Partition Functions For The Six-vertex Lattice Model

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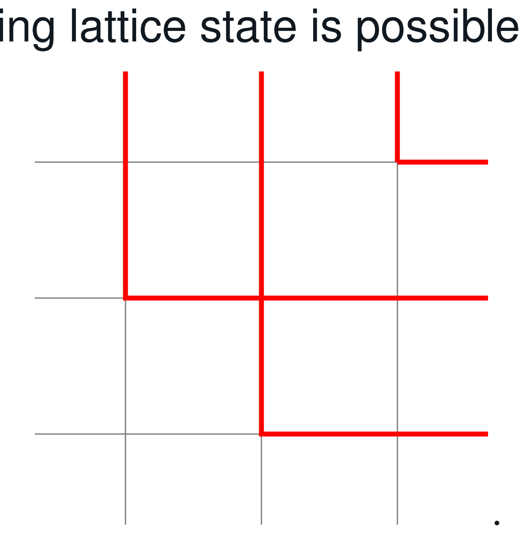
Background

- The six-vertex model was introduced in connection with the statistical mechanics of crystal lattices, but since then has found connections with enumerative combinatorics [3], and the theory of symmetric functions [1].
- Consider an $n \times n$ grid with paths entering the top and leaving from the right. The configuration of paths results in six types of vertices, each of which is assigned a multiplicative weight. Summing the weights of the vertices gives the weight of a particular state of the lattice.
- For instance, the following lattice state is possible



The partition function of the lattice model, denoted $Z_n(x_1, \ldots, x_n)$, is the sum over all weights of lattice states. **Computing the partition function associated with different boundary conditions and weights is a central problem in the field.**

- We will be interested in a class of weights which satisfy a functional equation known as the Yang-Baxter equation. This makes the partition function 'morally symmetric'. Weights which satisfy the Yang-Baxter equation will be called integrable.

Partition Function for Constant-Field Weights

- Constant-field weights are a class of integrable weights with applications to combinatorics.
- These weights can be parametrized by the following R-matrix:

$$R_i^j = \begin{pmatrix} q_1x_i - q_2y_j & & & \\ & x_i(q_1-q_2)k & x_i - y_j & \\ & q_1q_2(x_i-y_j) & y_j(q_1-q_2)k^{-1} & \\ & & & q_1x_i - q_2y_j \end{pmatrix},$$

- We adapt the method from [3] to find the partition function Z_n of the $n \times n$ lattice model with standard domain wall boundary conditions (DWBC) for constant-field weights.

We prove that Z_n is uniquely defined by the following properties:

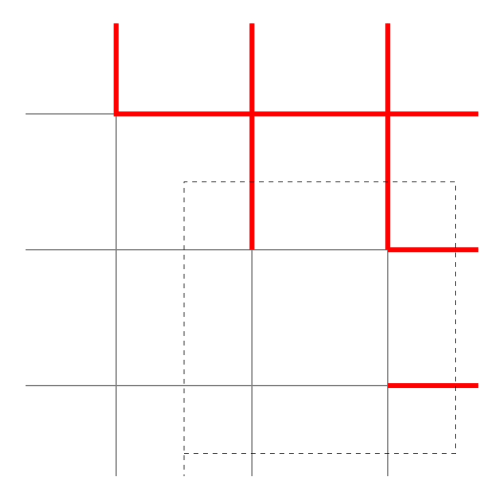
i. Symmetric in x_i's and y_j's
ii. Z_n is a polynomial of degree at most N in each x_i and y_j.
iii. Base case: $Z_1 = c_1(x_1, y_1)$
iv. Z_n can be computed recursively by a forbidden vertex argument when $x_1 = y_n$. In this case, the weight of a path leaving the first row in the first column becomes 0, and hence the only contributing terms to the partition function come from when the first path turns on the first row. This allows us to derive a determinant expression for the partition function.

The Determinant Expression

- We derive the following recurrence relation:

$$Z_n|_{x_1=y_1} = c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i, y_1) a(x_1, y_i) Z_{n-1}(\mathbf{x}_{2,n}; \mathbf{y}_{2,n})$$

- By specializing $x_1 = y_1$, we notice that



is the only admissible lattice state. Z_n can be expressed as follows:

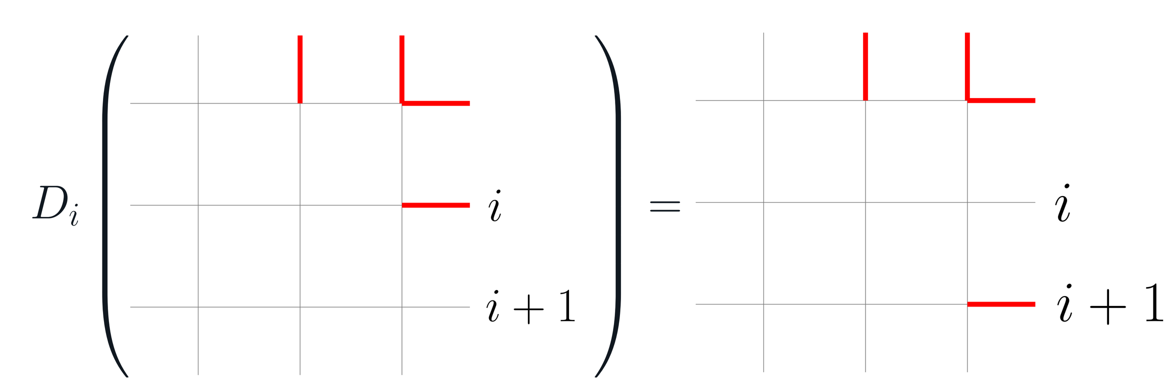
$$Z_n = (-1)^{n-1} \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j} b(x_i, x_j) b(y_i, y_j))} \det \left(\frac{c_1(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

The general recurrence relation can also be written using Lagrange interpolation, from which the determinant expression follows naturally

$$Z_n = \sum_{i=1}^{n} c_1(x_i, y_1) \prod_{k \neq 1} a(x_i, y_k) \prod_{k \neq i} a(x_k, x_i) \frac{x_k - y_1}{x_k - x_i} Z_{n-1}(\hat{x}_i; \hat{y}_1)$$

Defining the Lattice Algebra

- We construct an algebra based on the Demazure operator [2] in order to find the partition functions of lattice models with arbitrary boundaries.
- The Demazure operator acts on a partition function Z by switching the positions of adjacent rows i and $i+1$, illustrated as follows:



- The operator is defined as

$$\mathcal{D}_i(Z_n) = \frac{R_{i+1,i}(a_1)s_i - R_{i+1,i}(c_1)}{R_{i+1,i}(b_2)} Z_n$$

where s_i transposes rows i and $i+1$.

- Parameterizing weights such that

i. $R_{i+1,i}(a_1) = z_i - tz_{i+1}$
ii. $R_{i+1,i}(b_2) = z_i - z_{i+1}$
iii. $R_{i+1,i}(c_1) = (1-t)z_i$

the operator acts as the well-known Demazure-Lusztig operator, which obeys the braid relation such that

$$\mathcal{D}_i \mathcal{D}_{i+1} \mathcal{D}_i = \mathcal{D}_{i+1} \mathcal{D}_i \mathcal{D}_{i+1}.$$

This guarantees our general operator can be specialized to show that it is well defined.

Lattice Algebra Results

- We discovered a more general Demazure operator than the operators existing in literature.
- **We found that a system of lattice weights are *Demazure integrable* if they satisfy the following conditions:**

i. $R_{i+1,i}(a_1) R_{i,i+1}(a_1) = R_{i+1,i}(c_1) R_{i+1,i}(c_2) - R_{i+1,i}(b_1) R_{i+1,i}(b_2)$
ii. $R_{i+1,i}(a_2) = R_{i+1,i}(c_1) + R_{i+1,i}(b_2) = R_{i+1,i}(a_1)$
iii. $R_{i+1,i}(a_1) - R_{i+1,i}(c_2) = R_{i+1,i}(b_1)$

- We demonstrate a way to relate arbitrary row and column partition for Z_n to the Z_n domain wall boundary case for the lattice model, which allows parition functions for arbitrary boundary conditions to be computed effectively.
- We developed a new way to calculate any partition function by using recursion and Demazure operators. We can relate Z_n to the domain wall boundary case partition function for Z_{n-1} by applying the modified Demazure operators that act on rows and columns.

Conclusion

- Lagrange Interpolation can be effective in computing the partition functions of graphs with domain wall boundary conditions, while Demazure operators allow us to extend these results to models with arbitrary boundary conditions.
- Lagrange interpolation requires data about a large number of points of the partition function, which other methods, such as the functional method do not.
- Demazure operators are particularly effective. However, for the full power of this method to be realized, we must restrict to the class of Demazure integrable weights.
- In the future, we plan on applying our methods to the colored models. For example, Demazure operators play a key role in finding the partition functions of colored models as seen in [2]. We would also like to show that the general Demazure operator satisfies the braid relation without specialization.

Acknowledgements

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References

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