ON THE PARTITION FUNCTION OF SIX VERTEX LATTICE MODELS THROUGH INTEPOLATION AND DEMAZURE OPERATORS

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Abstract. Six-vertex integrable lattice models are an important tool from statistical mechanics that can be applied to enumerative combinatorics and the theory of symmetric functions. In this paper we apply Lagrange Interpolation to find the partition function of lattice models with domain wall boundary conditions under free fermionic and constant field weights. Inspired by the Demazure-Lusztig operators, we define a lattice algebra that that allows us to extend our results for the domain wall boundary conditions case to lattice models with arbitrary boundary conditions.

Contents

1. Introduction

Six-vertex lattice models were first studied by Linus Pauling in connection to the statistical mechanics of hydrogen bonds in crystal lattices. However, since then, these models have found deep connections to a variety of areas of mathematics, such as enumerative combinatorics [Kup97], and representation theory [BBF09].

The central problem in this area is the computation and study of the partition function of a lattice model. Consider an $n \times n$ grid with paths entering from the top edges and leaving from the right edges (as represented in Figure 1) Each path can only move down, or to the right at each vertex. We allow paths to intersect at a vertex, but two paths may not travel along the same edge. Given these constraints, there are six possible states for each vertex in the $n \times n$ lattice, which we denote by $a_1, a_2, b_1, b_2, c_1, c_2$, as detailed below:

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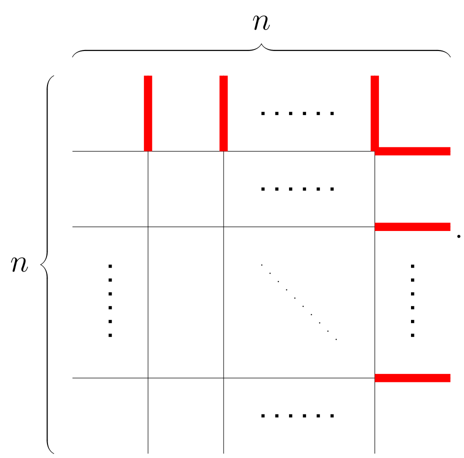
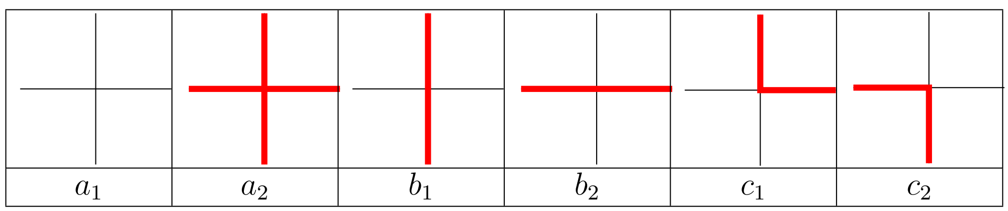


FIGURE 1. An $n \times n$ Lattice Model



We assign a weight to each possible vertex state, which may depend on the coordinate of the vertex in the grid. Hence, we will denote this weight by of the (i,j) vertex by either $a_1(i,j), a_2(i,j), b_1(i,j), c_1(i,j), c_2(i,j)$, depending on its state in the lattice. The weight of a state of the lattice is given by the product of the states of the vertices. We define the partition function of a lattice model to be the sum of the weights of all possible lattice states. While the weights of each vertex may be set arbitrarily, most literature focuses on the study of integrable weights, which satisfy the Yang-Baxter equation. These broadly fall into two classes: free-fermionic and non-free fermionic.

In the case of non-free fermionic weights, we study a more general parametrization of these weights than has previously appeared in the literature, and, extending the interpolation method due to Izergin and Korepin [Kup97], we compute the partition function for lattice models with free-fermionic weights. We also show that this method can be applied to Free-Fermionic weights for the first time, and obtain a recurrence relation for the partition function for specific free-fermionic weights.

Finally, we generalize the notion of the Demazure Operators in defined in [BBBG19], and define a new algebra that allows us to calculate partition functions of lattice models with arbitrary boundary conditions. Using this algebra we then define a new class of weights that allow us to utilize Demazure Operators properly.

2. Notation

Definition 2.1. A *six-vertex matrix* $u \in \mathrm{GL}_4(\mathbb{C})$ is an invertible 4×4 matrix of the form

$$u = \begin{pmatrix} a_1(u) & & & \\ & c_1(u) & b_1(u) & \\ & b_2(u) & c_2(u) & \\ & & & a_2(u) \end{pmatrix}. \tag{2.1}$$

Let $V \cong \mathbb{C}^2$ with the standard basis e_1, e_2. We view a six-vertex matrix as a matrix of an operator $u \in \mathrm{End}(V \otimes V)$ in basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. By abuse of notation, we denote both the matrix and the operator by u.

