Random walks in slightly changing environments

Bryan Park
Advisor: Amir Dembo

Abstract

A *Random Walk in Changing Environment* (RWCE) is a weighted random walk on a locally finite, connected graph G with random edge-weights at each time step. This includes self-interacting random walks, where the edge-weights depend on the history of the process. In general, even the basic question of recurrence or transience for RWCEs is difficult, especially when the underlying graph contains cycles. In this note, we derive a condition for recurrence or transience that is too restrictive for classical RWCEs but instead works for any graph G. Namely, we show that any bounded RWCE on G with "slightly" changing edge-weights inherits the recurrence or transience of the initial weighted graph.

1 Introduction

Let $\mathbb{N} := \{0, 1, 2, \dots\}$ and $G = (V, E)$ be any simple, undirected, locally finite, and connected graph. We begin with the following definition.

Definition 1 (RWCE)**.** A *Random Walk in Changing Environment* on a graph $G = (V, E)$ is a stochastic process $\{\langle X_t, w_t \rangle\}_{t \in \mathbb{N}}$ such that for any $y \in V$, we have

$$\mathbb{P}(X_{t+1} = y \mid \mathcal{F}_t) = \frac{w_t(X_t, y)}{\sum_{z \sim X_t} w_t(X_t, z)}$$

where $X_t \in V$, $w_t \in (0, \infty)^E$, and $\mathcal{F}_t = \sigma(X_0, \dots, X_t, w_0, \dots, w_t)$ for each $t \in \mathbb{N}$.

In words, at time $t \in \mathbb{N}$ the RWCE traverses a neighboring edge from X_t with probability proportional to its weight, which is the realization of some random variable. While the term RWCE was coined by Amir et al. in [1], many special cases have been extensively studied before. For instance, RWCEs include the large class of self-interacting random walks, where the weights depend on the history of the process. A well-known example is the linearly edge-reinforced random walk by Coppersmith and Diaconis [3] from the eighties. Other examples include the once-reinforced random walk [4] or the "true" self-avoiding walk with bond repulsion [7].

In general, even the basic question of recurrence or transience is difficult for RWCEs as the process is not Markovian. In fact, the various notions of recurrence and transience may not necessarily coincide in general. In this note, we adopt the definition below from [1].

Definition 2 (Recurrence/Transience/Mixed-Type)**.** An RWCE is *recurrent* if a.s. every vertex is visited infinitely often. It is *transient* if a.s. every vertex is visited finitely often. Otherwise, it is of *mixed-type*.

To aid the study of recurrence or transience, we further assume that the RWCE is *elliptic*.

Definition 3 (Elliptic RWCE)**.** Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be an RWCE on $G = (V, E)$. For each $\{x, y\} \in E$, assume that $\mathbb{P}(X_{t+1} = y \mid X_t = x)$, whenever $\mathbb{P}(X_t = x) > 0$, is bounded away from 0 as t varies. Then, we say that $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is *elliptic* (uniformly in time).

This is useful since any elliptic RWCE that a.s. visits some vertex infinitely (resp. finitely) often is also recurrent (resp. transient). To see this, fix some $x, y \in V$. Then, by ellipticity (and connectivity of G), whenever the RWCE is at x, the probability of eventually visiting y is at least p_{xy} for some constant $p_{xy} > 0$. Hence, if x is visited infinitely often, then a.s. y is visited infinitely often. The contrapositive implies that if y is visited finitely often, then a.s. x is visited finitely often.

A special case of ellipticity is when there are deterministic $w, w' \in (0,\infty)^E$ such that $w \le w_t \le w'$ for all $t \in \mathbb{N}$. We say that such an RWCE is *bounded*. Even assuming boundedness, however, most results on recurrence or transience rely on the underlying graph being a tree. For instance, consider the once-reinforced random walk, which is a bounded RWCE. In [2], Collevecchio et al. completely characterized the recurrence or transience of such processes on trees by introducing a quantity called the branching-ruin number. In contrast, on $\mathbb{Z}^2$ the question of recurrence remains completely open. We remark that partial progress has been made by Kious et al. [6] for graphs of the form $\mathbb{Z} \times \Gamma$ where Γ is finite.

The situation is in fact similar for a much wider class of bounded RWCEs. Namely, consider any RWCE on a tree T with increasing weights bounded above by w_∞. In [1], Amir et al. showed that if (T, w_∞) is recurrent, then the RWCE is also recurrent.[1] Such a general result, however, cannot hold if the underlying graph contains cycles. For instance, Amir et al. [1] constructed a self-interacting, bounded, and increasing RWCE on $\mathbb{Z}^2$ that serves as a counterexample. The case for transience is analogous, where Amir et al. proved a general result for trees [1] that is not expected to hold for arbitrary graphs. An important question that follows is whether the general result above can hold for any graph if the RWCE is not self-interacting. A partial answer was given in [5] where Dembo et al. confirmed the transient case for $\mathbb{Z}^d$, $d \ge 3$.

In this note, we derive a condition for recurrence or transience that is too restrictive for classical RWCEs (such as the once-reinforced random walk) but instead holds for any graph. Loosely speaking, we show that any bounded RWCE on G with "slightly" changing edge-weights inherits the recurrence or transience of the initial weighted graph. We discuss our result further in the following section.

2 Main Result

Our condition on "slight" changes is conveniently stated in terms of *resistances*, which are simply the reciprocal of weights. For any $t \in \mathbb{N}$, we write $r_t := 1/w_t$. Such resistances arise when viewing the weighted graph as an electrical network, which we will further exploit throughout this note. For now, the following is our main result.

Theorem 1. *Let $G = (V, E)$ be any graph and $w_0 \in (0,\infty)^E$ be deterministic such that (G, w_0) is recurrent (resp. transient). Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any bounded RWCE on G. If the random variable*

$$\sum_{t,e} |r_t(e) - r_{t+1}(e)|$$

is bounded, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is recurrent (resp. transient).

