

Introduction

• An n by n complex matrix Q is *unitary* if

$$QQ^* = I,$$

where Q^* is the conjugate transpose of Q. The group of n by n unitary matrices form the *n*-th unitary group U(n).

• Define the *unitary group* as $U := \operatorname{colim} U(n)$ where the transition maps are natural inclusions. The celebrated Bott periodicity theorem ([2], [3]) states that the homotopy groups of U are periodic.

Theorem 1. The stable homotopy of the classical groups is periodic; that is,

 $\pi_k(\mathsf{U}) = \pi_{k+2}(\mathsf{U}).$

Different proofs of the theorem:

- Bott's original proof by Morse theory.
- Atiyah's proof relating the Bott periodicity to topological K-theory [1].
- This project presents an algebro-geometric proof of Bott periodicity for the unitary group via the geometry of affine Grassmannians.

Infinite Grassmannians

• Let V be a finite-dimensional complex vector space.

Definition. The Grassmannian Gr(n; V) is the space of all *n*dimensional linear subspaces of V.

• Gr(n; V) has a structure of compact smooth manifold.

Example. Gr $(1; \mathbb{C}^n)$ is the space of all lines in \mathbb{C}^n passing through origin, which is the (n-1)-dimensional complex projective space.

Definition. Define the Grassmannian manifold of V to be $Gr(V) := \bigsqcup_{n=0}^{\infty} Gr(n; V)$. If $V = \mathbb{C}^k$, we write Gr(V) = Gr(k).

- We can glue all Grassmannian manifolds together to get an infinite dimensional Grassmannian as follows.
- Consider the ring of all complex Laurent series $\mathbb{C}((t))$ (or $\mathbb{C}((t))^k$ for some positive integer k), which is an infinite-dimensional complex vector space.

Definition. Define the complex points of the Sato Grassmannians as

 $\operatorname{Gr}(V)(\mathbb{C}) := \mathbb{Z} \times \bigcup_{n=0}^{\infty} \operatorname{Gr}(t^n \mathbb{C}[[t]]/\mathbb{C}[[t]])$

• There is also a notion of affine Grassmannians.

Definition.

 $\operatorname{Gr}_{\operatorname{GL}_n}(\mathbb{C}) := \bigcup \left\{ \mathbb{C}[[t]] \text{-submodules of } t^m \mathbb{C}[[t]]^n / t^{-m} \mathbb{C}[[t]]^n \right\}.$

BOTT PERIODICITY AND AFFINE GRASSMANNIANS

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Affine Grassmannian in Algebraic Geometry

- We now introduce affine Grassmannians as algebro-geometric objects, for details see [6].
- Let k be a field. Let k[[t]] and k((t)) be the rings of formal power series and the field of Laurent series with coefficients in k, respectively.

Definition. Let R be a k-algebra. Define an R-family of lattices Λ in $k((t))^n$ to be a finitely generated projective R[[t]]-submodule of $R((t))^n$ such that $\Lambda[t^{-1}] = R((t))^n.$

Definition. The affine Grassmannian Gr_{GL_n} (for GL_n) is the presheaf that sends each k-algebra R to the set of R-family of lattices.

- By a *presheaf* we mean a covariant functor from the category of affine kschemes to the category of sets.
- The following result forms the foundation for further discussion.

Theorem 2. The affine Grassmannian Gr_{GL} is represented by an indprojective scheme.

- The proof shows that there is a natural closed embedding $\operatorname{Gr}_{\operatorname{GL}_n} \to \operatorname{colim}_N \operatorname{Gr}(2nN).$
- Affine Grassmannians can be defined more generally over any reductive group scheme G over k[[t]].
- There are two equivalent characterizations:

Characterization 1. The *affine Grassmannian* Gr_G of <u>G</u> is given by $\mathsf{Gr}_G(R) := \{ (\mathcal{E}, \beta) \mid \mathcal{E} \text{ a } \underline{G} \text{-torsor on } D_R, \beta : \mathcal{E}|_{D_p^*} \simeq \mathcal{E}^0|_{D_p^*} \text{ a trivialization} \}$.