Definition 2.2. A *six-vertex model* R with n rows and m columns is a collection of the six-vertex matrices R_i^j for $1 \leq i \leq n$ and $1 \leq j \leq m$. We call them the *vertex matrices*, or vertex R-matrices. We call the non-zero matrix coefficients the *weights* of a given R-matrix.

The matrix R_i^j gives the weights for the i-th row and j-th column. For example, a weight of a vertex of type a_1 at 3-rd row and 7-th column has weight $a_1(R_3^7)$.

Remark 2.3. In literature, it is common to write $a_1(i,j), a_2(i,j)$, and so on for the vertex weights. We will not use this notation to avoid ambiguity with the weights of the cross vertices.

Definition 2.4. A six-vertex model R is said to be *integrable in rows* if for any $1 \leq k \leq m$ there exist six-vertex matrices $R_{i,j}$ with $1 \leq i \neq j \leq n$ such that they satisfy two the parametrized Yang-Baxter equations:

$$[[R_{i,j}, R_{i,k}, R_{j,k}]] = 0, \quad [[R_{i,j}, R_i^k, R_j^k]] = 0.$$

We call them *horizontal cross vertices*, or horizontal R-matrices.

Similarly, a six-vertex model is said to be *integrable in columns* if for any $1 \leq k \leq n$, there exist projective six-vertex matrices $R^{i,j}$ with $1 \leq i, j \leq m$ such that they satisfy two parametrized Yang-Baxter equations:

$$[[R^{i,j}, R^{i,k}, R^{j,k}]] = 0, \quad [[R^{i,j}, R_k^i, R_k^j]] = 0.$$

We call them *vertical cross vertices*, or vertical R-matrices.

The matrix $R_{i,j}$ gives the weights of the cross vertex matrix that exchange rows with labels i and j. Similarly for $R^{i,j}$ and columns.

Definition 2.5. *The standard partition function* Z_λ of a six-vertex model R with top boundary condition λ is the sum over the states with top boundary defined by λ as usual (suppose n is the size of the partition, then the lattice has $\lambda_1 + n - 1$ columns and n rows. The top boundary of columns $\lambda_i + n - i$ are full. The rest are empty.), the bottom boundary is empty, the left boundary is empty, the right boundary is full.

Definition 2.6. The *partition function with the domain wall boundary conditions* Z_n of size n is the standard partition function with $\lambda = (0, 0, \ldots, 0)$ of length n.

Given state of the lattice model S, denote by S_{ij} the vertex at the intersection of the spectral parameters x_i and y_j. Call a vertex a t-vertex if the top edge is shaded, a l-vertex if the left edge is shaded, a r-vertex if the right edge is shaded, and a b-edge if the bottom edge is shaded. For example, b_1 is a l-vertex and r-vertex, but not a r-vertex or an l-vertex.

3. The Free-Fermionic models

One class of the integrable six-vertex models is the free-fermionic six-vertex models. They are characterized by the property that all of the R-matrices u (including vertex, horizontal, and vertical cross vertices) satisfy the free-fermionic condition:

$$a_1(u)a_2(u) + b_1(u)b_2(u) = c_1(u)c_2(u). \tag{3.1}$$

3.1. **The Factorial Schur Model.** The factorial Schur model is the model from the following paper. Note that the model is not homogeneous as the weights depend on the column. The factorial Schur model is integrable ([BMN14]) in both rows and columns. We take $\{\alpha_j\}$ to be a sequence of the complex numbers.

$$R_i^j = \begin{pmatrix} 1 & & & \\ & 1 & t & \\ & z_i + \alpha_j & z_i(t+1) & \\ & & & z_i - t\alpha_j \end{pmatrix}.$$

Theorem 3.1. *For any $k \in \mathbb{N}$, we have*

$$Z_{(k)}(z_1, \alpha_k, \ldots, \alpha_1) = \prod_{j=1}^{k-1}(z_1 - \alpha_j).$$

Proof. There is one possible state S of the lattice that is nonzero. Namely, by boundary conditions $S_{1,k}$ is a t-vertex but not a l-vertex nor a b-vertex, hence it must be c_1. Then, $S_{1,k-1}$ must be b_1. Similarly, $S_{1,j}$ must all be b_1 for $j < k$. □

Theorem 3.2. *Given a partition $(\lambda_n, \ldots, \lambda_1)$, we have the recurrence relation*

$$Z_{(\lambda_n,\ldots,\lambda_1)}(z_1, \ldots, z_n, \alpha_{\lambda_n+n-1}, \ldots, \alpha_1)|_{z_n=-\alpha_{\lambda_1}}$$
$$= c_1(2, \lambda_1)\left(\prod_{i=1}^{\lambda_1-1} a_2(i, \alpha_{\lambda_1})\right)\left(\prod_{\substack{1\le i\le n,\\ 0\le j<\lambda_1}} b_2(i,j)\right) Z_{(\lambda_n-\lambda_1,\ldots,\lambda_2-\lambda_1)}(z_1, \ldots, z_{n-1}; \alpha_{\lambda_n+n-1}, \ldots, \lambda_1 + 1).$$

