A Four-Manifold Invariant Arising From Involutive Heegaard Floer Homology

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Abstract

Following an analogous construction to Ozsváth-Szabó's mixed invariant in ordinary Heegaard Floer theory, we define a mixed invariant using the cobordism maps on involutive Heegaard Floer homology. In particular, our construction produces a map $\Phi^I_{X,\mathfrak{s}}$ for any spin four-manifold $(X, \mathfrak{s})$ with connected (or empty) boundary such that $b_2^+(X) > 3$, and we show that this construction is independent of choices whenever $b_2^+(X) > 5$. We derive several properties of $\Phi^I_{X,\mathfrak{s}}$, including its relation to the Ozsváth-Szabó mixed invariant and its behavior under stabilizations.

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1 Introduction

In [1], Hendricks and Manolescu introduced involutive Heegaard Floer homology, a variant of Heegaard Floer homology which accounts for the natural conjugation symmetry on Heegaard diagrams given by swapping α and β curves and reversing the orientation of the splitting surface. They showed that, as in the original theory, their invariant fits into the framework of a (suitably modified) TQFT – namely, a spin cobordism between three-manifolds induces a map between their involutive Heegaard Floer homologies. It was shown in [2] that these cobordism maps are well-defined.

Many of the significant four-dimensional applications of Ozsváth and Szabó's original theory came through their *mixed invariant* $\Phi_{X,\mathfrak{s}}$ (defined in [3]) which, like the Seiberg-Witten invariant $SW_{X,\mathfrak{s}}$, is associated to a four-manifold X with $b_2^+ > 1$ and equipped with a spin^c structure $\mathfrak{s}$ (in fact, $\Phi_{X,\mathfrak{s}}$ and $SW_{X,\mathfrak{s}}$ are conjecturally equal). Such applications include re-proofs of the indecomposability of symplectic four-manifolds and the symplectic Thom conjecture.

In this paper, we construct a natural analogue of the mixed invariant in the setting of involutive Heegaard Floer theory – an *involutive mixed invariant* $\Phi^I_{X,\mathfrak{s}}$ – associated to spin four-manifolds $(X, \mathfrak{s})$ with $b_2^+ > 3$. At present, we have only managed to show that it is independent of choices made in its construction when $b_2^+ > 5$, though we conjecture that it is well-defined for $b_2^+ > 3$.

For closed spin four-manifolds, the involutive mixed invariant takes a particularly simple form. In the following theorem statement, we are treating elements $U^m + QU^n \in \mathbb{Z}_2[U,Q]/Q^2 \cong HFI^-(S^3)$ as vectors of the form

(U^m, U^n).

Theorem 1. *Let $(X, \mathfrak{s})$ be a closed spin four-manifold with $b_2^+ > 5$. Then*

$$\Phi^I_{X,\mathfrak{s}} = \begin{pmatrix} \Phi_{X,\mathfrak{s}} & 0 \\ \Psi_{X,\mathfrak{s}} & \Phi_{X,\mathfrak{s}} \end{pmatrix},$$

where $\Phi_{X,\mathfrak{s}}$ is the Ozsváth-Szabó mixed invariant, and $\Psi_{X,\mathfrak{s}}$ is a U-equivariant homomorphism. By grading considerations, $\Psi_{X,\mathfrak{s}}$ is nonzero only if $\Phi_{X,\mathfrak{s}}$ is zero.

Using the calculation of the involutive cobordism map induced by $(S^2 \times S^2, \mathfrak{s}_0)$ due to Hendricks, Hom, Stoffregen, and Zemke (see [2], Proposition 14.1), where $\mathfrak{s}_0$ is the unique spin structure on $S^2 \times S^2$, we obtain the following stabilization formula:

Theorem 2. *Let $(X, \mathfrak{s})$ be a closed spin four-manifold with $b_2^+(X) > 4$. Then, we have that:*

$$\Phi^I_{X\#(S^2\times S^2), \mathfrak{s}\#\mathfrak{s}_0} = Q\Phi_{X,\mathfrak{s}}.$$

In fact, the formula we prove in section 4.2 is more general, as it applies to manifolds with boundary and to b_2^+ as low as 3.

This research remains a work in progress. It is the author's intention to complete a more thorough investigation of $\Phi^I_{X,\mathfrak{s}}$ and its properties in the future.

1.1 Outline

In section 2, we review the basics of Heegaard Floer theory – including the definition of the Heegaard Floer homology groups, the formal properties of the theory as a 3+1 dimensional TQFT, and the mixed invariant construction – and then review the involutive version of the theory due to Hendricks and Manolescu. In section 3, we define the involutive mixed map for cobordisms and the involutive mixed invariant for four-manifolds (with connected or empty boundary). In section 4, we prove several properties of the invariant, including its relationship to Ozsváth-Szabó's invariant, its behavior under stabilization, and its particular form for closed manifolds. Finally, in section 5, we prove a weak form of the adjunction inequality and discuss possible directions for future application of the invariant.

1.2 Acknowledgements

I am grateful to my advisor, Ciprian Manolescu, for suggesting this idea and providing invaluable guidance, and to Irving Dai for teaching me most of what I know about Heegaard Floer theory. I am also grateful to Gary Guth for helpful discussions. Finally, I am grateful to Lernik Asserian for organizing SURIM, without which the project wouldn't have been possible.

2 Background on Heegaard Floer Theory

2.1 Heegaard Floer homology

Heegaard Floer homology is an invariant for closed, oriented three-manifolds, first introduced in [4]. The theory associates to a (pointed) Heegaard splitting $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ of a three-manifold Y an $\mathbb{F}[U]$-chain complex (where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$), denoted $CF^-(\mathcal{H})$ with differential ∂^-. More specifically, $CF^-(\mathcal{H})$ is the free $\mathbb{F}[U]$-module generated by the intersection points of the two tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \dots \beta_g$ inside of the symmetric space $\mathrm{Sym}^g(\Sigma_g)$, and ∂^- is computed by counting pseudoholomorphic strips (whenever the reduced moduli space of such strips $\widetilde{\mathcal{M}}(\phi)$ has dimension 0):

$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1\}} \#\widetilde{\mathcal{M}}(\phi) \cdot U^{n_z(\phi)} \cdot y.$$

In the above formula, $\pi_2(x, y)$ denotes the set of homology classes of Whitney disks from x to y, $n_z(\phi)$ counts intersection points of ϕ with $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ (which is always positive, by basic properties of holomorphic curves), and the condition $\mu(\phi) = 1$ ensures that $\widetilde{\mathcal{M}}$ has dimension 0 (μ is the *Maslov index*). The homology of this chain complex is the minus flavor of *Heegaard Floer homology* $HF^-(\mathcal{H})$. Note that, if $b_1(Y) > 0$, one must first modify the Heegaard splitting to be "admissible" in order to ensure this is well-defined, see [4].