[1] A weighted graph (G, w) is recurrent (resp. transient) if the weighted random walk on (G, w) is recurrent (resp. transient).

If the final condition above (or the *boundedness condition*) is satisfied, note that $\sum_{e\in E} |r_t(e) - r_{t+1}(e)|$ is finite for each $t \in \mathbb{N}$ and tends to zero as $t \to \infty$. This is the reason we say that the weights are changing only "slightly." Moreover, in the special case where $\{r_t\}_{t\in\mathbb{N}}$ is increasing (resp. decreasing) and bounded above (resp. below) by r_∞, we remark that it suffices for $\sum_e |r_0(e) - r_\infty(e)|$ to be bounded.

2.1 Proof Overview

We begin with an overview of our proof. Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any RWCE on $G = (V, E)$ satisfying the conditions of Theorem 1. Fix any $s \in V$ which we consider as the origin of V and also the source of an electrical network on G. To show recurrence (resp. transience), it suffices by ellipticity to show that s is a.s. visited infinitely (resp. finitely) often.

Our proof consists of two main parts. First, we show Theorem 1 in the special case where s has a single neighbor. To do so, we use the standard technique [1] of constructing a sup/submartingale of the form $\{f_t(X_t)\}_{t\in\mathbb{N}}$ and then using the optional stopping theorem to bound the probability of return. In particular, the construction works on any graph as we consider the maximum/minimum ratio of vertex-voltages across a single time step. Since s has a single neighbor, for any $v \neq s$, the voltage at v is positive and thus ratios are well-defined.

Next, when s has multiple neighbors, we attach a new vertex s' to s and construct a new RWCE whose recurrence or transience implies the recurrence or transience of the original RWCE. Then, we apply the first part above to this new RWCE by considering s' as the origin of G'. This concludes the proof.

3 The Single Neighbor Case

Throughout this section, we assume that the origin s has a single neighbor. We aim to show the following special case of Theorem 1.

Lemma 1. *Let $G = (V, E)$ be any graph and $w_0 \in (0, \infty)^E$ be deterministic such that (G, w_0) is recurrent (resp. transient). Assume the origin $s \in V$ has a single neighbor and let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any bounded RWCE on G. If the random variable*

$$\sum_{t,e} |r_t(e) - r_{t+1}(e)|$$

is bounded, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is recurrent (resp. transient).

Indeed, choosing the origin is arbitrary and it suffices for a vertex of degree one to exist. As mentioned above, we will construct a sup/submartingale and then apply the optional stopping theorem to derive a condition for recurrence or transience. Then, we will show that this condition is satisfied assuming the condition of Lemma 1.

3.1 Sup/submartingales

We first construct the desired sup/submartingale. Assuming that the origin s and the RWCE $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ are given, we introduce some notation involving electrical networks. For $n \geq 0$, let $V_n := \{v \in V : d(s, v) \leq n\}$ and $\partial V_n := \{v \in V : d(s, v) = n\}$ where d is the shortest-path distance on G. Let $G_n = (V_n, E_n)$ denote the subgraph induced in G by V_n. For $n \geq 1, t \in \mathbb{N}$, and $u \in V_n$, let $v_{n,t}(u)$ denote the (random) voltage of u in (G_n, w_t) when s is grounded and ∂V_n is kept at voltage 1. If $u \notin V_n$, define $v_{n,t}(u) := 1$.

The key connection between random walks on graphs and electrical networks is that whenever w_t is fixed, $v_{n,t}(u)$ equals the probability that the weighted random walk $\{Z_k\}_{k\in\mathbb{N}}$ on (G_n, w_t) with $Z_0 = u \in V_n$ will hit ∂V_n before s. In particular, both quantities are *harmonic*, meaning $\{v_{n,t}(Z_{k\wedge\theta})\}_{k\in\mathbb{N}}$ is a martingale with respect to $\{Z_k\}_{k\in\mathbb{N}}$ where $\theta := \inf\{k \in \mathbb{N} : Z_k \in \{s\} \cup \partial V_n\}$. In our case, the analogous process for $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is $\{v_{n,t}(X_{t\wedge\tau})\}_{t\in\mathbb{N}}$ where $\tau = \inf\{t \in \mathbb{N} : X_t \in \{s\} \cup \partial V_n\}$. Unfortunately, for arbitrary $u \in V_n$ the sequence $\{v_{n,t}(u)\}_{t\in\mathbb{N}}$ is not necessarily monotone and thus $\{v_{n,t}(X_{t\wedge\tau})\}_{t\in\mathbb{N}}$ is not a sup/submartingale.

To bypass this difficulty, for each $t \in \mathbb{N}$ we consider the maximum/minimum of the ratio $v_{n,t+1}(u)/v_{n,t}(u)$ over all $u \in V_n \setminus \{s\}$. For $n \geq 1$ and $t \in \mathbb{N}$, let

$$\alpha_{n,t} := \max_{u\in V_n\setminus\{s\}} \frac{v_{n,t+1}(u)}{v_{n,t}(u)} \geq 1,$$
$$\beta_{n,t} := \min_{u\in V_n\setminus\{s\}} \frac{v_{n,t+1}(u)}{v_{n,t}(u)} \leq 1,$$

where the inequalities follow by considering $u \in \partial V_n$. Also, the quantities are well-defined (positive and finite) since s has a single neighbor which gives $v_{n,t} > 0$ on $V_n \setminus \{s\}$ for all $t \in \mathbb{N}$. Finally, recall that $\tau = \inf\{t \in \mathbb{N} : X_t \in \{s\} \cup \partial V_n\}$ and $\mathcal{F}_t = \sigma(X_0, \ldots, X_t, w_0, \ldots, w_t)$ for each $t \in \mathbb{N}$. The following is our desired sup/submartingale.