Proof. Note that if $z_n = -\alpha_{\lambda_1}$, we have $b_2(n, \lambda_1) = 0$. Hence, we can consider only states in which $S_{n\lambda_1} \neq b_2$. In this case, we must have $S_{n\lambda_1} = c_1$. Furthermore, since a path exits at every row, $S_{k\lambda_1}$ must all be r-vertices. However, they are also u, b-vertices since the path entering at the λ_1 column must reach $S_{n\lambda_1}$. Furthermore, since $S_{1\lambda_1}$ is not a l-vertex, all previous vertices in the nth row must be empty. Finally, it is clear that $S_{nk} = b_2$ for all $k > \lambda_1$. Hence. collecting the multiplicative factors from these vertices, we obtain that the partition function is equal to

$$c_1(2, \lambda_1)\left(\prod_{i=1}^{\lambda_1-1} a_2(i, \alpha_{\lambda_1})\right)\left(\prod_{\substack{1\le i\le n,\\ 0\le j<\lambda_1}} b_2(i,j)\right) Z_{(\lambda_n-\lambda_1,\ldots,\lambda_2-\lambda_1)}(z_1, \ldots, z_{n-1}; \alpha_{\lambda_n+n-1}, \ldots, \lambda_1 + 1).$$

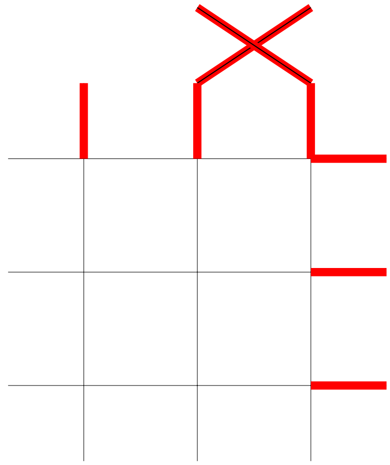
□

3.2. **Domain Wall Boundary Conditions.** Given Domain Wall Boundary Conditions, a Train Argument is sufficient to show that the partition function is symmetric in the columns of the lattice.

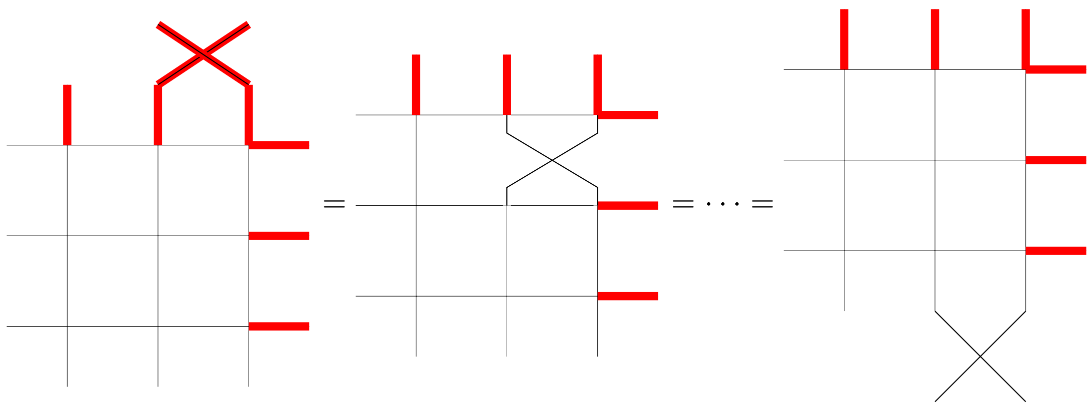
Theorem 3.3. $Z_{(0,\ldots,0)}(z, \alpha_n, \ldots, \alpha_1)$ *is symmetric in α_i's.*

Proof. Note that, at the cost of a multiplicative factor of 1, we can attach a cross-vertex to the lattice as follows:

$$Z_{(0,\ldots,0)}(z, \alpha_n, \ldots, \alpha_1) =$$



However, since we know that these weights satisfy the Yang-Baxter equation, we have



where each subsequent equality comes from an application of the Yang-Baxter equation. On the right-hand side, we can simply remove this vertex at no multiplicative cost, and hence, we have that Z is symmetric in α_i. □

Corollary 3.4.

$$Z_{(0,\ldots,0)}(z_1, \ldots, z_n, \alpha_n, \ldots, \alpha_1)|_{z_n = -\alpha_j}$$
$$= c_1(2, \lambda_1) \left(\prod_{i=1}^{\lambda_1 - 1} a_2(i, \alpha_{\lambda_1}) \right) \left(\prod_{\substack{1 \le i \le n, \\ 0 \le j < \lambda_1}} b_2(i, j) \right) Z_{(0,\ldots,0)}(z_1, \ldots, z_{n-1}; \alpha_n, \ldots, \hat{\alpha_j}, \ldots, \alpha_1).$$

In particular, this gives us n points, which is only one less than we need to interpolate the polynomial.