There are several versions of the Heegaard Floer chain complex (and consequently several versions of homology): Localizing at U gives $CF^\infty(\mathcal{H})$; taking the cokernel of the localization map gives $CF^+(\mathcal{H})$; taking the quotient $CF^-(\mathcal{H})/U \cdot CF^-(\mathcal{H})$ yields $\widehat{CF}(\mathcal{H})$.

The choice of basepoint $z \in \Sigma_g$ partitions the set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ into classes, each of which corresponds to a spin^c structure on Y. This partition is such that there can exist a Whitney disk between x and y only if they are in the same class, so both the chain complexes and the homology groups split according to spin^c structures on Y. For a spin^c structure $\mathfrak{s}$, we denote the corresponding chain complex and homology group, respectively, by $CF^\circ(\mathcal{H}, \mathfrak{s})$ and $HF^\circ(\mathcal{H}, \mathfrak{s})$, where $\circ \in \{\widehat{\ }, +, -, \infty\}$.

It turns out that the isomorphism class of $HF^\circ(\mathcal{H}, \mathfrak{s})$ is independent of the Heegaard splitting $\mathcal{H}$ and depends only upon the diffeomorphism class of Y and the choice of spin^c structure $\mathfrak{s}$ [4]. In fact, the construction is natural in the following sense [5]:

Proposition 1. *For fixed Y, $\mathfrak{s}$, and z, and any two Heegaard splittings $\mathcal{H}$, $\mathcal{H}'$ compliant with those choices, there is a distinguished chain homotopy equivalence (unique up to chain homotopy)*

$$\Phi(\mathcal{H}, \mathcal{H}') : CF^\circ(\mathcal{H}, \mathfrak{s}) \to CF^\circ(\mathcal{H}', \mathfrak{s})$$

such that, for any $\mathcal{H}, \mathcal{H}', \mathcal{H}''$,

(1) $\Phi(\mathcal{H}, \mathcal{H}) \sim Id_{CF^\circ(\mathcal{H},\mathfrak{s})}$,

(2) $\Phi(\mathcal{H}', \mathcal{H}'') \circ \Phi(\mathcal{H}, \mathcal{H}') \sim \Phi(\mathcal{H}, \mathcal{H}'')$,

where $\sim$ indicates that the maps are chain homotopic.

We may therefore unambiguously write $HF^\circ(Y, \mathfrak{s})$, defined as the colimit of the above transitive system (or rather, the induced system of maps on homology).

The Heegaard Floer homology groups fit into various long exact sequences. The most important one for our purposes is the one induced by localization:

$$\xrightarrow{\delta} HF^-(Y, \mathfrak{s}) \xrightarrow{i_*} HF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi_*} HF^+(Y, \mathfrak{s}) \xrightarrow{\delta} \qquad (1)$$

HF^-_{red} is defined as the kernel of i_* and HF^+_{red} is defined as the cokernel of π_*. Note that δ induces an isomorphism between the two.

2.2 Cobordism maps

The content of this section can be found in [3]. Let Y_1, Y_2 be oriented three-manifolds, and W a cobordism from Y_1 to Y_2 (i.e. W is a smooth four-manifold with boundary $-Y_1 \sqcup Y_2$). Choosing a spin^c structure $\mathfrak{s}$ on W (as well as a path γ connecting basepoints, suppressed from the notation), there is a well-defined induced map on Floer homology

$$F^\circ_{W,\mathfrak{s}} : HF^\circ(Y_1, \mathfrak{s}|_{Y_1}) \to HF^\circ(Y_2, \mathfrak{s}|_{Y_2}).$$

The map $F^\circ_{W,\mathfrak{s}}$ depends only on the diffeomorphism type of W and the spin^c structure $\mathfrak{s}$.

This construction is functorial in the following sense: given W_1 a cobordism from Y_1 to Y_2 and W_2 a cobordism from Y_2 to Y_3, as well as spin^c structures $\mathfrak{s}_1 \in \mathrm{Spin}^c(W_1)$ and $\mathfrak{s}_2 \in \mathrm{Spin}^c(W_2)$ which agree on Y_2, we have:

$$F^\circ_{W_2,\mathfrak{s}_2} \circ F^\circ_{W_1,\mathfrak{s}_1} = \sum_{\{\mathfrak{s}\in \mathrm{Spin}^c(W):\mathfrak{s}|_{W_i}=\mathfrak{s}_i, i=1,2\}} F^\circ_{W,\mathfrak{s}}$$

where $W := W_1 \cup_{Y_2} W_2$. In other words, the composition of the two cobordism maps is the sum of all the cobordism maps induced by gluing the two cobordisms together and choosing a spin^c structure which restricts to the original spin^c structures on each of the two cobordisms.

Using their cobordism maps, Ozsváth and Szabó show that the Heegaard Floer homology groups associated to torsion spin^c structures admit an absolute $\mathbb{Q}$-grading. These absolute gradings are such that the cobordism maps satisfy the following grading shift formula (see [3], Theorem 7.1):

Proposition 2. *Let $(W, \mathfrak{s})$ be a cobordism from Y_1 to Y_2 such that $\mathfrak{s}$ restricts to being torsion on both boundary components. Then $F^\circ_{W,\mathfrak{s}}$ has a grading shift of*

$$\frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

2.2.1 The mixed invariant

Naively, one may wish to define an invariant of closed four-manifolds by removing two four-balls, treating it as a cobordism from S^3 to S^3, and looking at the induced map on Floer homology. Unfortunately, Proposition 3 below (from [3]), together with the fact that $HF_{\mathrm{red}}(S^3) = 0$, implies that all of these cobordism maps must vanish.