Lemma 2. *Fix $n \geq 1$ and let*

$$A_t = \frac{v_{n,t}(X_t)}{\prod_{k=0}^{t-1} \alpha_{n,k}}, \quad B_t = \frac{v_{n,t}(X_t)}{\prod_{k=0}^{t-1} \beta_{n,k}}$$

for $t \in \mathbb{N}$. Then, $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$ is a supermartingale and $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$ is a submartingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$.

Proof. It suffices to prove the supermartingale case as the submartingale case is identical. First, note that

$$A_{t+1} = \frac{v_{n,t+1}(X_{t+1})}{\prod_{k=0}^{t} \alpha_{n,k}} \leq \frac{v_{n,t}(X_{t+1})}{\prod_{k=0}^{t-1} \alpha_{n,k}}$$

by construction. Next, if $t < \tau$, we have $(t + 1) \wedge \tau = t + 1$ and thus

$$\mathbb{E}[A_{(t+1)\wedge\tau} \mid \mathcal{F}_t] \leq \mathbb{E}\left[\frac{v_{n,t}(X_{t+1})}{\prod_{k=0}^{t-1} \alpha_{n,k}} \mid \mathcal{F}_t\right] = \frac{1}{\prod_{k=0}^{t-1} \alpha_{n,k}} \mathbb{E}\left[v_{n,t}(X_{t+1}) \mid \mathcal{F}_t\right] = A_{t\wedge\tau}.$$

If $t \geq \tau$, then

$$\mathbb{E}[A_{(t+1)\wedge\tau} \mid \mathcal{F}_t] = \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{k=0}^{\tau-1} \alpha_{n,k}} \mid \mathcal{F}_t\right] = \frac{v_{n,\tau}(X_\tau)}{\prod_{k=0}^{\tau-1} \alpha_{n,k}} = A_{t\wedge\tau}$$

as desired and we conclude our proof. □

3.2 Optional Stopping Theorem

We now apply the optional stopping theorem to the sup/submartingale constructed above. For the results of this section, we remark that it suffices to assume ellipticity instead of boundedness of the given RWCE. We begin with the supermartingale $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$.

Lemma 3. *Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any elliptic RWCE on $G = (V, E)$ and assume the origin $s \in V$ has a single neighbor. For each $n \geq 1$, assume there is $a_n \in \mathbb{R}$ such that $\prod_{t=0}^{\infty} \alpha_{n,t} \leq a_n < \infty$ almost surely. If $\limsup_{n\to\infty} a_n < \infty$ and $v_{n,t}(u) \to 0$ almost surely as $n \to \infty$ for any $t \in \mathbb{N}$ and $u \in V$, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is recurrent.*

Proof. By ellipticity, it suffices to show that s is a.s. visited infinitely often. Since $\alpha_{n,t} \geq 1$, note that the conditions of the lemma hold for any subprocess $\{\langle X_t, w_t\rangle\}_{t\geq t'}$ where $t' > 0$. Hence, it suffices to show that $X_t = s$ for some $t \in \mathbb{N}$ assuming $X_0 = u \neq s$. Fix some $n \geq 1$ and recall the supermartingle $\{A_{t\wedge\tau}\}_{t\in\mathbb{N}}$ from Lemma 2. Since $|A_{t\wedge\tau}| \leq 1$ for all $t \in \mathbb{N}$, the optional stopping theorem gives $\mathbb{E}[A_\tau] \leq \mathbb{E}[A_0]$. Hence,

$$\mathbb{E}[v_{n,0}(X_0)] \geq \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{t=0}^{\tau-1} \alpha_{n,\tau}}\right] \geq \frac{\mathbb{P}(X_\tau \in \partial V_n)}{a_n}$$

which can be rearranged as $\mathbb{P}(X_\tau \in \partial V_n) \leq a_n \cdot v_{n,0}(u)$. Taking the limit superior on both sides of the inequality gives our desired result. □

Next, we proceed similarly with the submartingale $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$.

Lemma 4. *Let $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ be any elliptic RWCE on $G = (V, E)$ and assume the origin $s \in V$ has a single neighbor x. For each $n \geq 1$, assume there is $b_n \in \mathbb{R}$ such that a.s. $\prod_{t=0}^{\infty} \beta_{n,t} \geq b_n > 0$. If $\liminf_{n\to\infty} b_n > 0$ and $\inf_{n,t} v_{n,t}(x) > 0$, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is transient.*

Proof. By ellipticity, it suffices to show that s is a.s. visited finitely often. We will show that given $X_t = x$, assuming $\mathbb{P}(X_t = x) > 0$, the probability of never returning to s again is at least some positive constant independent of t. This suffices since whenever the process visits s, it must visit x the next step.

First consider when $X_0 = x$. Fix some $n \geq 1$ and recall the submartingale $\{B_{t\wedge\tau}\}_{t\in\mathbb{N}}$ from Lemma 2. Since $|B_{t\wedge\tau}| \leq 1/b_n < \infty$, the optional stopping theorem gives $\mathbb{E}[B_\tau] \geq \mathbb{E}[B_0]$. Hence,

$$\mathbb{E}[v_{n,0}(X_0)] \leq \mathbb{E}\left[\frac{v_{n,\tau}(X_\tau)}{\prod_{t=0}^{\tau-1} \beta_{n,\tau}}\right] \leq \frac{\mathbb{P}[X_\tau \in \partial V_n]}{b_n}$$

which can be rearranged as $\mathbb{P}[X_\tau \in \partial V_n] \geq b_n \cdot v_{n,0}(x) \geq b_n \cdot \inf_{n,t} v_{n,t}(x)$. Taking the limit inferior on both sides as $n \to \infty$, we get $\mathbb{P}[\text{never return to } s \text{ again} \mid X_0 = x] \geq K$ for some $K > 0$.