4. The Constant-Field Models

Another class of the integral six-vertex models is the constant-field six-vertex models (or, interacting, Baxter-type, etc.). In such models we have $a_1(u) = a_2(u) = a(u)$, and $b_2(u) = \Gamma\, b(u)$, and $b_1(u) = b(u)$ for a universal constant $\Gamma \in \mathbb{C}^\times$. Hence, we have only parameters a, b, c_1, c_2 defining weights in the constant-field models.

4.1. **The Constant-Field Model.** We can get the following generalization of the rational constant-field model. This model is not known in the literature in this generality, but the special cases are used in the alternating sign matrices and the stochastic six-vertex models. Again, let us define matrices

$$R(x_i, y_j) = \begin{pmatrix} q_1x_i - q_2y_j & & & \\ & x_i(q_1-q_2)k & x_i - y_j & \\ & q_1q_2(x_i - y_j) & y_j(q_1-q_2)k^{-1} & \\ & & & q_1x_i - q_2y_j \end{pmatrix},$$

where $k, q_1, q_2 \in \mathbb{C}^\times$ are non-zero constants. Then $R_i^j = R(x_i, y_j)$, $R_{i,j} = R(x_i, x_j)$, and $R^{i,j} = R(y_i, y_j)$.

Lemma 4.1. *Within each possible state of the lattice model with DWBC, there exists an equal number of b_1 and b_2 vertices.*

Proof. Let S be a possible state of a $n \times n$ lattice model with DWBC, such that n paths enter from the top and exit from the right, at each vertex having only the options of going down or going right. S can always be interpreted as a state without the intersection of paths, since the intersection of two paths require a a_2 vertex, which can be interpreted as the top path going right and the left path going down, so as to avoid intersection.

Label the entering paths $0, 1, \ldots, n-1$ from right to left, the exits $0, 1, \ldots, n-1$ from top to bottom. The lack of intersections implies that the i^{th} path must leave through the i^{th} exit. We examine each path individually. Single paths contain only b_1, b_2, c_1, c_2 vertices. The i^{th} path must travel through $2i+1$ vertices. Let $N(b_1) =$ the number of b_1 vertices in a particular path. Similarly define $N(b_2)$, $N(c_1)$, $N(c_2)$. We have the following relations:

(1) $N(b_1) + N(c_1) = i + 1$
(2) $N(b_2) + N(c_2) = i$
(3) $N(c_1) = N(c_2) + 1$

(1) and (2) are evident because the i^{th} path must go right $i+1$ times and down i times. (3) is true because each c_1 and c_2 change the orientation of the path between going down and going right, and overall the path starts out going down and must end going right. The above implies $N(b_1) = N(b_2)$ within individual paths. Finally, observe that when paths are assembled together to form a possible state of the lattice, b_1 and b_2 vertices remain invariant. □

Remark 4.2. Due to the above lemma, the following set of weights is equivalent to the weights of the constant-field model, as it preserves $b_1(x_i, y_j)b_2(x_{i'}, y_{j'}) = q_1q_2(x_i - y_j)(x_{i'} - y_{j'})$ and thus gives an equivalent partition function.

$$R(x_i, y_j) = \begin{pmatrix} q_1x_i - q_2y_j & & & \\ & x_i(q_1-q_2)k & \sqrt{q_1q_2}(x_i - y_j) & \\ & \sqrt{q_1q_2}(x_i - y_j) & y_j(q_1-q_2)k^{-1} & \\ & & & q_1x_i - q_2y_j \end{pmatrix}$$

Theorem 4.3. *The partition function for the constant-field model satisfies and is uniquely determined by the following properties:*

(1) *Z_n is respectively symmetric in x_i's and y_j's. This is true by the Yang-Baxter equation and train argument.*
(2) *Z_n is a polynomial of degree at most N in each x_i and y_j.*
(3) *$Z_1 = c_1(x_1, y_1)$*
(4) *Recurrence relation:*

$$Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n)|_{x_1=y_1} = c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i - y_1)a(x_1 - y_i)Z_{n-1}(x_2, \ldots, x_n; y_2, \ldots, y_n).$$

Lemma 4.4 (Symmetry)**.** *$Z(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is symmetric in x_i and y_i*

Proof. The weights satisfy Yang-Baxter, and $a_1 = a_2$ so the train argument works unchanged. □

Lemma 4.5. *As a polynomial in x_i, Z_n has degree at most n.*

Proof. Each of the weights are polynomials in x_i of degree at most 1. Note that only the ith row have weights which depend on x_i, and each row has at most n vertices. As such, each term has degree at most n in each x_i. Thus, Z_n has degree at most n as a polynomial in x_i. □