Proposition 3. *Let W be a cobordism with $b_2^+(W) > 0$. Then $F^\infty_{W,\mathfrak{s}} = 0$ for any $\mathfrak{s} \in Spin^c(W)$.*

However, Ozsváth and Szabó refine their construction to develop an interesting invariant, called the *mixed invariant* of $(X, \mathfrak{s})$ (see [3]): Assuming $b_2^+ > 1$, we can find a hypersurface $N \subset X$ which separates X into two subcobordisms, say, X_1 from S^3 to N and X_2 from N to S^3, such that $b_2^+(X_i) > 0$ and the map $\delta : H_2(N;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ from the Meier-Vietoris sequence is trivial (this latter condition ensures that the spin^c structure on X can be uniquely recovered from the spin^c structures on X_1 and X_2). Such an N is called an *admissible cut*.

Now, the cobordism maps $F^\infty_{X_1,\mathfrak{s}|_{X_1}}$, $F^\infty_{X_2,\mathfrak{s}|_{X_2}}$ must be trivial by Proposition 3. Thus, looking at the long exact sequence (1), it follows that $\mathrm{im}(F^-_{X_1,\mathfrak{s}|_{X_1}}) \subset HF^-(N, \mathfrak{s}|_N)$ is in fact contained in $HF^-_{\mathrm{red}}(N, \mathfrak{s}|_N)$. Meanwhile, the triviality of $F^\infty_{X_2,\mathfrak{s}|_{X_2}}$ implies that $F^+_{W_2,\mathfrak{s}|_{X_2}}$ descends to $HF^+_{\mathrm{red}}(N, \mathfrak{s}|_N)$. Thus, we may define the composition

$$\Phi_{X,\mathfrak{s}} := F^+_{X_2,\mathfrak{s}|_{X_2}} \circ \delta^{-1} \circ F^-_{X_1,\mathfrak{s}|_{X_1}}$$

to get a "mixed map" from $HF^-(S^3)$ to $HF^+(S^3)$, which is our desired invariant. It turns out that $\Phi_{X,\mathfrak{s}}$ is independent of our choice of admissible cut, and depends only on the diffeomorphism type of X and on $\mathfrak{s}$.

2.3 Involutive Heegaard Floer homology

There is a natural symmetry on Heegaard diagrams given by "conjugating" the Heegaard data: $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \mapsto \overline{\mathcal{H}} = (-\Sigma_g, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ (which induces isomorphisms $HF^\circ(Y, \mathfrak{s}) \to HF^\circ(Y, \bar{\mathfrak{s}})$). The naturality of the Heegaard Floer chain complex – in the sense of Proposition 1 – provides a distinguished chain homotopy equivalence $\Phi(\mathcal{H}, \overline{\mathcal{H}})$ corresponding to the above change in Heegaard splitting. This gives rise to a homotopy involution ι on the Heegaard Floer chain complex, which is used by Hendricks and Manolescu in [1] to define *involutive Heegaard Floer homology* $HFI^\circ(Y)$. More specifically, the involutive Heegaard Floer homology of the pair (Y, ϖ), where ϖ is a conjugation class of spin^c structures on Y, is the homology of the mapping cone:

$$CF^\circ(Y, \varpi) \xrightarrow{Q(1+\iota)} Q \cdot CF^\circ(Y, \varpi)[-1]$$

where Q is a formal variable with $Q^2 = 0$. Here, we are defining $CF^\circ(Y, \varpi)$ to be the sum

$$\bigoplus_{\mathfrak{s} \in \varpi} CF^\circ(Y, \mathfrak{s}).$$

Note that ϖ contains either one or two elements, and it contains one if and only if it corresponds to a spin structure.

2.3.1 Involutive cobordism maps

Using a handle-by-handle construction similar to that of Ozsváth and Szabó, Hendricks and Manolescu also construct cobordism maps in involutive Heegaard Floer homology for pairs (W, ϖ), where ϖ is a conjugation class of spin^c structures on W (see Proposition 4.9 in [1]). They show that these maps have very similar formal properties to those from the original theory. However, they do not show that the maps are independent of the choices made in their construction. This was later shown specifically for maps induced by *spin* cobordisms in [2]. As such, most of our constructions in this paper are limited to spin structures (that is, self-conjugate spin^c structures).

3 An Involutive Mixed Map

We begin with an algebraic lemma.

Lemma 1. *Suppose we have the following diagram in which both squares commute, and the middle row is exact:*

$$\begin{array}{ccccc} & & A_2 & \xrightarrow{a_2} & A_3 \\ & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle 0} \\ B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle F_2} & & \\ C_1 & \xrightarrow{c_1} & C_2 & & \end{array}$$

Then $F_2 \circ F_1 = 0$.

Proof. The proof is a standard diagram chase. The commutativity of the upper right square implies that the image of F_1 lies inside the kernel of b_2, which by exactness is equal to the image of b_1. The commutativity of the lower left square implies that F_2 vanishes on the image of b_1, hence F_2 vanishes on the image of F_1.

□

Using this, we may prove a vanishing property for the map on HFI^∞ under certain topological conditions, in analogy with Lemma 8.2 of [3]:

Lemma 2. *Let W be a smooth cobordism from Y_1 to Y_2 with $b_2^+ > 1$, and let ϖ be a conjugation class of spin^c structures on W. Then the map*

$$F^{I,\infty}_{W,\mathfrak{s}} : HFI^\infty(Y_1, \varpi|_{Y_1}) \to HFI^\infty(Y_2, \varpi|_{Y_2})$$

vanishes.

Proof. We first diagonalize the intersection form of W over $\mathbb{Q}$. Due to the condition $b_2^+ > 1$, at least two of the diagonal entries will be positive. After multiplying by a sufficiently large positive integer, it follows that there exists a pair of $\mathbb{Z}$-homology classes β_1 and β_2 which satisfy $\beta_i^2 > 0$ $(i = 1, 2)$ and $\beta_1 \cdot \beta_2 = 0$. Let Σ_i be a smoothly embedded surface in W representing β_i (for $i = 1, 2$), and let Q_i denote the boundary of a tubular neighborhood of Σ_i. Note that since $\beta_1 \cdot \beta_2 = 0$, we can choose Σ_1 and Σ_2 such that they are disjoint (pairs of canceling intersection points can be surgered out, cf. proof of Lemma 8.6 in [3]), and hence so are Q_1 and Q_2 by choosing small enough tubular neighborhoods.