When $X_t = x$, we can construct a submartingale similar to $B_{t\wedge\tau}$ by viewing X_t as the initial vertex and (G, w_t) as the initial graph. Since $\beta_{n,t} \leq 1$, the same method gives $\mathbb{P}[\text{never return to } s \text{ again} \mid X_t = x] \geq K$ as desired and we conclude our proof. □

3.3 Bounding Voltage-Ratios

Having Lemma 3 and 4, we want to use these results to prove Lemma 1. For this purpose, we estimate $\alpha_{n,t}$ and $\beta_{n,t}$ by deriving an upper bound for $|v_{n,t+1}(u)/v_{n,t}(u) - 1|$. We begin with the following expression for $|v_{n,t+1}(u) - v_{n,t}(u)|$.

Lemma 5. *For any $n \geq 1$, $t \in \mathbb{N}$, and $u \in V_{n-1} \setminus \{s\}$ we have*

$$v_{n,t+1}(u) - v_{n,t}(u) = \frac{1}{\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n)} \sum_{e=\{x,y\}\in E_n} (r_t(e) - r_{t+1}(e)) \cdot i^{n,t+1}_{u,\{s\}\cup\partial V_n}(x, y) \cdot i^{n,t}_{s,\partial V_n}(x, y)$$

where $\mathcal{R}_{n,t}(a \leftrightarrow b)$ is the effective resistance between vertices a, b in (G_n, w_t). Also, $i^{n,t}_{v,S}$ is the unit current in (G_n, w_t) from v (which is grounded) to $S \subseteq V_n \setminus \{v\}$. Finally, $i^{n,t}_{v,S}(x, y)$ is the amount of the current $i^{n,t}_{v,S}$ across $\{x, y\}$ from x to y.

Proof. Note that all random variables in the claim are determined given w_t and w_{t+1}. The key idea is to represent $v_{n,t+1}(u)$ in terms of the current $i_1 := i^{n,t+1}_{u,s\cup\partial V_n}$. Namely, we claim that

$$v_{n,t+1}(u) = \sum_{y\in\partial V_n} \sum_{x\in V_n} i_1(x, y). \tag{1}$$

In words, the right-hand side of (1) is the total amount of current in i_1 that flows into ∂V_n. Recall that the probabilistic interpretation of $i_1(x, y)$ is given by the weighted random walk on (G_n, w_{t+1}) that begins at u and runs until hitting $s \cup \partial V_n$. Namely, $i_1(x, y)$ equals the expected net number of crossings of $\{x, y\}$ in the given direction during the random walk. In particular, it is zero if $x \nsim y$. Taking $x \sim y$ as specified in the summation above, if $x \in s \cup \partial V_n$ we also have $i_1(x, y) = 0$ as $\{x, y\}$ is never crossed. Otherwise, if $x \in V_{n-1} \setminus \{s\}$, we can cross $\{x, y\}$ exactly once during the random walk as it will terminate after crossing. Hence, $i_1(x, y)$ equals the probability that the random walk terminates after crossing $\{x, y\}$. It follows that the right-hand side of (1) is simply the probability that the weighted random walk on (G_n, w_{t+1}) beginning at u will hit ∂V_n before s. By the probabilistic interpretation of voltage, this is exactly $v_{n,t+1}(u)$.

The rest of our proof is routine algebra of flows, which we explain below. First, by Kirchhoff's current law we extend (1) to get

$$v_{n,t+1}(u) - v_{n,t}(u) = \sum_{y\in V_n} v_{n,t}(y) \sum_{x\in V_n} i_1(x, y).$$

As current is antisymmetric, we further obtain

$$\begin{aligned} v_{n,t+1}(u) - v_{n,t}(u) &= \frac{1}{2} \sum_{x,y\in V_n} (v_{n,t}(y) - v_{n,t}(x)) \cdot i_1(x, y) \\ &= \frac{1}{\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n)} \sum_{e=\{x,y\}\in E_n} r_t(e) \cdot i_0(x, y) \cdot i_1(x, y) \end{aligned}$$

where $i_0 := i^{n,t}_{s,\partial V_n}$ and the second equality is by Ohm's law.

To conclude, it suffices to show that

$$L := \sum_{e=\{x,y\}\in E_n} r_{t+1}(e) \cdot i_1(x, y) \cdot i_0(x, y) = 0.$$

We evaluate L by essentially reversing the above process. Let $\phi(x)$ denote the voltage of $x \in V_n$ induced by i_1. Then, by Ohm's law we have

$$\begin{aligned} L &= \sum_{e=\{x,y\}\in E_n} (\phi(y) - \phi(x)) \cdot i_0(x, y) \\ &= \frac{1}{2} \sum_{x,y\in V_n} (\phi(y) - \phi(x)) \cdot i_0(x, y) \\ &= \sum_{y\in V_n} \phi(y) \sum_{x\in V_n} i_0(x, y) \end{aligned}$$

where the second and third equalities follow since current is antisymmetric. By Kirchoff's current law, we can simplify further to obtain

$$L = \phi(s) \sum_{x \in V_n} i_0(x,s) + \sum_{y \in \partial V_n} \phi(y) \sum_{x \in V_n} i_0(x,y).$$

Note that $\phi(y) = \phi(s)$ for any $y \in \partial V_n$ and i_0 is a unit flow. Hence, we get $L = -\phi(s) + \phi(s) = 0$ as desired and conclude our proof. □

We now crucially use the assumption that s has a single neighbor to get the following corollary.

Corollary 1. *Assume that s has a single neighbor x. Then, for any $n \geq 1$, $t \in \mathbb{N}$, and $u \in V_n \setminus \{s\}$, we have*

$$\left| \frac{v_{n,t+1}(u)}{v_{n,t}(u)} - 1 \right| \leq w_t(s,x) \sum_{e \in E} |r_t(e) - r_{t+1}(e)|.$$

Proof. Since the right-hand side of Lemma 4 involves unit currents, taking absolute values gives

$$|v_{n,t+1}(u) - v_{n,t}(u)| \leq \frac{1}{\mathcal{R}_{n,t(s \leftrightarrow \partial V_n)}} \sum_{e \in E} |r_t(e) - r_{t+1}(e)|.$$

Moreover, the inequality trivially holds for $u \in \partial V_n$. Finally, since s has a single neighbor x, we see that $v_{n,t}(u) \geq v_{n,t}(x) = r_t(s,x)/\mathcal{R}_{n,t(s \leftrightarrow \partial V_n)}$. Combining the two inequalities gives our desired result. □

3.4 Proof of Lemma 1

We are now ready to prove Lemma 1.

