### Introduction

Motivating Fact: Elliptic Curves over \( \mathbb{C} \rightleftharpoons \text{Complex Tori} \)

1. Elliptic Curves over \( \mathbb{C} \):
   - Solutions in \( \mathbb{C}^2 \) to the Weierstrass Form
   \( y^2 = x^3 + Ax + B \)
   for some \( A, B \in \mathbb{C} \), plus a point of infinity \( \infty \).
2. Complex Tori:
   - A lattice \( L \subset \mathbb{C} \) gives the complex torus \( \mathbb{C}/L \).

### Main Objective

Let \( E \) be an elliptic curve over \( \mathbb{C} \). Then, \( E \cong \mathbb{C}/L \) for some lattice \( L \). The standard method to show this uses the Weierstrass elliptic function. That is,

\[
z \mapsto \left( \wp(z) - \frac{1}{3} \wp'(z) \right)
\]

gives a surjective homomorphism between \( \mathbb{C} \) and \( E \) with kernel \( L \).

**Objective:**
- Find a natural, geometric isomorphism between \( E \) and \( \mathbb{C}/L \).
- Show that \( \wp \) is essentially the exponential map on \( E \).

### Toy Problem (Circle)

Q. How can we show \( S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \)?

A. Construct a surjective homomorphism from a line to a circle!

\[
\exp(z) = \lim_{n \to \infty} \left( 1 + \frac{z}{n} \right)^n
\]

### Defining The Exponential Map on Elliptic Curves

Let \( E \) be an elliptic curve over \( \mathbb{C} \).

1. Consider \( E \) in \( \mathbb{C}^2 \) and pick a point \( P \in E \).
2. Make \( P \) the identity by the new group law

\[
P + E P = P_1 + E P_2 - E P
\]

3. Take the tangent space to \( P \) and associate it with \( \mathbb{C} \).
4. Define the exponential map

\[
\exp(z) = \left( 2^z \right)^f \left( \frac{z}{2\pi} \right)
\]

where \( f \) projects \( z/2\pi \in \mathbb{C} \) to the elliptic curve, and \( \left( 2^z \right)^f \) means that \( f/2^z \) is added to itself \( 2^z \) times with respect to \( +_E \).

**Claim:** This map is a well-defined surjective homomorphism whose kernel is a lattice \( L \), so it induces an isomorphism between \( E \) and \( \mathbb{C}/L \), and in fact \( \exp(z) = \left( \wp(z), \frac{1}{3} \wp'(z) \right) \).

### Numerical Work

Exponential map for \( y^2 = x^3 - 1 \) with identity point \((1,0)\). RGB values indicate argument, opacity indicates norm (zeros in white). The graphs match the behavior of \( \wp(z) \) and \( \wp'(z) \), respectively, as expected.

### Outline of Proof of Claim

- Rewrite \( 2^n f(z) \) as a telescoping sum.
- Derive estimates showing that \(+_E \) is "approximately the same as Euclidian addition" near \( P \).
- Use these estimates to show that the sequence defining \( \exp \) converges (i.e. \( \exp \) is well-defined) and that \( \exp \) is a homomorphism.
- Show that the image of the exponential map is an open and closed subset of a connected set, so \( \exp \) is surjective.
- Because \( E \) (endowed with an appropriate topology) is compact, conclude by arguing the kernel of \( \exp \) must be a lattice \( L \) in order for \( \mathbb{C}/\ker \exp \) to be compact.
- Compare poles and zeros, take a quotient, and use Liouville’s Theorem to show that the exponential map is equal to \( z \mapsto \left( \wp(z), \frac{1}{3} \wp'(z) \right) \).

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**Fig. 1:** Complex Torus

**Fig. 2:** Group Law

**Fig. 3:** Exponential Map

**Fig. 4:** Plots