Fix a path from Y_1 to Q_1 away from Q_2, and take a regular neighborhood to decompose W into a cobordism W_1 from Y_1 to $Y_1\#Q_1$ followed by a cobordism W_2 from $Y_1\#Q_1$ to Y_2. Note that $\Sigma_1 \subset W_1$ and $\Sigma_2 \subset W_2$, so $b_2^+(W_i) > 0$ for $i = 1, 2$. By Lemma 8.2 of [3], the map $F_i := F^\infty_{W_i,\varpi|_{W_i}}$ vanishes for $i = 1, 2$. By Proposition 4.9 of [1], we have the following commutative diagram:

$$\begin{array}{ccccc}
QHF^\infty(Y_1,\varpi|_{Y_1})[-1] & \xrightarrow{(g^\infty_{Y_1})_*} & HFI^\infty(Y_1,\varpi|_{Y_1}) & \xrightarrow{(h^\infty_{Y_1})_*} & HF^\infty(Y_1,\varpi|_{Y_1}) \\
\downarrow 0 & & \downarrow F_1^I & & \downarrow 0 \\
QHF^\infty(Y_1\#Q_1,\varpi|_{Y_1\#Q_1})[-1] & \xrightarrow{(g^\infty_{Y_1\#Q_1})_*} & HFI^\infty(Y_1\#Q_1,\varpi|_{Y_1\#Q_1}) & \xrightarrow{(h^\infty_{Y_1\#Q_1})_*} & HF^\infty(Y_1\#Q_1,\varpi|_{Y_1\#Q_1}) \\
\downarrow 0 & & \downarrow F_2^I & & \downarrow 0 \\
QHF^\infty(Y_2,\varpi|_{Y_2})[-1] & \xrightarrow{(g^\infty_{Y_2})_*} & HFI^\infty(Y_2,\varpi|_{Y_2}) & \xrightarrow{(h^\infty_{Y_2})_*} & HF^\infty(Y_2,\varpi|_{Y_2})
\end{array}$$

where $F_i^I := F^{I,\infty}_{W_i,\varpi|_{W_i}}$. From the composition law (Proposition 4.11 of [1]) and the fact that $\delta H^1(Y_1\#Q_1;\mathbb{Z}) = 0 \subset H^2(W,\partial W;\mathbb{Z})$, it follows that

$$F^{I,\infty}_{W,\varpi} = F_2^I \circ F_1^I,$$

which, by Lemma 1 applied to the above diagram, must vanish. □

In analogy with Ozsváth-Szabó's definition of an admissible cut for a cobordism, we make the following definition:

Definition 1. *An involutively admissible cut of W is a smoothly embedded three-manifold $N \hookrightarrow W$ which splits W into two pieces W_1, W_2 each with $b_2^+ > 1$, such that $\delta H^1(N;\mathbb{Z}) \subset H^2(W, \partial W;\mathbb{Z})$ is trivial.*

Just as any W with $b_2^+ > 1$ admits an admissible cut [3], we have the following:

Proposition 4. *Any cobordism W with $b_2^+ > 3$ admits an involutively admissible cut.*

Proof. As in the proof of Lemma 2 above, we diagonalize the intersection form of W over $\mathbb{Q}$ to find four mutually disjoint positive self-intersection surfaces Σ_i, where $i \in \{1, 2, 3, 4\}$. Take Q_i the boundary of a tubular neighborhood of Σ_i such that the Q_i for $i \in \{1, 2, 3, 4\}$ are also mutually disjoint. Fix paths γ_1 from Y_1 to Q_1 and γ_2 from Y_1 to Q_2 such that γ_i avoids Q_j for all $j \neq i$, and such that γ_1 and γ_2 avoid each other. Use γ_1 and γ_2 to form the connect sum $Y_1 \# Q_1 \# Q_2$, and in this way decompose W into W_1 from Y_1 to $Y_1 \# Q_1 \# Q_2$ and W_2 from $Y_1 \# Q_1 \# Q_2$ to Y_2 (note that we are treating Y_1 in the sum $Y_1 \# Q_1 \# Q_2$ as a slight perturbation of Y_1 into the collar neighborhood $Y_1 \times [0, 1] \cong \text{nbhd}(Y_1) \subset W$). Both W_1 and W_2 have $b_2^+ > 1$ (namely, W_1 contains Σ_1 and Σ_2, while W_2 contains Σ_3 and Σ_4), and one can check that $\delta H^1(Y_1 \# Q_1 \# Q_2) = 0$ inside $H^2(W, \partial W)$. Thus, $N := Y_1 \# Q_1 \# Q_2$ is an involutively admissible cut. $\square$

If W has an involutively admissible cut N, then we can construct the *involutive mixed map* $F^{I,\text{mixed}}_{W,\varpi} : HFI^-(Y_1, \varpi|_{Y_1}) \to HFI^+(Y_2, \varpi|_{Y_2})$ in exactly the same way as Ozsváth-Szabó's mixed map for regular Heegaard Floer homology. We review the construction here for the sake of being explicit. Let W_1 and W_2 be the two cobordisms into which N cuts W. In the following commutative diagram,

$$\begin{array}{ccccccccc}
\longrightarrow & HFI^-(Y_1, \varpi|_{Y_1}) & \xrightarrow{i_*} & HFI^\infty(Y_1, \varpi|_{Y_1}) & \xrightarrow{j_*} & HFI^+(Y_1, \varpi|_{Y_1}) & \longrightarrow \\
& \downarrow F^{I,-}_{W_1,\varpi|_{W_1}} & & \downarrow F^{I,\infty}_{W_1,\varpi|_{W_1}} & & \downarrow F^{I,+}_{W_1,\varpi|_{W_1}} & \\
\longrightarrow & HFI^-(N, \varpi|_N) & \xrightarrow{i_*} & HFI^\infty(N, \varpi|_N) & \xrightarrow{j_*} & HFI^+(N, \varpi|_N) & \longrightarrow
\end{array}$$

we have that $F^{I,\infty}_{W_1,\varpi|_{W_1}} = 0$ by Lemma 2. Thus, by commutativity of the left square, we must have that $i_* \circ F^{I,-}_{W_1,\varpi|_{W_1}} = 0$, so that im $F^{I,-}_{W_1,\varpi|_{W_1}} \subset \ker i_* = HFI^-_{\text{red}}(N, \varpi|_N)$. Meanwhile, in the commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow HFI^-(N,\varpi|_N) & \xrightarrow{i_*} & HFI^\infty(N,\varpi|_N) & \xrightarrow{j_*} & HFI^+(N,\varpi|_N) \longrightarrow \\
\downarrow F^{I,-}_{W_2,\varpi|_{W_2}} & & \downarrow F^{I,\infty}_{W_2,\varpi|_{W_2}} & & \downarrow F^{I,+}_{W_2,\varpi|_{W_2}} \\
\longrightarrow HFI^-(Y_2,\varpi|_{Y_2}) & \xrightarrow{i_*} & HFI^\infty(Y_2,\varpi|_{Y_2}) & \xrightarrow{j_*} & HFI^+(Y_2,\varpi|_{Y_2}) \longrightarrow
\end{array}$$