3.4.1 Showing Recurrence

We begin with the recurrent case.

Proof of Lemma 1 (Recurrence). We aim to use Lemma 3. First, we check that for any $t \in \mathbb{N}$ and $u \in V_n$, we have $v_{n,t}(u) \to 0$ almost surely as $n \to \infty$. Let $d(s,u) = \ell$ and $(x_0, \ldots, x_\ell)$ be a path from s to u. Then, for $n > \ell$ we have

$$\begin{aligned} v_{n,t}(u) &= \frac{1}{\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n)} \sum_{k=0}^{\ell-1} i^{n,t}_{s,\partial V_n}(x_k, x_{k+1}) r_t(x_k, x_{k+1}) \\ &\leq \frac{1}{\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n)} \sum_{k=0}^{\ell-1} r_t(x_k, x_{k+1}). \end{aligned}$$

Next, by the boundedness condition there exists $C_1 > 0$ such that $\sum_e \delta_e \leq C_1$ almost surely where $\delta_e := \sum_{t=0}^{\infty} |r_t(e) - r_{t+1}(e)|$ for $e \in E$. Hence, it follows that $|r_0(e) - r_t(e)| \leq \delta_e \leq C_1$ and thus $r_t(e) \leq r_0(e) + C_1$. Since r_0 is deterministic, $\sum_{k=0}^{\ell-1} r_t(x_k, x_{k+1})$ is bounded and it suffices to show that a.s. $\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n) \to \infty$ as $n \to \infty$. Let $i := i^{n,t}_{s,\partial V_n}$. Then, by Thomson's principle, we a.s. have

$$\mathcal{R}_{n,0}(s \leftrightarrow \partial V_n) \leq \sum_{e \in E_n} i^2(e) r_0(e) \leq \sum_{e \in E_n} i^2(e)(r_t(e) + \delta_e) \leq \mathcal{R}_{n,t}(s \leftrightarrow \partial V_n) + C_1.$$

Since (G, w_0) is recurrent, it follows that a.s. $\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n) \to \infty$ for each $t \in \mathbb{N}$ as desired.

Next, we show that the condition on $\alpha_{n,t}$ holds. With the crucial assumption that the RWCE is bounded, there exists $C_2 > 0$ such that $w_t(s, x) \leq C_2$ where x is the unique neighbor of s. By Corollary 1, we get

$$\prod_{t=0}^{\infty} \alpha_{n,t} \leq \prod_{t=0}^{\infty} \left(1 + \sum_{e \in E} w_t(s,x)|r_t(e) - r_{t+1}(e)|\right) \leq \exp\left(\sum_{t,e} w_t(s,x)|r_t(e) - r_{t+1}(e)|\right) \leq e^{C_1 C_2}.$$

Therefore, we can choose $a_n = e^{C_1 C_2}$ in Lemma 2 for each $n \geq 1$. This concludes our proof. □

3.4.2 Showing Transience

By similar methods, we next prove the transient case.

Proof of Lemma 1 (Transience). We aim to use Lemma 4. We first check that $\inf_{n,t} v_{n,t}(x) > 0$. Since x is the unique neighbor of s, recall that $v_{n,t}(x) = r_t(s,x)/\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n)$ for $n \geq 1$. Also, by the boundedness condition, there exists $C_1 > 0$ such that a.s. $|r_0(e) - r_t(e)| \leq \delta_e \leq C_1$ where $\delta_e := \sum_{t=0}^{\infty} |r_t(e) - r_{t+1}(e)|$. Letting $i := i^{n,0}_{s,\partial V_n}$, Thomson's principle gives

$$\mathcal{R}_{n,t}(s \leftrightarrow \partial V_n) \leq \sum_{e \in E_n} i^2(e) r_t(e) \leq \sum_{e \in E_n} i^2(e)(r_0(e) + \delta_e) \leq \mathcal{R}_{n,0}(s \leftrightarrow \partial V_n) + C_1.$$

Moreover, as the RWCE is bounded, there exists $C_2 > 0$ such that $w_t(s,x) \leq C_2$ for all $t \in \mathbb{N}$. Hence,

$$v_{n,t}(x) \geq \frac{1/C_2}{\mathcal{R}_{n,0}(s \leftrightarrow \partial V_n) + C_1} \geq \frac{1/C_2}{\lim\limits_{n \to \infty} \mathcal{R}_{n,0}(s \leftrightarrow \partial V_n) + C_1}$$

since $\mathcal{R}_{n,0}(s \leftrightarrow \partial V_n)$ is increasing in n. As (G, w_0) is transient, we conclude that $\inf_{n,t} v_{n,t}(x) > 0$ as desired.

Next, we show that the condition on $\beta_{n,t}$ holds. Note that

$$\beta_{n,t} \geq v_{n,t+1}(x) \geq \inf_{n,t} v_{n,t}(x)$$

for any $n \geq 1$ and $t \in \mathbb{N}$. Moreover, let $S := \{t \in \mathbb{N} : \sigma_t > 1/(2C_2)\}$ where $\sigma_t = \sum_{e \in E} |r_t(e) - r_{t+1}(e)|$. Since $\sum_{t=0}^{\infty} \sigma_t \leq C_1$ a.s., it follows that $|S| \leq 2C_1 C_2$ almost surely. Beginning with Corollary 1, we have

$$\prod_{t=0}^{\infty} \beta_{n,t} \geq \prod_{t \in S} \beta_{n,t} \cdot \prod_{t \notin S} (1 - C_2 \sigma_t) \geq \prod_{t \in S} \beta_{n,t} \cdot \exp\left(-\sum_{t \notin S} \frac{C_2 \sigma_t}{1 - C_2 \sigma_t}\right).$$