Lemma 4.6 (Recurrence Relation)**.** *If we let $x_1 = y_1$, we have*

$$Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n)|_{x_1=y_1} = c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i - y_1)a(x_1 - y_i)Z_{n-1}(x_2, \ldots, x_n; y_2, \ldots, y_n).$$

Proof. If $x_1 = y_1$, then $b_1(x_1, y_1) = b_2(x_1, y_1) = 0$. Hence, all states S such that $S_{1,1} = b_2$ have weight zero. Thus, we only consider states with $S_{1,1} = c_1$ (since this is the only other possible state given the Domain Wall Boudary Conditions). Hence, $S_{1.2}$ must be a r-vertex and a u-vertex, since it neighbours $S_{1,1}$ and the Domain-Wall boudary conditions. As such, $S_{1,2} = a_2$. Now, we can inductively argue that $S_{1,k} = a_2$ for $k > 1$. Furthermore, we can see that since no path enters $S_{2,1}$ and paths move only leftwards and downwards, we have $S_{k,1} = a_1$ for $k > 1$. On the other hand, the remaining undetermined vertices are simply a $(n-1) \times (n-1)$ lattice with Domain Wall Boundary conditions. Hence, we can write

$$Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n)|_{x_1=y_1} = c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i - y_1)a(x_1 - y_i)Z_{n-1}(x_2, \ldots, x_n; y_2, \ldots, y_n)$$

as desired. □

Remark 4.7. In the Izergin-Korepin paper, they exploit the fact that they have set $c_1 = [1]$ to guarantee that the polynomial must have degree at most $n-1$ in which the above points are sufficient to guarantee uniqueness. However, in the general case, we need one additional point. This is the point of the following lemma.

Remark 4.8. A degenerate case arises if we set $q_1 = q_2$. In this case, the partition function vanishes for any n, since every state of the lattice model which satisfies boundary conditions must contain at least one c_1 vertex, and $c_1(x_i, y_i) = 0$ for all i, j if $q_1 = q_2$. Hence, the partition function becomes a polynomial of degree 0 in all x_i.

Lemma 4.9 (Uniqueness by Lagrange Interpolation)**.** *There exists a unique polynomial of degree at most n in each variable $x_1, \ldots, x_n$ such that $Z_1(x_1; y_1) = x_1(q_1 - q_2)$, and*

$$Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = c_1(x_n, y_1) \prod_{i=1}^{n-1} b_2(x_i, y_1) Z_{n-1}(x_1, \ldots, x_{n-1}, y_2, \ldots, y_n).$$

and

$$Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n)|_{x_1=y_j} = c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i - y_1) a(x_1 - y_i) Z_{n-1}(x_2, \ldots, x_n; y_2, \ldots, y_n),$$

for $j = 1, \ldots, n$.

Proof. We will show this by induction on n. By definition, $Z_1(x_1; y_1)$ is uniquely determined. Now, suppose that Z_{n-1} is uniquely determined by the recurrence relations for some positive integer n. Now, let $y_{n+1} = q_1^{-1} q_2 y_1$. Then, denote

$$\varphi_j(x) = \prod_{i \neq j}^{n+1} \frac{(x - y_i)}{(y_j - y_i)}$$

and observe that by Lagrange interpolation, since Z_n has degree at most n in x_1, we can write

$$\begin{aligned} Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = &\sum_{j=1}^{n} \left(c_1(x_1, y_1) \prod_{i=2}^{n} a(x_i - y_1) a(x_1 - y_i) Z_{n-1}(x_2, \ldots, x_n; y_2, \ldots, y_n) \right) \varphi_j(x_1) \\ &+ \left(c_1(x_n, y_1) \prod_{i=1}^{n-1} b_2(x_i, y_1) Z_{n-1}(x_1, \ldots, x_{n-1}, y_2, \ldots, y_n) \right) \varphi_{j+1}(x_1), \end{aligned}$$

and hence we can uniquely determine $Z_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$.

□

Theorem 4.10. *The partition function for the constant-field model is given by the Izergin-Korepin determinant,*

$$Z_n(x_1, \ldots, x_n, y_1, \ldots, y_n, q_1, q_2) = (-1)^{n-1} \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j} b(x_i, x_j) b(y_i, y_j))} \det \left(\frac{c_1(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

Note that the argument is inspired by [Kup97]

Proof. Let

$$Z_n'(x_1, \ldots, x_n, y_1, \ldots, y_n, q_1, q_2) = (-1)^{n-1} \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j} b(x_i, x_j) b(y_i, y_j))} \det \left(\frac{c_1(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

We show that Z_n' satisfies the four properties mentioned above.