Lemma 2 once again implies that $F^{I,\infty}_{W_2,\varpi|_{W_2}} = 0$. Commutativity of the right square gives that $F^{I,+}_{W_2,\varpi|_{W_2}} \circ \pi_* = 0$, so $\ker \delta_* = \text{im } \pi_* \subset \ker F^{I,+}_{W_2,\varpi|_{W_2}}$. Therefore $F^{I,+}_{W_2,\varpi|_{W_2}}$ descends to a well-defined map on $HFI^+(N,\varpi|_N)/\ker \delta_* = HFI^+_{\text{red}}(N,\varpi|_N)$. Note that the boundary map δ induces an isomorphism $\overline{\delta} : HFI^+_{\text{red}}(N,\varpi|_N) \to HFI^-_{\text{red}}(N,\varpi|_N)$, so we can define:

$$F^{I,\text{mixed}}_{W,\varpi} := \overline{F}^{I,+}_{W_2,\varpi|_{W_2}} \circ \overline{\delta}^{-1} \circ F^{I,-}_{W_1,\varpi|_{W_1}}.$$

At this point, before showing this is well-defined, we restrict ourselves to the case where $\mathfrak{s}$ is self-conjugate (i.e. spin). It is proved in [2] that the involutive cobordism maps are independent of handle decomposition for $\mathfrak{s}$ spin, so the Proposition which follows implies that $F^{I,\text{mixed}}_{W,\mathfrak{s}}$ is a genuine invariant. Indeed, as in the case of the original mixed map on Heegaard Floer homology, our construction is independent of the choice of cut for sufficiently large b_2^+.

Proposition 5. *Let $(W,\mathfrak{s})$ be a spin cobordism with $b_2^+(W) > 5$. The map $F^{I,mixed}_{W,\mathfrak{s}}$ does not depend on the choice of involutively admissible cut N.*

Proof. Suppose first that N and N' be disjoint involutively admissible cuts. Then the conclusion follows from the same argument as [Ozsváth-Szabó], Lemma 8.6 – namely, by analyzing the commutative diagram:

$$\begin{array}{ccccc}
 & & HFI^+_{\text{red}}(N,\mathfrak{s}|_N) & \longrightarrow & HFI^+_{\text{red}}(N',\mathfrak{s}|_{N'}) \longrightarrow HFI^+(Y_2,\mathfrak{s}|_{Y_2}) \\
 & & \downarrow \overline{\delta} & & \downarrow \overline{\delta} \\
HFI^-(Y_1,\mathfrak{s}|_{Y_1}) & \longrightarrow & HFI^-_{\text{red}}(N,\mathfrak{s}|_N) & \longrightarrow & HFI^-_{\text{red}}(N',\mathfrak{s}|_{N'})
\end{array}$$

Here, all horizontal maps are induced by cobordisms.

Now, let N and N' be arbitrary involutively admissible cuts, with N breaking W up into the pieces W_1 and W_2 and N' breaking up W into the pieces W_1' and W_2'. Let Σ_1, Σ_2 be a disjoint pair of positively self-intersecting surfaces in W_1, and let Q_1, Q_2 denote the boundaries of their tubular neighborhoods, respectively. Then $Y_1 \# Q_1 \# Q_2$ is an involutively admissible cut which, by the disjoint case above, gives the same mixed map as N. Meanwhile, let Σ_1', Σ_2' be a disjoint pair of positively self-intersecting surfaces in W_1', with neighborhood boundaries Q_1', Q_2', respectively. Then $Y_1 \# Q_1' \# Q_2'$ is an involutively admissible cut which gives the same mixed map as N'.

On the other hand, by the assumption that $b_2^+ \geq 6$, we can choose an additional disjoint pair of positively self-intersecting surfaces $\Sigma_3, \Sigma_4 \subset W$ which are disjoint from all of the surfaces in the set $\{\Sigma_1, \Sigma_2, \Sigma_1', \Sigma_2'\}$. Letting Q_3, Q_4 denote the boundaries of the tubular neighborhoods of Σ_3 and Σ_4, respectively, we observe that $Y_2 \# Q_3 \# Q_4$ gives another involutively admissible cut which is disjoint from both $Y_1 \# Q_1 \# Q_2$ and $Y_1 \# Q_1' \# Q_2'$, and therefore yields the same mixed map as both of them. It follows that N and N' both yield the same mixed map as $Y_2 \# Q_3 \# Q_4$, and hence the same mixed map as one another.

□

Note that, unfortunately, Ozsváth and Szabó's strategy of using blow-ups to prove invariance of their mixed map for $b_2^+ = 2$ does not transfer over to prove invariance of our mixed map for $4 \leq b_2^+ \leq 5$, since the blow-up of a spin manifold is no longer spin.

Now, let $(X, \mathfrak{s})$ be a spin four-manifold with connected boundary $\partial X = Y$. Assume that $b_2^+(X) > 3$. Removing an open ball from X, we obtain a cobordism $\mathring{X}$ from S^3 to Y, for which there exists a well-defined mixed map:

$$F^{I,\mathrm{mix}}_{\mathring{X},\mathfrak{s}} : \mathit{HFI}^-(S^3) \cong \mathbb{Z}_2[U,Q]/Q^2 \to \mathit{HFI}^+(Y, \mathfrak{s}|_Y).$$

Definition 2. *The involutive mixed invariant of* $(X, \mathfrak{s})$ *is* $\Phi^I_{X,\mathfrak{s}} := F^{I,mix}_{\mathring{X},\mathfrak{s}}$. *For a closed four-manifold* X *with* $b_2^+ > 3$*, we define* $\Phi^I_{X,\mathfrak{s}} := \Phi^I_{X \setminus \mathring{B}^4,\mathfrak{s}}$.