Since $1/(1 - C_2 \sigma_t) \leq 2$ if $t \notin S$, we conclude that a.s.

$$\begin{aligned}\prod_{t=0}^{\infty} \beta_{n,t} &\geq \left(\inf_{n,t} v_{n,t}(x)\right)^{\lceil 2C_1 C_2 \rceil} \exp\left(-2C_2 \sum_{t \notin S} \sigma_t\right) \\ &\geq \left(\inf_{n,t} v_{n,t}(x)\right)^{\lceil 2C_1 C_2 \rceil} e^{-2C_1 C_2}.\end{aligned}$$

Choosing the final value as b_n in Lemma 3 for all $n \geq 1$, we conclude our proof. □

4 The Multiple Neighbor Case

We now consider the general case where the origin s has multiple neighbors. As mentioned in section 2.1, the idea is to attach a new vertex s' to s and construct a new RWCE on the new graph.

4.1 Desired Properties

Here, we describe the desired properties of the new RWCE. Recall that s is the origin of G and $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ is a bounded RWCE on G that satisfies the boundedness condition. Also, w_0 is deterministic. First attach a vertex s' to s to get $G' = (V', E')$ where $V' = V \cup \{s'\}$ and $E' = E \cup \{s, s'\}$. We aim to construct a new bounded RWCE $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ on G' whose recurrence (resp. transience) implies the recurrence (resp. transience) of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$. Then, viewing s' as the origin of G', we can apply Lemma 1 to $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ if it also satisfies the boundedness condition.

Note that the restriction of $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ to G induces a natural RWCE on G. If this induced RWCE is equal in distribution to $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$, we claim that we have our desired implication of recurrence or transience. More concretely, let N_t be the number of edges in E traversed by $(X'_0, \ldots, X'_t)$ for each $t \in \mathbb{N}$. Also define stopping times $\tau_k = \inf\{t \in \mathbb{N} : N_t = k\}$ for $k \in \mathbb{N}$. Then, we say the *RWCE induced by* $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ on G is $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ where $Y_k = X'_{\tau_k}$ and $\omega_k = w'_{\tau_k} \upharpoonright_E$ for each $k \in \mathbb{N}$. In particular, the vertex sequence $\{Y_k\}_{k\in\mathbb{N}}$ simply tracks the edges in E crossed by $\{X'_t\}_{t\in\mathbb{N}}$.

We now explain how the desired implications follow if $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is bounded and $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ equals $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ in distribution. First consider when $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is recurrent and thus a.s. visits s' infinitely often. If s is visited finitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$, then the only way s' can be visited infinitely often in $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is by alternating between s and s' infinitely many times in a row. However, this happens with probability zero as $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is bounded and the probability of jumping from s to s' is bounded above by some number less than 1. Hence, s is a.s. visited infinitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ which implies recurrence of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$. Next, assume that $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is transient and thus a.s visits s finitely often. Since we only remove vertices when obtaining $\{Y_k\}_{k\in\mathbb{N}}$ from $\{X'_t\}_{t\in\mathbb{N}}$, it follows that s is a.s. visited finitely often in $\{\langle Y_k, \omega_k\rangle\}_{k\in\mathbb{N}}$ which implies transience of $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$.

4.2 Formal Construction

Here, we construct our desired $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$. We determine the random variables sequentially, beginning with X'_0, then w'_0, then X'_1, then w'_1, and so on. The key idea is to determine w'_t as if we were determining w_{N_t} given $(Y_0, Y_1, \ldots, Y_{N_t}, \omega_0, \ldots, \omega_{N_t-1})$ as the history. If $\{X'_{t-1}, X'_t\} = \{s, s'\}$, however, then $N_t = N_{t-1}$ and in this case we freeze the weights by letting $w'_t = w'_{t-1}$. Indeed, we unfreeze afterwards as soon as an edge in E is crossed.

For notational simplicity, let $\mathcal{W}_E := (0,\infty)^E$ and $w'_{t,E} := w'_t \upharpoonright_E$ for any $w'_t \in \mathcal{W}^{E'}$. We now give the measure-theoretic construction of $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$. Let $X'_0 = X_0$ in distribution and let $w'_0 \upharpoonright_E = w_0$. Also let $w'_t(s, s') = 1$ for all $t \in \mathbb{N}$. Then, it remains to define the conditional probabilities $\mathbb{P}(w'_{t+1,E} \in A \mid \mathcal{G}'_t)$ for each $t \in \mathbb{N}$ and $A \in \mathcal{B}(\mathcal{W}_E)$ where $\mathcal{G}'_t := \sigma(X'_0, \ldots, X'_{t+1}, w'_0, \ldots, w'_t)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the underlying probability space and $\widehat{\mathbf{P}}_{w_{t+1}|\mathcal{G}_t}(\cdot,\cdot) : \mathcal{B}(\mathcal{W}_E) \times \Omega \to [0,1]$ denote the RCPD of w_{t+1} given $\mathcal{G}_t := \sigma(X_0, \ldots, X_{t+1}, w_0, \ldots, w_t)$ for $t \in \mathbb{N}$. The RCPD exists since E is countable and $\mathcal{W}_E$ is a Polish space. By the Doob-Dynkin functional representation, we have

$$\widehat{\mathbf{P}}_{w_{t+1}|\mathcal{G}_t}(A, \omega) = f_{t,A}(X_0(\omega), \ldots, X_{t+1}(\omega), w_0(\omega), \ldots, w_t(\omega))$$

for any $A \in \mathcal{B}(W_E)$ and $\omega \in \Omega$ where $f_{t,A} : V^{t+2} \times (\mathcal{W}_E)^{t+1} \to [0,1]$ is some measurable function. Moreover, we know that $f_{t,\cdot}(X_0, \ldots, X_{t+1}, w_0, \ldots, w_t)$ is a.s. a probability measure on $(\mathcal{W}_E, \mathcal{B}(\mathcal{W}_E))$.