(1) Symmetry in x_i's and y_j's

$$\frac{Z_n'(x_1,\ldots,x_k,x_{k+1},\ldots,x_n,y_1,\ldots,y_n,q_1,q_2)}{Z_n'(x_1,\ldots,x_{k+1},x_k,\ldots,x_n,y_1,\ldots,y_n,q_1,q_2)} = \frac{b(x_{k+1},x_k)}{b(x_k,x_{k+1})}\frac{\det\left(\frac{c_1(x_i,y_j)}{a(x_i,y_j)b(x_i,y_j)}\right)}{\det(M)}$$
$$= \frac{(x_{k+1}-x_k)}{(x_k-x_{k+1})}\frac{\det\left(\frac{c_1(x_i,y_j)}{a(x_i,y_j)b(x_i,y_j)}\right)}{\det(M)} = 1$$

Where

$$\det(M) = \begin{vmatrix} \frac{c_1(x_1,y_1)}{a(x_1,y_1)b(x_1,y_1)} & \cdots & \frac{c_1(x_1,y_k)}{a(x_1,y_k)b(x_1,y_k)} & \frac{c_1(x_1,y_{k+1})}{a(x_1,y_{k+1})b(x_1,y_{k+1})} & \cdots & \frac{c_1(x_1,y_n)}{a(x_1,y_n)b(x_1,y_n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{c_1(x_{k+1},y_1)}{a(x_{k+1},y_1)b(x_{k+1},y_1)} & \cdots & \frac{c_1(x_{k+1},y_k)}{a(x_{k+1},y_k)b(x_{k+1},y_k)} & \frac{c_1(x_{k+1},y_{k+1})}{a(x_{k+1},y_{k+1})b(x_{k+1},y_{k+1})} & \cdots & \frac{c_1(x_{k+1},y_n)}{a(x_{k+1},y_n)b(x_{k+1},y_n)} \\ \frac{c_1(x_k,y_1)}{a(x_k,y_1)b(x_k,y_1)} & \cdots & \frac{c_1(x_k,y_k)}{a(x_k,y_k)b(x_k,y_k)} & \frac{c_1(x_k,y_{k+1})}{a(x_k,y_{k+1})b(x_k,y_{k+1})} & \cdots & \frac{c_1(x_k,y_n)}{a(x_k,y_n)b(x_k,y_n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}$$

$$= - \begin{vmatrix} \frac{c_1(x_1,y_1)}{a(x_1,y_1)b(x_1,y_1)} & \cdots & \frac{c_1(x_1,y_k)}{a(x_1,y_k)b(x_1,y_k)} & \frac{c_1(x_1,y_{k+1})}{a(x_1,y_{k+1})b(x_1,y_{k+1})} & \cdots & \frac{c_1(x_1,y_n)}{a(x_1,y_n)b(x_1,y_n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{c_1(x_k,y_1)}{a(x_k,y_1)b(x_k,y_1)} & \cdots & \frac{c_1(x_k,y_k)}{a(x_k,y_k)b(x_k,y_k)} & \frac{c_1(x_k,y_{k+1})}{a(x_k,y_{k+1})b(x_k,y_{k+1})} & \cdots & \frac{c_1(x_k,y_n)}{a(x_k,y_n)b(x_k,y_n)} \\ \frac{c_1(x_{k+1},y_1)}{a(x_{k+1},y_1)b(x_{k+1},y_1)} & \cdots & \frac{c_1(x_{k+1},y_k)}{a(x_{k+1},y_k)b(x_{k+1},y_k)} & \frac{c_1(x_{k+1},y_{k+1})}{a(x_{k+1},y_{k+1})b(x_{k+1},y_{k+1})} & \cdots & \frac{c_1(x_{k+1},y_n)}{a(x_{k+1},y_n)b(x_{k+1},y_n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{c_1(x_n,y_1)}{a(x_n,y_1)b(x_n,y_1)} & \cdots & \frac{c_1(x_n,y_k)}{a(x_n,y_k)b(x_n,y_k)} & \frac{c_1(x_n,y_{k+1})}{a(x_n,y_{k+1})b(x_n,y_{k+1})} & \cdots & \frac{c_1(x_n,y_n)}{a(x_n,y_n)b(x_n,y_n)} \end{vmatrix}.$$
$$= -\det\left(\frac{c_1(x_i,y_j)}{a(x_i,y_j)b(x_i,y_j)}\right)$$

Notice the determinant expression is antisymmetric.

(2) Z_n' is a polynomial of degree at most N in each x_i and y_j.

Let $N = \prod_{i,j=1}^n a(x_i,y_j)b(x_i,y_j)$, $D = \det\left(\frac{c_1(x_i,y_j)}{a(x_i,y_j)b(x_i,y_j)}\right)$, $E = \prod_{i<j} b(x_i,x_j)b(y_i,y_j))$. Then $Z_n' = \frac{ND}{E}$. Both ND and E are multivariate polynomials in x_i's and y_j's. Since $E = \sqrt{q_1q_2}^{n(n-1)}\prod_{i<j}(x_i-x_j)(y_i-y_j))$, it is divisible by D, which is antisymmetric in x_i's and y_j's. All powers of $\sqrt{q_1q_2}^{n(n-1)}$ from the numerator and denominator cancel. Hence Z_n is a multivariate polynomial.