4 Properties of the Involutive Mixed Invariant

4.1 Relationship to the Ozsváth-Szabó Mixed Invariant

In Proposition 4.9 of [1], Hendricks and Manolescu show that the involutive cobordism maps commute with the long exact sequence relating involutive Heegaard Floer homology to ordinary Heegaard Floer homology. The goal of this section is to prove a similar fact for the mixed invariant.

Theorem 3. *Let $(X, \mathfrak{s})$ be a spin four-manifold with boundary Y. The following diagram commutes:*

$$\begin{array}{ccccc} QHF^-(S^3)[-1] \cong Q\mathbb{Z}_2[U][-1] & \xrightarrow{j} & HFI^-(S^3) \cong \mathbb{Z}_2[U,Q]/Q^2 & \xrightarrow{p} & HF^-(S^3) \cong \mathbb{Z}_2[U] \\ \downarrow \Phi_{X,\mathfrak{s}} & & \downarrow \Phi^I_{X,\mathfrak{s}} & & \downarrow \Phi_{X,\mathfrak{s}} \\ QHF^+(Y, \mathfrak{s}|_Y)[-1] & \xrightarrow{g_*} & HFI^+(Y, \mathfrak{s}|_Y) & \xrightarrow{h_*} & HF^+(Y, \mathfrak{s}|_Y) \end{array}$$

where $j : Q\mathbb{Z}_2[U] \hookrightarrow \mathbb{Z}_2[U,Q]/Q^2$ is the natural inclusion, $p : \mathbb{Z}_2[U,Q]/Q^2 \twoheadrightarrow \mathbb{Z}_2[U]$ is the projection given by setting $Q = 0$, g_* is induced by inclusion $g : QCF^+(Y, \mathfrak{s}|_Y)[-1] \to CFI^+(Y, \mathfrak{s}|_Y)$, and h_* is induced by projection $h : CFI^+(Y, \mathfrak{s}|_Y) \to CF^+(Y, \mathfrak{s}|_Y)$.

Since we know that the cobordism maps commute, our main focus is on showing the appropriate commutativity for the inverse of the boundary map on HFI_{red} and HF_{red}.

Lemma 3. *The following diagram commutes for any three-manifold Z and spin-structure $\mathfrak{t}$:*

$$\begin{array}{ccccc} QHF^+(Z, \mathfrak{t})[-1] & \xrightarrow{g_*^+} & HFI^+(Z, \mathfrak{t}) & \xrightarrow{h_*^+} & HF^+(Z, \mathfrak{t}) \\ \downarrow \delta & & \downarrow \delta^I & & \downarrow \delta \\ QHF^-(Z, \mathfrak{t})[-1] & \xrightarrow{g_*^-} & HFI^-(Z, \mathfrak{t}) & \xrightarrow{h_*^-} & HF^-(Z, \mathfrak{t}) \end{array}$$

where g_*° is induced by inclusion and h_*° is induced by projection.

Proof. This follows from basic homological algebra applied to the following commutative diagram, in which all maps are the obvious chain maps:

$$\begin{array}{ccccc} QCF^{-}(Z,\mathfrak{t})[-1] & \longrightarrow & CFI^{-}(Z,\mathfrak{t}) & \longrightarrow & CF^{-}(Z,\mathfrak{t}) \\ \downarrow & & \downarrow & & \downarrow \\ QCF^{\infty}(Z,\mathfrak{t})[-1] & \longrightarrow & CFI^{\infty}(Z,\mathfrak{t}) & \longrightarrow & CF^{\infty}(Y,\mathfrak{t}) \\ \downarrow & & \downarrow & & \downarrow \\ QCF^{+}(Z,\mathfrak{t})[-1] & \longrightarrow & CFI^{+}(Z,\mathfrak{t}) & \longrightarrow & CF^{+}(Z,\mathfrak{t}) \end{array}$$

□

Lemma 4. *Let Z be a three-manifold and $\mathfrak{t}$ be a spin structure on Z. Consider the sequence in Floer homology*

$$QHF^{\pm}(Z,\mathfrak{t})[-1] \xrightarrow{g_*^{\pm}} HFI^{\pm}(Z,\mathfrak{t}) \xrightarrow{h_*^{\pm}} HF^{\pm}(Z,\mathfrak{t}).$$

Then g_*^- and h_*^- (resp. g_*^+ and h_*^+) restrict (resp. descend) to give well-defined maps

$$g_*^{\pm} : QHF^{\pm}_{red}(Z,\mathfrak{t})[-1] \to HFI^{\pm}_{red}(Z,\mathfrak{t}),$$
$$h_*^{\pm} : HFI^{\pm}_{red}(Z,\mathfrak{t}) \to HF^{\pm}_{red}(Z,\mathfrak{t}).$$

Proof. We will prove the statement for $g_*^{\pm}$ only. The proofs for $h_*^{\pm}$ are nearly identical. We begin with g_*^-. The minus-flavor reduced Floer groups can be characterized as the image of the appropriate boundary operator. So we may write any $x \in QHF^-(Z,\mathfrak{t})[-1]$ as $x = \delta y$ for some $y \in QHF^+(Z,\mathfrak{t})[-1]$. By Lemma 3, we have that

$$g_*^-(\delta y) = \delta^I g_*^+(y) \in \text{im } \delta^I = HFI^-_{\text{red}}(Z,\mathfrak{t}).$$

This proves the claim about g_*^-. Now, to show the claim for g_*^+, we must show that g_*^+ sends elements of $\ker \delta$ to elements of $\ker \delta^I$. Indeed, if $x \in \ker \delta$, then, again by Lemma 3, we have

$$\delta^I g_*^+(x) = g_*^-(\delta x) = g_*^-(0) = 0.$$

This concludes the proof. □

We will now prove Theorem 3.