To conclude, when $N_{t+1} = N_t + 1$, we require

$$\mathbb{P}(w'_{t+1,E} \in A \mid \mathcal{G}'_t) = f_{N_t,A}(Y_0, \ldots, Y_{N_t+1}, \omega_0, \ldots, \omega_{N_t}). \tag{2}$$

Otherwise, if $N_{t+1} = N_t$, then given $\mathcal{G}'_t$ we require $w'_{t+1} = w'_t$. Note that if we know

$$(X_0, \ldots, X_t, w_0, \ldots, w_{t-1}) \stackrel{d}{=} (Y_0, \ldots, Y_t, \omega_0, \ldots, \omega_{t-1})$$

for $t \le k+1$, then (2) is a well-defined probability for $t \le \tau_{k+1} - 1$. We will show this equality in distribution while proving $\{\langle X_t, w_t \rangle\}_{t\in\mathbb{N}} \stackrel{d}{=} \{\langle Y_t, \omega_t \rangle\}_{t\in\mathbb{N}}$ in the following section.

4.3 Verifying the Construction

Here, we justify our construction through the following lemma.

Lemma 6. *We have $\{\langle X_t, w_t \rangle\}_{t\in\mathbb{N}} \stackrel{d}{=} \{\langle Y_t, \omega_t \rangle\}_{t\in\mathbb{N}}$. In particular,* (2) *is well-defined for all $t \in \mathbb{N}$.*

Proof. By Kolmogorov's extension theorem for Polish spaces, it suffices to show that

$$(X_0, w_0, \ldots, X_{t-1}, w_{t-1}, X_t) \stackrel{d}{=} (Y_0, \omega_0, \ldots, Y_{t-1}, \omega_{t-1}, Y_t), \tag{3}$$

$$(X_0, w_0, \ldots, X_{t-1}, w_{t-1}, X_t, w_t) \stackrel{d}{=} (Y_0, \omega_0, \ldots, Y_{t-1}, \omega_{t-1}, Y_t, \omega_t) \tag{4}$$

for each $t \in \mathbb{N}$. We proceed by strong induction on $t \in \mathbb{N}$. If $t = 0$, both claims follow since $\tau_0 = 0$. Next, assume that both claims hold for all $t \le k$ where $k \in \mathbb{N}$. We first show that (3) also holds for $t = k+1$. By Carathéodory's extension theorem, it suffices to show that

$$\mathbb{P}\left(\{X_i = x_i\}_{i=0}^{k+1}, \{w_j \in E_j\}_{j=0}^{k}\right) = \mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right)$$

for any $x_0, \ldots, x_{k+1} \in V$ and $E_0, \ldots, E_t \in \mathcal{B}(\mathcal{W}_E)$. Beginning with the right-hand side, we have

$$\begin{aligned}\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right) &= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j}\right] \\ &= \mathbb{E}\left[\prod_{i=0}^{k} 1_{Y_i = x_i} \prod_{j=0}^{k} 1_{\omega_j \in E_j} \mathbb{E}\left[1_{Y_{k+1} = x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right]\right]\end{aligned}$$

where $\mathcal{F}'_k = \sigma(X'_0, \ldots, X'_t, w'_0, \ldots, w'_k)$. If $Y_k \ne s$, then we have $\mathbb{E}\left[1_{Y_{k+1}=x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right] = \omega_k(Y_k, x_{k+1})/\omega_k(Y_k)$. Otherwise, if $Y_k = s$, then we have

$$\mathbb{E}\left[1_{Y_{k+1}=x_{k+1}} \mid \mathcal{F}'_{\tau_k}\right] = \sum_{k=0}^{\infty} \frac{1}{(\omega_k(s)+1)^k} \cdot \frac{\omega_k(s, x_{k+1})}{\omega_k(s)+1} = \frac{\omega_k(s, x_{k+1})}{\omega_k(s)}.$$

Hence, we can write

$$\mathbb{P}\left(\{Y_i = x_i\}_{i=0}^{k+1}, \{\omega_j \in E_j\}_{j=0}^{k}\right) = \mathbb{E}[g_k(Y_0, \ldots, Y_k, \omega_0, \ldots, \omega_k)]$$

where

$$g_k(Y_0,\ldots,Y_k,\omega_0,\ldots,\omega_k) = \frac{\omega_k(Y_k,x_{k+1})}{\omega_k(Y_k)} \cdot \prod_{i=0}^{k} 1_{Y_i=x_i} \prod_{j=0}^{k} 1_{\omega_j\in E_j}.$$

By the inductive hypothesis, we have

$$\mathbb{E}[g_k(Y_0,\ldots,Y_k,\omega_0,\ldots,\omega_k)] = \mathbb{E}[g_k(X_0,\ldots,X_k,w_0,\ldots,w_k)].$$

Working backwards, we see that

$$\mathbb{E}[g_k(X_0,\ldots,X_k,w_0,\ldots,w_k)] = \mathbb{P}\left(\{X_i=x_i\}_{i=0}^{k+1},\{w_j\in E_j\}_{j=0}^{k}\right).$$

This gives (3) for $t \le k+1$. It follows that (2) is well-defined for $t \le \tau_{k+1}-1$.

