# Bott Periodicity and Affine Grassmannians


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## Introduction

- An $n$ by $n$ complex matrix $Q$ is *unitary* if
$$QQ^* = I,$$
where $Q^*$ is the conjugate transpose of $Q$. The group of $n$ by $n$ unitary matrices form the $n$-th unitary group $U(n)$.
- Define the *unitary group* as $\mathsf{U} := \operatorname{colim} \mathsf{U}(n)$ where the transition maps are natural inclusions. The celebrated Bott periodicity theorem ([2], [3]) states that the homotopy groups of U are periodic.

**Theorem 1.** The stable homotopy of the classical groups is periodic; that is,
$$\pi_k(\mathsf{U}) = \pi_{k+2}(\mathsf{U}).$$

Different proofs of the theorem:

- Bott's original proof by Morse theory.
- Atiyah's proof relating the Bott periodicity to topological $K$-theory [1].
- This project presents an **algebro-geometric proof** of Bott periodicity for the unitary group via the geometry of affine Grassmannians.

## Infinite Grassmannians

- Let $V$ be a finite-dimensional complex vector space.

**Definition.** The Grassmannian $\mathsf{Gr}(n;V)$ is the space of all $n$-dimensional linear subspaces of $V$.

- $\mathsf{Gr}(n;V)$ has a structure of compact smooth manifold.

**Example.** $\mathsf{Gr}(1;\mathbb{C}^n)$ is the space of all lines in $\mathbb{C}^n$ passing through origin, which is the $(n-1)$-dimensional complex projective space.

**Definition.** Define the Grassmannian manifold of $V$ to be $\mathsf{Gr}(V) := \coprod_{n=0}^{\infty} \mathsf{Gr}(n;V)$. If $V = \mathbb{C}^k$, we write $\mathsf{Gr}(V) = \mathsf{Gr}(k)$.

- We can glue all Grassmannian manifolds together to get an infinite dimensional Grassmannian as follows.
- Consider the ring of all complex Laurent series $\mathbb{C}((t))$ (or $\mathbb{C}((t))^k$ for some positive integer $k$), which is an infinite-dimensional complex vector space.

**Definition.** Define the complex points of the Sato Grassmannians as
$$\mathsf{Gr}(V)(\mathbb{C}) := \mathbb{Z} \times \bigcup_{n=0}^{\infty} \mathsf{Gr}(t^n\mathbb{C}[[t]]/\mathbb{C}[[t]])$$

- There is also a notion of affine Grassmannians.

**Definition.**
$$\mathsf{Gr}_{\mathsf{GL}_n}(\mathbb{C}) := \bigcup_{m=1}^{\infty} \left\{\mathbb{C}[[t]]\text{-submodules of } t^m\mathbb{C}[[t]]^n/t^{-m}\mathbb{C}[[t]]^n\right\}.$$

## Affine Grassmannian in Algebraic Geometry

- We now introduce affine Grassmannians as algebro-geometric objects, for details see [6].
- Let $k$ be a field. Let $k[[t]]$ and $k((t))$ be the rings of formal power series and the field of Laurent series with coefficients in $k$, respectively.

**Definition.** Let $R$ be a $k$-algebra. Define an *$R$-family of lattices* $\Lambda$ in $k((t))^n$ to be a finitely generated projective $R[[t]]$-submodule of $R((t))^n$ such that $\Lambda[t^{-1}] = R((t))^n$.

**Definition.** The affine Grassmannian $\mathsf{Gr}_{\mathsf{GL}_n}$ (for $\mathsf{GL}_n$) is the presheaf that sends each $k$-algebra $R$ to the set of $R$-family of lattices.

- By a *presheaf* we mean a covariant functor from the category of affine $k$-schemes to the category of sets.
- The following result forms the foundation for further discussion.