Proof. We denote by W the cobordism $\mathring{X}$ from S^3 to Y, by N an involutively admissible cut of W, and by W_i, $i \in \{1, 2\}$, the two pieces into which N cuts W. We expand out the diagram stated in the theorem using the definitions of $\Phi^I_{X,\mathfrak{s}}$ and $\Phi_{X,\mathfrak{s}}$, with no assumption of commutativity.

$$
\begin{array}{ccccc}
QHF^-(S^3)[-1] & \xrightarrow{\;j\;} & HFI^-(S^3) & \xrightarrow{\;p\;} & HF^-(S^3) \\
\Big\downarrow {\scriptstyle F^-_{W_1,\mathfrak{s}|_{W_1}}} & & \Big\downarrow {\scriptstyle F^{I,-}_{W_1,\mathfrak{s}|_{W_1}}} & & \Big\downarrow {\scriptstyle F^-_{W_1,\mathfrak{s}|_{W_1}}} \\
QHF^-_{\mathrm{red}}(N,\mathfrak{s}|_N)[-1] & \xrightarrow{\;k^-_*\;} & HFI^-_{\mathrm{red}}(N,\mathfrak{s}|_N) & \xrightarrow{\;\ell^-_*\;} & HF^-_{\mathrm{red}}(N,\mathfrak{s}|_N) \\
\Big\downarrow {\scriptstyle \overline{\delta}^{-1}} & & \Big\downarrow {\scriptstyle \overline{\delta}^{-1}} & & \Big\downarrow {\scriptstyle \overline{\delta}^{-1}} \\
QHF^+_{\mathrm{red}}(N,\mathfrak{s}|_N)[-1] & \xrightarrow{\;k^+_*\;} & HFI^+_{\mathrm{red}}(N,\mathfrak{s}|_N) & \xrightarrow{\;\ell^+_*\;} & HF^+_{\mathrm{red}}(N,\mathfrak{s}|_N) \\
\Big\downarrow {\scriptstyle \overline{F}^+_{W_2,\mathfrak{s}|_{W_2}}} & & \Big\downarrow {\scriptstyle \overline{F}^{I,+}_{W_2,\mathfrak{s}|_{W_2}}} & & \Big\downarrow {\scriptstyle \overline{F}^+_{W_2,\mathfrak{s}|_{W_2}}} \\
QHF^+(Y,\mathfrak{s}|_Y) & \xrightarrow{\;g_*\;} & QHFI^+(Y,\mathfrak{s}|_Y) & \xrightarrow{\;h_*\;} & HF^+(Y,\mathfrak{s}|_Y)
\end{array} \tag{2}
$$

where the functions $k^{\pm}_*$ and $\ell^{\pm}_*$ were shown to exist in Lemma 4. Now, the commutativity of the top and bottom pairs of squares follows from Proposition 4.9 of [1]. Meanwhile, the commutativity of the middle pair follows from Lemma 3, along with the fact that $\overline{\delta}$ is an isomorphism in each column. The commutativity of each square implies the commutativity of the entire diagram. □

By virtue of the surjectivity of p above, Theorem 3 immediately implies

Corollary 1. *If* $\Phi^I_{X,\mathfrak{s}} = 0$*, then* $\Phi_{X,\mathfrak{s}} = 0$*.*

4.2 Stabilization Formula

The involutive cobordism map for a twice punctured $S^2 \times S^2$ with its unique spin structure $\mathfrak{s}_0$ was computed in [2]:

Theorem 4. *The cobordism map*

$$F^{I,-}_{S^2\times S^2,\mathfrak{s}_0} : HFI^-(S^3) \to HFI^-(S^3)$$

is multiplication by Q*.*

Recall that the *stabilization* of a four-manifold X (possibly with boundary) is the (interior) connect sum $X\#(S^2 \times S^2)$. Theorem 4 is useful for computing the involutive cobordism maps induced by stabilizations of four-manifolds. We will use it to compute the involutive mixed invariant of four-manifolds after one stabilization.

Let $(X, \mathfrak{s})$ be a smooth, spin four-manifold with $b_2^+ \geq 3$ and boundary $\partial X = Y$ (replace X with $\mathring{X}$ if X is closed). Choose an admissible cut N (in the sense of [3], Definition 8.3) of X such that $b_2^+(X_2) > 1$ and $b_2^+(X_1) > 0$. Moreover, fix a map, in the category of sets,

$$\mathfrak{h} : HFI^-(N, \mathfrak{s}|_N) \to HFI^-_{\mathrm{red}}(N, \mathfrak{s}|_N)$$

which is the identity on $HFI^-_{\mathrm{red}}(N, \mathfrak{s}|_N)$.

Now, we define the involutive mixed invariant of the quadruple $(X, \mathfrak{s}, N, \mathfrak{h})$ to be:

$$\Phi^I_{X,\mathfrak{s},N,\mathfrak{h}} := F^{I,+}_{X_2,\mathfrak{s}|_{X_2}} \circ \overline{\delta}^{-1} \circ \mathfrak{h} \circ F^{I,-}_{X_1,\mathfrak{s}|_{X_1}}.$$

Note that, in the case that X has $b_2^+ > 3$ and we choose N to be an involutively admissible cut, this is precisely the involutive mixed invariant of $(X, \mathfrak{s}, N)$, which in turn is independent of N if $b_2^+ > 5$.

We are now ready to state the stabilization formula.

Theorem 5. *Let $(X, \mathfrak{s})$ be a smooth, spin four-manifold with connected boundary Y. Assume that $b_2^+(X) \geq 3$, and fix a choice of N and $\mathfrak{h}$ as above. Let X' denote X stabilized once via an interior connect sum with $S^2 \times S^2$, and let $\mathfrak{s}' := \mathfrak{s}\#\mathfrak{s}_0$. Assume that the gluing for the stabilization occurred in the interior of X_1. Then:*

$$\Phi^I_{X',\mathfrak{s},N} = Q \cdot \Phi^I_{X,\mathfrak{s},N,\mathfrak{h}}.$$

Proof. Note that $b_2^+(S^2 \times S^2) = 1$, so in particular $b_2^+(X') = b_2^+(X) + 1$. Due to the choice of N and the choice of where the stabilization occurred, N is an involutively admissible cut of X'. By definition, we have that

$$\Phi^I_{X',\mathfrak{s},N} = F^{I,+}_{X_2,\mathfrak{s}|_{X_2}} \circ \overline{\delta}^{-1} \circ F^{I,-}_{X_1\#(S^2\times S^2),\mathfrak{s}'|_{X_1\#(S^2\times S^2)}}.$$