We show that Z_n' is a polynomial of degree at most N in each x_i and y_j. Fix a choice of x_i (or respectively y_j). Note that in x_i, N gives $2n$ powers in the numerator and E gives $n-1$ powers in the denominator. Each term in the expansion of D gives 1 power in the numerator and 2 powers in the denominator.

(3) Base Case: $Z_1 = c_1(x_1,y_1)$

(4) Recurrence Relation:

$Z_n(x_1,\ldots,x_n;y_1,\ldots,y_n)|_{x_k=y_l} = c_1(x_k,y_l)\prod_{i=2}^n a(x_i,y_l)a(x_k,y_i)Z_{n-1}(\hat{x_k};\hat{y_l})$

We first prove the case with $k = l = 1$. Let $Y_n(x_1,\ldots,x_n;y_1,\ldots,y_n,q_1,q_2) = \det\left(\frac{c(x_i,y_j)}{a(x_i,y_j)b(x_i,y_j)}\right)$.

$$
\begin{aligned}
&Z_n(x_1,\dots,x_n;y_1,\dots,y_n)|_{x_1=y_1}\\
&= (-1)^{n-1}\prod_{i=2}^{n}\frac{a(x_1,y_i)a(x_i,y_1)b(x_1,y_i)b(x_i,y_1)}{b(x_1,x_i)b(y_1,y_i)}\frac{\prod_{i,j=2}^{n}a(x_i,y_j)b(x_i,y_j)}{\prod_{2\le i,j\le n}b(x_i,x_j)b(y_i,y_j)}Y_n(x_1,\dots,x_n;y_1,\dots,y_n,q_1,q_2)\\
&= (-1)^{n-1}\prod_{i=2}^{n}\frac{a(x_1,y_i)a(x_i,y_1)b(x_1,y_i)b(x_i,y_1)}{b(x_1,x_i)b(y_1,y_i)}\frac{\prod_{i,j=2}^{n}a(x_i,y_j)b(x_i,y_j)}{\prod_{2\le i,j\le n}b(x_i,x_j)b(y_i,y_j)}\frac{c(x_1,y_1)}{a(x_1,y_1)b(x_1,y_1)}\\
&\qquad\qquad \dots Y_{n-1}(x_2,\dots,x_n;y_2,\dots,y_n,q_1,q_2)\\
&= (-1)^{n-1}c(x_1,y_1)\prod_{i=2}^{n}a(x_1,y_i)a(x_i,y_1)\prod_{i=2}^{n}\frac{b(x_1,y_i)b(x_i,y_1)}{b(x_1,x_i)b(y_1,y_i)}Y_{n-1}(x_2,\dots,x_n;y_2,\dots,y_n,q_1,q_2)\\
&= (-1)^{n-2}c(x_1,y_1)\prod_{i=2}^{n}a(x_1,y_i)a(x_i,y_1)Y_{n-1}(x_2,\dots,x_n;y_2,\dots,y_n,q_1,q_2)
\end{aligned}
$$

□

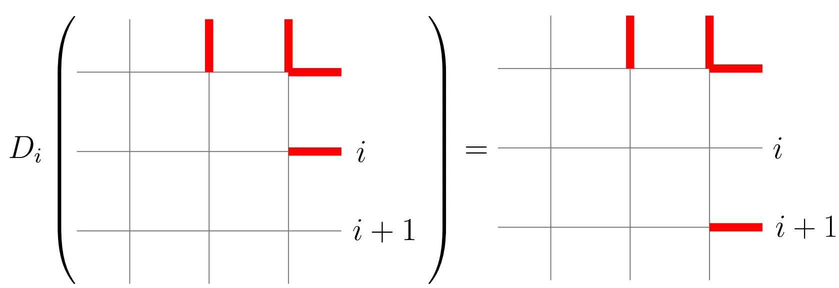
5. Demazure Operators and Lattice Algebra

Suppose we have a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, and the two n-tuples $\gamma = (n-1, \dots, 0)$ and $\rho = (n, n-1, \dots, 1)$. We can define a lattice with n rows ordered from 1 to n, and $n+\lambda_1$ columns labeled from left to right, beginning with $n+\lambda_1$. Then a partition λ implies that we have incoming paths at columns labeled $\lambda_i + \rho_i$.