**Theorem 2.** The affine Grassmannian $\mathsf{Gr}_{\mathsf{GL}_n}$ is represented by an ind-projective scheme.

- The proof shows that there is a natural closed embedding
$$\mathsf{Gr}_{\mathsf{GL}_n} \to \operatorname{colim}_N \mathsf{Gr}(2nN).$$
- Affine Grassmannians can be defined more generally over any reductive group scheme $G$ over $k[[t]]$.
- There are two equivalent characterizations:

**Characterization 1.** The *affine Grassmannian* $\mathsf{Gr}_{\underline{G}}$ *of* $\underline{G}$ is given by
$$\mathsf{Gr}_{\underline{G}}(R) := \left\{(\mathcal{E},\beta) \mid \mathcal{E} \text{ a } \underline{G}\text{-torsor on } D_R, \beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ a trivialization}\right\}.$$

**Characterization 2.** The *affine Grassmannian* $\mathsf{Gr}_{\underline{G}}$ is the fpqc quotient $[\underline{G}(k((t)))/\underline{G}(k[[t]])]$.

- There is also an algebro-geometric notion of Sato Grassmannians.

**Definition.** We say that a topological vector space is *linearly compact* if it is the topological dual of a discrete vector space. A topological vector space is *locally linearly compact* if it admits a basis of neighborhoods of 0 of linearly compact subspaces. A *lattice* in a Tate vector space $V$ is a linearly compact open subspace of $V$.

**Definition.** Let $V$ be a Tate vector space. The *Sato Grassmannian* $\mathsf{Gr}(V)$ is the ind-scheme
$$\mathsf{Gr}(V) := \varinjlim_{L_1\subset L_2} \mathsf{Gr}(L_2/L_1),$$
where the direct limit is indexed by $L_1 \subset L_2$ that are two lattices in $V$.

- A standard example of a Tate vector space is $V = k((t))^n$ with the usual $t$-adic topology.
- It contains $\Lambda_0 = k[[t]]^n$ as a standard lattice, and lattices in $k((t))^n$ are precisely subspaces that are commensurable to $\Lambda_0$.
- The proof of the Bott periodicity boils down to showing that the closed embedding
$$\mathsf{Gr}_{\mathsf{GL}_n} \to \mathsf{Gr}(V)$$
induces isomorphisms on lower homotopy groups at the level of $\mathbb{C}$-points.

## Outline of the Proof

$$\begin{array}{ccccccccc}
\mathsf{Gr}_{\mathsf{GL}_1} & \longrightarrow & \mathsf{Gr}_{\mathsf{GL}_2} & \longrightarrow & \cdots & \longrightarrow & \mathsf{Gr}_{\mathsf{GL}_n} & \longrightarrow & \cdot \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathsf{Gr}(\mathbb{C}((t)))(\mathbb{C}) & \longrightarrow & \mathsf{Gr}(\mathbb{C}((t))^2)(\mathbb{C}) & \longrightarrow & \cdots & \longrightarrow & \mathsf{Gr}(\mathbb{C}((t))^n)(\mathbb{C}) & \longrightarrow & \cdots
\end{array}$$

**Lemma.** The closed embedding
$$\mathsf{Gr}_{\mathsf{GL}_n} \to \mathsf{Gr}(V^n)$$
induces isomorphisms on homotopy groups on the level of $\mathbb{C}$ up to dimension $2n-2$ (with the analytic topology).

- The proof uses the Schubert decomposition of the affine Grassmannians and a dimension calculation.
- To conclude we look at the following commutative diagram.

$$\begin{array}{ccccc}
\Omega U(n) & \longrightarrow & \Omega U & \longrightarrow & BU \times \mathbb{Z} \\
\uparrow\sim & & \uparrow\sim & & \| \\
\mathsf{Gr}_{\mathsf{GL}_n}(\mathbb{C}) & \longrightarrow & \mathsf{Gr}_{\mathsf{GL}}(\mathbb{C}) & \longrightarrow & \mathsf{Gr}(V)(\mathbb{C})
\end{array}$$

- The relationship between the loop space and affine Grassmannian is established by Pressley-Segal [5] and, much more generally, by Quillen [4].

## Future Work

- Generalize this proof to $O$ and $Sp$.
- Generalize this closed embedding and consider possible implications in algebraic $K$-theory.
- Say something in the motivic setting.

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