Now, since the map $F^{I,-}_{X_1\#(S^2\times S^2),\mathfrak{s}'|_{X_1\#(S^2\times S^2)}}$ maps into $HFI^-_{\mathrm{red}}(N, \mathfrak{s}|_N)$, applying $\mathfrak{h}$ makes no difference on the overall result, so we may write:

$$\Phi^I_{X',\mathfrak{s},N} = F^{I,+}_{X_2,\mathfrak{s}|_{X_2}} \circ \overline{\delta}^{-1} \circ \mathfrak{h} \circ F^{I,-}_{X_1\#(S^2\times S^2),\mathfrak{s}'|_{X_1\#(S^2\times S^2)}}.$$

Finally, by the composition law for cobordism maps and Theorem 4, we get that:

$$\begin{aligned} F^{I,-}_{X_1\#(S^2\times S^2),\mathfrak{s}'|_{X_1\#(S^2\times S^2)}} &= F^{I,-}_{X_1,\mathfrak{s}|_{X_1}} \circ F^{I,-}_{S^2\times S^2,\mathfrak{s}_0} \\ &= QF^{I,-}_{X_1,\mathfrak{s}|_{X_1}}. \end{aligned}$$

Plugging this back in gives the result. □

The stabilization formula takes a particularly simple form for closed manifolds.

Corollary 2. *Let $(X,\mathfrak{s})$ be a closed spin four-manifold with $b_2^+ \geq 3$. Choose N as above, and let X' denote X once stabilized, where we performed the stabilization in X_1. Then:*

$$\Phi^I_{X,\mathfrak{s},N} = Q\Phi_{X,\mathfrak{s}}$$

where $\Phi_{X,\mathfrak{s}}$ is the Ozsváth-Szabó mixed invariant.

Proof. The proof of Theorem 3 implies that $\Phi_{X,\mathfrak{s}}$ is the only part of $\Phi^I_{X,\mathfrak{s},N,\mathfrak{h}}$ which doesn't die after multiplication by Q (see the beginning of the next subsection). With this observation, the result follows immediately from Theorem 5.

□

4.3 $\Phi^I_{X,\mathfrak{s}}$ for closed four-manifolds

For closed spin four-manifolds, Theorem 3 implies that $\Phi^I_{X,\mathfrak{s}}$ takes the form

$$\Phi^I_{X,\mathfrak{s}} = \begin{pmatrix} \Phi_{X,\mathfrak{s}} & 0 \\ \Psi_{X,\mathfrak{s}} & \Phi_{X,\mathfrak{s}} \end{pmatrix}$$

where we are treating elements of the form $U^n + QU^m \in \mathbb{Z}_2[U,Q]/Q^2$ as vectors (U^n, U^m), and $\Psi_{X,\mathfrak{s}}$ is some U-equivariant homomorphism which is defined by the above formula. In other words, we have that:

$$\Phi^I_{X,\mathfrak{s}}(U^n + QU^m) = \Phi_{X,\mathfrak{s}}(U^n) + Q\Phi_{X,\mathfrak{s}}(U^m) + Q\Psi_{X,\mathfrak{s}}(U^n),$$

for some $\Psi_{X,\mathfrak{s}}$. By U-equivariance, it is enough to know $\Phi_{X,\mathfrak{s}}(1)$ and $\Psi_{X,\mathfrak{s}}(1)$ to completely determine $\Phi^I_{X,\mathfrak{s}}$. We make the following basic observation:

Proposition 6. *If* $\Phi_{X,\mathfrak{s}} \neq 0$*, then* $\Psi_{X,\mathfrak{s}} = 0$*.*

Proof. If they were both nonzero, $\Phi_{X,\mathfrak{s}}(1)$ and $Q\Psi_{X,\mathfrak{s}}(1)$ would lie in different towers, hence have different grading parities (and, in particular, different gradings). But they should both have grading equal to the grading shift of $\Phi^I_{X,\mathfrak{s}}$, which is equal to $d(\mathfrak{s})$, the dimension of the Seiberg-Witten moduli space. Therefore, at least one of them must be zero. □

More generally, the above argument shows that

Proposition 7. *If* X *is closed and*

$$d(\mathfrak{s}) := \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)}{4}$$

is an even integer, then $\Phi^I_{X,\mathfrak{s}} = \Phi_{X,\mathfrak{s}}$*, extended* Q*-equivariantly. In particular, this is true for all closed four-manifolds* X *for which* $b_2^+(X) - b_1(X)$ *is odd.*

However, $\Psi_{X,\mathfrak{s}}$ is not always zero, as the stabilization formula shows. Due to the general difficulty of computing involutive cobordism maps, we have yet to find any examples of four-manifolds with nontrivial $\Psi_{X,\mathfrak{s}}$ which are not obtained through stabilization.

5 Limitations and Plans for Future Work

5.1 A note about adjunction

There are several technical impediments to obtaining a nontrivial adjunction inequality – for example, one comparable to [6] Theorem 4.7 – with the hypothesis that $\Phi^I_{X,\mathfrak{s}} \neq 0$ replacing $\Phi_{X,\mathfrak{s}} \neq 0$. For instance, in order to address the case where $[\Sigma]^2 \geq 0$, the standard proof involves blowing up (i.e. taking a connect sum with $\overline{\mathbb{CP}^2}$) to reduce to the case $[\Sigma]^2 = 0$. However, the blow-up of a spin four-manifold is no longer spin, so we would not have a well-defined involutive mixed invariant in that case.

Moreover, the proof of the ordinary adjunction inequality in the case that X is of simple type relies on an adjunction relation (first proven in [7], and proven in greater generality in [8]). However, it is hard to make sense of what an adjunction relation would look like in the involutive theory since the H_1 action on ordinary Heegaard Floer homology does not naturally extend to involutive Heegaard Floer homology.

5.2 A note about Seiberg-Witten theory

In [9], Baraglia showed that the mod 2 Seiberg-Witten invariants for spin structures on closed four-manifolds are topologically determined, and in fact vanish for $b_2^+ > 3$. As the Ozsváth-Szabó mixed invariant conjecturally agrees with the Seiberg-Witten invariant, it seems likely that $\Phi^I_{X,\mathfrak{s}}$ cannot distinguish between homeomorphic closed four-manifolds after one stabilization, in light of Theorem 5. However, by the work of Kang [10], it is known that the involutive cobordism maps $F^{I,+}$ are capable of distinguishing between homeomorphic four-manifolds with non-empty boundary after one stabilization. There is some hope that this is the case for $\Phi^I_{X,\mathfrak{s}}$ as well.

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