Characterization 2. The *affine Grassmannian* Gr_G is the fpqc quotient $[\underline{G}(k((t)))/\underline{G}(k[[t]])].$

• There is also an algebro-geometric notion of Sato Grassmannians.

Definition. We say that a topological vector space is *linearly compact* if it is the topological dual of a discrete vector space. A topological vector space is *locally linearly compact* if it admits a basis of neighborhoods of 0 of linearly compact subspaces. A *lattice* in a Tate vector space V is a linearly compact open subspace of V.

Definition. Let V be a Tate vector space. The Sato Grassmannian Gr(V) is the ind-scheme

 $\mathsf{Gr}(V) := \varinjlim_{L_1 \subset L_2} \mathsf{Gr}(L_2/L_1),$

where the direct limit is indexed by $L_1 \subset L_2$ that are two lattices in V.

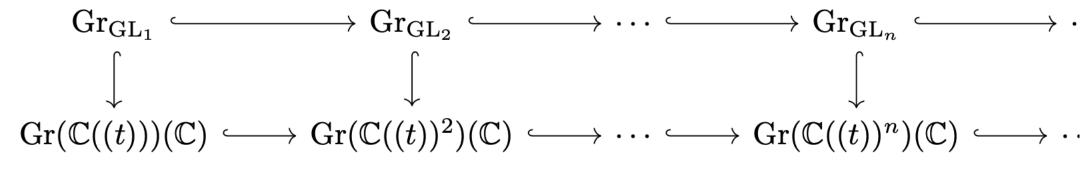
- A standard example of a Tate vector space is $V = k((t))^n$ with the usual tadic topology.
- It contains $\Lambda_0 = k[[t]]^n$ as a standard lattice, and lattices in $k((t))^n$ are precisely subspaces that are commensurable to Λ_0 .
- The proof of the Bott periodicity boils down to showing that the closed embedding

 $\operatorname{Gr}_{\operatorname{GL}_n} \to \operatorname{Gr}(V)$

induces isomorphisms on lower homotopy groups at the level of \mathbb{C} -points.



Outline of the Proof



Lemma. The closed embedding

 $\operatorname{Gr}_{\operatorname{GL}_n} \to \operatorname{Gr}(V^n)$

induces isomorphisms on homotopy groups on the level of \mathbb{C} up to dimension 2n-2 (with the analytic topology).

- The proof uses the Schubert decomposition of the affine Grassmannians and a dimension calculation.
- To conclude we look at the following commutative diagram.

$\Omega U(n)$ —	$\longrightarrow \Omega U$ ———	\rightarrow
$\sim \uparrow$	$\uparrow \sim$	
	$ ightarrow Gr_{GL}(\mathbb{C}) \longrightarrow$	>

• The relationship between the loop space and affine Grassmannian is established by Pressley-Segal [5] and, much more generally, by Quillen [4].

Future Work

- Generalize this proof to O and Sp.
- Generalize this closed embedding and consider possible implications in algebraic *K*-theory.
- Say something in the motivic setting.

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References

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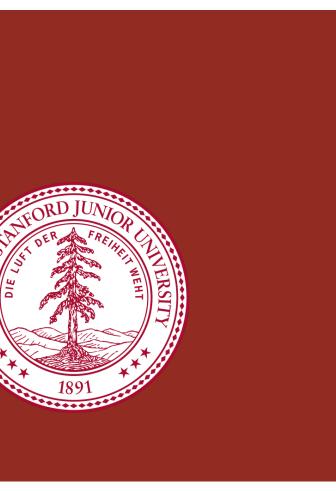
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$\rightarrow BU \times \mathbb{Z}$ $\operatorname{Gr}(V)(\mathbb{C})$