Next, we show that (4) also holds for $t=k+1$. Again, it suffices to show that

$$\mathbb{P}\left(\{X_i=x_i\}_{i=0}^{k+1},\{w_j\in E_j\}_{j=0}^{k+1}\right) = \mathbb{P}\left(\{Y_i=x_i\}_{i=0}^{k+1},\{\omega_j\in E_j\}_{j=0}^{k+1}\right)$$

for any $x_0,\ldots,x_{k+1}\in V$ and $E_0,\ldots,E_{k+1}\in\mathcal{B}(\mathcal{W}_E)$. Beginning with the right-hand side, we have

$$\begin{aligned}\mathbb{P}\left(\{Y_i=x_i\}_{i=0}^{k+1},\{\omega_j\in E_j\}_{j=0}^{k+1}\right) &= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i=x_i}\prod_{j=0}^{k+1} 1_{\omega_j\in E_j}\right]\\ &= \mathbb{E}\left[\prod_{i=0}^{k+1} 1_{Y_i=x_i}\prod_{j=0}^{k} 1_{\omega_j\in E_j}\mathbb{E}\left[1_{\omega_{k+1}\in A_{k+1}} \mid \mathcal{G}'_{\tau_{k+1}-1}\right]\right].\end{aligned}$$

Since (2) is well-defined for $t\le\tau_{k+1}-1$, we have

$$\mathbb{E}[1_{\omega_{k+1}\in E_{k+1}} \mid \mathcal{G}'_{\tau_{k+1}-1}] = f_{k,E_{k+1}}(Y_0,\ldots,Y_{k+1},\omega_0,\ldots,\omega_k).$$

Hence, we can write

$$\mathbb{P}\left(\{Y_i=x_i\}_{i=0}^{k+1},\{\omega_j\in E_j\}_{j=0}^{k+1}\right) = \mathbb{E}[h_k(Y_0,\ldots,Y_{k+1},\omega_0,\ldots,\omega_k)]$$

where

$$h_k(Y_0,\ldots,Y_{k+1},\omega_0,\ldots,\omega_t) = f_{k,E_{k+1}}(Y_0,\ldots,Y_{k+1},\omega_0,\ldots,\omega_k)\cdot\prod_{i=0}^{k+1} 1_{Y_i=x_i}\prod_{j=0}^{k} 1_{\omega_j\in E_j}.$$

Since (3) holds for $t=k+1$, we have

$$\mathbb{E}[h_k(Y_0,\ldots,Y_{k+1},\omega_0,\ldots,\omega_k)] = \mathbb{E}[h_k(X_0,\ldots,X_{k+1},w_0,\ldots,w_k)].$$

Working backwards, we see that

$$\mathbb{E}[h_k(X_0,\ldots,X_{k+1},w_0,\ldots,w_k)] = \mathbb{P}\left(\{X_i=x_i\}_{i=0}^{k+1},\{w_j\in E_j\}_{j=0}^{k+1}\right).$$

This gives (4) for $t\le k+1$. By induction, we conclude our proof. □

4.4 Proof of Theorem 1

We are now ready to prove Theorem 1 in full generality.

Proof of Theorem 1. We aim to use Lemma 1. We will check the necessary conditions for G' and $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ constructed above. First, choose s' as the origin of G'. Then, s' has a single neighbor s and w'_0 is deterministic. Since $w_t(s,s') = 1$ for all $t \in \mathbb{N}$, combining this with Lemma 6 it follows that $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ is bounded. Finally, since the weights are frozen when $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ traverses along $\{s, s'\}$, it follows that

$$\sum_{t,e} |r'_t(e) - r'_{t+1}(e)| \overset{a.s.}{=} \sum_{k,e} |r'_{\tau_k}(e) - r'_{\tau_{k+1}}(e)| \overset{d}{=} \sum_{k,e} |r_k(e) - r_{k+1}(e)|$$

where the second equality is by Lemma 6. Hence, $\sum_{t,e} |r'_t(e) - r'_{t+1}(e)|$ is also bounded. To conclude, by Lemma 1, we see that $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$ inherits the recurrence or transience of (G', w'_0). Moreover, note that

$$\mathcal{R}'(s' \leftrightarrow \partial V'_{n+1}) = 1 + \mathcal{R}(s \leftrightarrow \partial V_n)$$

where $\mathcal{R}'$ is the effective resistance function on (G', w'_0) with s' as the origin and $\mathcal{R}$ is the effective resistance function on (G, w_0) with s as the origin. Hence, it follows that (G, w_0) and (G', w'_0) are either both recurrent or both transient. To conclude, if (G, w_0) is recurrent, it follows that (G', w'_0), then $\{\langle X'_t, w'_t\rangle\}_{t\in\mathbb{N}}$, then $\{\langle X_t, w_t\rangle\}_{t\in\mathbb{N}}$ are also recurrent where the last implication was discussed in section 4.1. The case is the same for transience and we conclude our proof. □

5 Acknowledgements

I am thankful to my advisor Professor Amir Dembo, for his various comments and suggestions that guided this research. I am also thankful to Dr. Lernik Asserian, the director of SURIM, for making the program possible. Finally, this work was built on ideas of Souvik Ray: We are preparing a manuscript together for submission of this work.

References

[1] G. Amir, I. Benjamini, O. Gurel-Gurevich, G. Kozma. Random walk in changing environment. *Stoch. Process. Their Appl.*, 130(12), 7463–7482, 2020.

[2] A. Collevecchio, D. Kious, V. Sidoravicius. The branching-ruin number and the critical parameter of once-reinforced random walk on trees. *Comm. Pure Appl. Math.* 73(1), 210–236, 2020.

[3] D. Coppersmith, P. Diaconis. Random walks with reinforcement. *Unpublished manuscript*, 1986.

[4] B. Davis. Reinforced random walk. *Probab. Theory Relat. Fields*, 84, 203–229, 1990.

[5] A. Dembo, R. Huang, B. Morris, Y. Peres. Transience in growing subgraphs via evolving sets. *Ann. Inst. H. Poincaré Probab. Statist.*, 53(3), 1164–1180, 2017.

[6] D. Kious, B. Schapira, A. Singh. Once reinforced random walk on $\mathbb{Z} \times \Gamma$. *Ann. Inst. H. Poincaré Probab. Statist.*, 57(4), 2219–2242, 2021.

[7] B. Toth. The "true" self-avoiding walk with bond repulsion on $\mathbb{Z}$: Limit theorems. *Ann. Probab.*, 23(4), 1523–1556, 1995.