Involutive Heegaard Floer Theory in Dimension Four

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Abstract

The work of Donaldson and Floer in the 1980s demonstrated that a 3+1 dimensional topological quantum field theory (TQFT) is an optimal framework for studying four-dimensional smooth manifolds. In this project, we use the TQFT structure of involutive Heegaard Floer homology, a modern construction due to Hendricks and Manolescu [1], to produce a novel invariant of smooth four-manifolds.

Heegaard Floer Homology as a TQFT

In addition to assigning four $\mathbb{Z}_2[U]$ modules – $\widehat{HF}(Y), HF^-(Y), HF^+(Y)$, and $HF^\infty(Y)$ – to every closed, oriented three-manifold Y, Heegaard Floer homology also assigns maps

$$F_W^\circ : HF^\circ(Y_1) \to HF_\circ(Y_2)$$

to every (oriented) four-dimensional cobordism W from Y_1 to Y_2.

These maps are **functorial**, in the sense that if W_1 is a cobordism from Y_1 to Y_2 and W_2 is a cobordism from Y_2 to Y_3, then

$$F^\circ_{W_1 \cup_{Y_2} W_2} = F_{W_2} \circ F_{W_1}.$$

The Ozsváth-Szabó Mixed Invariant

By removing two four-balls from a closed four-manifold X, we can view X as a cobordism from S^3 to S^3. Unfortunately, for any X with $b_2^+(X) > 0$, all of the corresponding maps

$$F_X^\circ : HF^+(S^3) \to HF^+(S^3)$$

vanish, so they are not useful invariants.

We can salvage the situation as follows:

- Cut X in half along a three-manifold N which satisfies some nice properties (called an admissible cut).
- Glue together the maps

$$F^-_{X_1,\mathfrak{s}} : HF^-(S^3) \to HF^-(N, \mathfrak{s}),$$
$$F^+_{X_2,\mathfrak{s}} : HF^+(N, \mathfrak{s}) \to HF^+(S^3).$$

The result is the **Ozsváth-Szabó mixed invariant** $\Phi_{X,\mathfrak{s}}$.

Applications of $\Phi_{X,\mathfrak{s}}$

- $\Phi_{X,\mathfrak{s}}$ can detect exotic smooth structures.
- The nonvanishing of $\Phi_{X,\mathfrak{s}}$ implies that embedded surfaces Σ in X satisfy the **adjunction inequality**:

$$\langle c_1(\mathfrak{s}), [\Sigma]\rangle + [\Sigma]^2 \leq 2g(\Sigma) - 2.$$

- The fact that $\Phi_{X,k} \neq 0$ for X a symplectic manifold with canonical spin^c structure k has several consequences, including the indecomposability of symplectic manifolds and the symplectic Thom conjecture.

Involutive Heegaard Floer Homology

Every oriented three-manifold Y admits a **Heegaard splitting** – that is, a decomposition into two handlebodies glued along their boundary.

- Choose a self-indexing Morse function $f : Y \to \mathbb{R}$ on the 3-manifold.
- Considering the two manifolds-with-boundary $f^{-1}([0, 3/2])$ and $f^{-1}([3/2, 3])$.

The information which informs how this gluing takes place can be stored in the form of a **Heegaard diagram** (shown below).

