider the identity components of the two spaces. Note that $\mathrm{Gr}(V^n)(\mathbb{C})$ is the colimit of finite Grassmannians and thus breaks to the union of Schubert cells of all finite-dimensional Grassmannians. Furthermore, the Schubert cell decomposition of $\mathrm{Gr}_{\mathrm{GL}_n}$ in Section 3 exhibits $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ as a CW subcomplex of $\mathrm{Gr}(V^n)(\mathbb{C})$. The Schubert cells of $\mathrm{Gr}(V^n)(\mathbb{C}) \simeq \mathrm{Gr}(V)(\mathbb{C})$ are indexed by increasing integer sequences $\{s_i\}$ that have a finite difference from $\mathbb{Z}$ (see [Pre86, §7.4]). Then the Schubert cells of $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ corresponds to the indices S such that $S + n \subset S$. Now consider that if a cell has index S satisfying

$$n + k + \mathbb{Z}_+ \subset S \subset k + \mathbb{Z}_+,$$

then such a cell is necessarily inside $\mathrm{Gr}_{\mathrm{GL}_n}$. Now consider all other cells Gr_S in both spaces; a simple calculation following from Lemma 4.2 shows that they all have complex dimensions of at least n. Apply the the cellular approximation theorem to conclude. □

Remark 4.3. The first proof of this argument is given by Pressley-Segal [Pre86, §8.8]. However, their construction of the closed embedding is not compatible with dimensions. One form of the diagonal argument is needed.

Remark 4.4. The same proof directly implies two other homotopy equivalences

$$\Omega(\mathrm{U}/\mathrm{Sp}) \simeq \mathbb{Z} \times B\mathrm{Sp}, \qquad \Omega(\mathrm{U}/\mathrm{O}) = \mathbb{Z} \times B\mathrm{O},$$

as parts of the orthogonal and symplectic cases of the Bott periodicity.

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