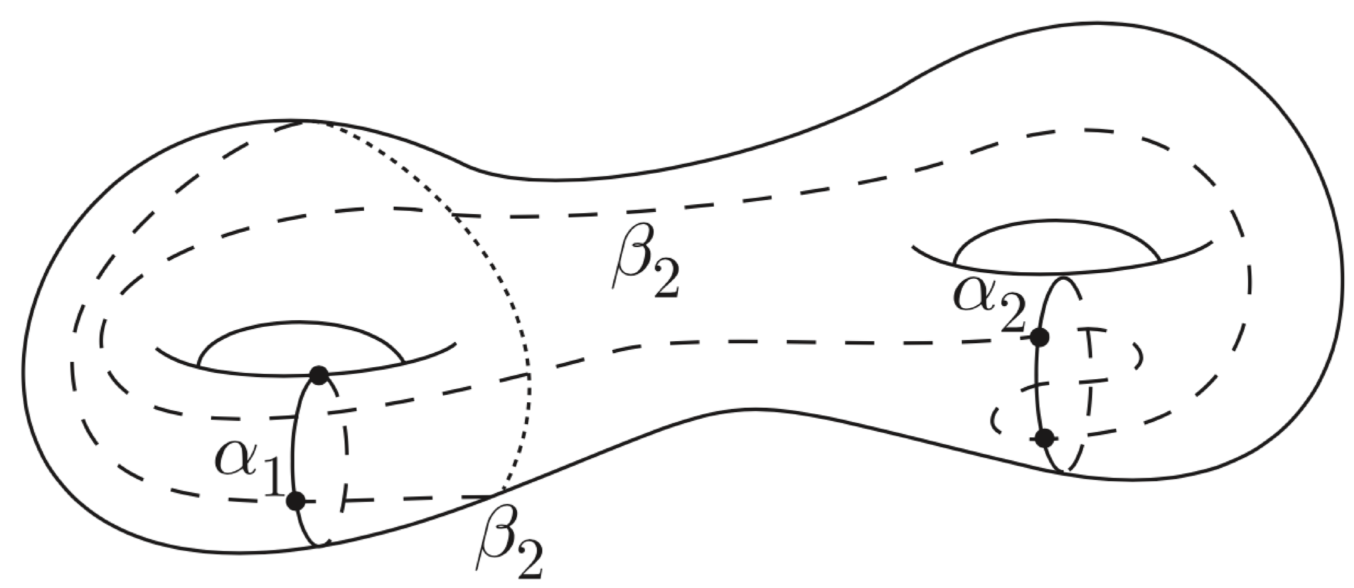

Fig. 1: A genus 2 Heegaard Splitting of S^3 [3].

The construction of Heegaard Floer homology for a three-manifold Y relies on a choice of Morse function $f : Y \to \mathbb{R}$, which induces a Heegaard splitting (Σ, α, β).

Involutive Heegaard Floer homology incorporates information about the natural symmetry on this data given by transitioning from f to $-f : Y \to \mathbb{R}$.

- This can be thought of as a path of Heegaard splittings from (Σ, α, β) to $(-\Sigma, \beta, \alpha)$.

Involutive Heegaard Floer homology also admits a TQFT structure, assigning maps:

$$F^\circ_{W,\mathfrak{s}} : HF^\circ(Y_1, \mathfrak{s}|_{Y_1}) \to HF^\circ(Y_2, \mathfrak{s}|_{Y_2})$$

to every spin cobordism $(W, \mathfrak{s})$.

These maps reveal structure which ordinary Heegaard Floer homology cannot see.

- Kang used them the discover a pair of contractible four-manifolds X_1, X_2 which are homeomorphic but not diffeomorphic even after one stabilization [2].
- Meanwhile, ordinary Heegaard Floer invariants vanish under stabilization.

An Involutive Mixed Invariant

Modeling Ozsváth and Szabó's construction in the setting of involutive Heegaard Floer theory, I defined an **involutive mixed invariant** $\Phi^I_{X,\mathfrak{s}}$, which for closed spin four-manifolds $(X, \mathfrak{s})$ (with $b_2^+(X) > 3$) takes the form

$$\Phi^I_{X,\mathfrak{s}} = \begin{pmatrix} \Phi_{X,\mathfrak{s}} & 0 \\ \Psi_{X,\mathfrak{s}} & \Phi_{X,\mathfrak{s}} \end{pmatrix}$$

where $\Phi_{X,\mathfrak{s}}$ is the Ozsváth-Szabó invariant, and $\Psi_{X,\mathfrak{s}}$ is a novel U-equivariant homomorphism.

Stabilization Formula for $\Phi^I_{X,\mathfrak{s}}$

The involutive mixed invariant satisfies a stabilization formula:

$$\Phi^I_{X\#(S^2\times S^2),\mathfrak{s}} = Q\Phi^I_{X,\mathfrak{s}}.$$

In particular, this implies that, unlike the Ozsváth-Szabó invariant, $\Phi^I_{X,\mathfrak{s}}$ need not vanish after stabilization. It also implies that the off-diagonal homomorphism, $\Psi_{X,\mathfrak{s}}$, need not be trivial.

Potential Applications of $\Phi^I_{X,\mathfrak{s}}$ and Future Goals

- $\Phi^I_{X,\mathfrak{s}}$ might detect exotic smooth structures which $\Phi_{X,\mathfrak{s}}$ misses.
- $\Phi^I_{X,\mathfrak{s}} \neq 0$ is more general than $\Phi_{X,\mathfrak{s}} \neq 0$. As such, showing adjunction in the case that $\Phi^I_{X,\mathfrak{s}} \neq 0$ would be a powerful result.
- Constructing a closed four-manifold X with $\Psi_{X,\mathfrak{s}} \neq 0$ which is *not* obtained through stabilization.

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