Definition 5.1. We define the *horizontal Demazure operator* as

$$\mathcal{D}_i = \frac{R_{i+1,i}(a_1)s_i - R_{i+1,i}(c_1)}{R_{i+1,i}(b_2)},$$

where s_i is a permutation operator that transposes the spectral parameters of rows i and $i+1$. The action of the horizontal Demazure operator on an integrable lattice model is illustrated below.



We define a similar action on the columns.

Definition 5.2. Let the *vertical Demazure operator* be defined as

$$\mathcal{D}^j = \frac{R^{j+1,j}(a_1)s_i - R^{j+1,j}(b_1)}{R^{j+1,j}(c_1)},$$

where s_i is a permutation operator that transposes the spectral parameters for columns j and $j+1$.

Demazure operators describe the recursive relationship between similar partition functions by repeated use of the train argument. Thus the Demazure operators exist for nontrivial, integrable lattice models with non-degenerate horizontal or vertical R-matrices of size at least 2×2.

Using the Demazure operators, we can express the partition functions of lattice models with arbitrary boundary conditions as a sequence of Demazure operators acting on the partition function of a lattice model with DWBC.

For example, let us calculate the partition function, Z^*, of a 3×3 lattice model with paths leaving at $\mu = (3, 1)$, and entering at $\lambda + \rho = (2, 1)$ for a column partition λ and a row partition μ. We calculate the partition function of this model in terms of the partition function of the 3×3 lattice with domain wall boundary conditions on the rightmost two columns and top two rows. We denote this partition function as Z. Since the value of $R_i^j(a_1)$ in the Factorial Schur weights is equal to 1, the partition function of this lattice model is equal to that of the 2×2 model with DWBC. Therefore, using the train argument, we have that $R_{2,3}(a_1)Z = R_{2,3}(c_1)s_2Z + R_{2,3}(b_2)s_2Z^*$. Hence solving this equation for Z^* gives the Demazure operator applied to the lattice model with $2{\times}2$ domain wall boundary conditions.

Interestingly, Demazure operators can be directly related to Schur polynomials. The Schur polynomials can be written as a sum of Demazure atoms, and the derivation, which uses the Demazure character formula, can be found in [BBBG19].

As the application of Demazure operators entails the attachment of R-matrices to a lattice, the choice of weights largely influences whether the method is successful or not. However, with non-degenerate horizontal and vertical R-matrices, the partition function can always be reduced to the Demazure operators acting on a base case.

Theorem 5.3. *For any nontrivial column partition λ acting upon an integrable lattice model of size at least 2×2, the partition function over λ can be reduced to the Demazure operators acting on a lattice with domain wall boundary conditions.*

Proof. Let $\lambda = (\lambda_{n_s-1}, \dots, \lambda_1, \lambda_0)$ be an arbitrary column partition where n_s denotes the number of shaded edges in λ, and let $\mu_i = \lambda_i + \rho_i$ where $\rho_i = i+1$ for $0 \leq i \leq n_s - 1$. We can begin from the domain wall boundary conditions acting on a lattice and reach our arbitrary column partition with the application of Demazure operators via the operator $\mathscr{D}$ such that

$$\mathscr{D}(\mu, \rho) = \prod_{i=0}^{n_s-1} \prod_{j=1}^{\mu_i - \rho_i} \mathcal{D}_{\mu_i - j}.$$

To verify the validity of the operator $\mathscr{D}$, pick an arbitrary $\mu_i \in \mu$; ρ_i must be shifted to μ_i, meaning that we must apply $\mu_i - \rho_i$ Demazure operators. It must be the case that at least $\mu_i - \rho_i$ unshaded edges reside to the left of the shaded edge at μ_i, because $\mu_{i+1} - \rho_{i+1} \geq \mu_i - \rho_i$, and the column μ_{i+1} will be operated on before μ_i due to the ordering of multiplication. Therefore, we can apply $\mu_i - \rho_i$ Demazure operators, as required. Hence, the operator works as expected, so we can reach our column partition by reducing it to a scalar multiple of the Demazure operators acting on a lattice with domain wall boundary conditions. ∎

Theorem 5.4. *For any nontrivial row partition μ acting upon an integrable lattice model of size at least 2×2, the partition function over μ can be reduced to the Demazure operators acting on a lattice with domain wall boundary conditions.*

Proof. The proof is analogous to the proof of Theorem 6.2, except we operate over the rows of the lattices as opposed to the columns. $\square$

5.1. **Lattice Algebra.** We now define the Lattice Algebra that allows us to compute partition functions, and define a new class of weights.

Let $\mathcal{D}_i$ denote the same Demazure operator acting on lattice models, as defined above. Let Z^f be the partition function of a lattice model with the ends of both its i-th and $i+1$-th rows filled, and let Z^e be a similar partition function of a lattice model with both of these row ends being empty.

We define an algebra with the Demazure operator that satisfies the following properties:

(1) $\mathcal{D}_i(Z^e) = vZ^e$
(2) $\mathcal{D}_i(Z^f) = \frac{1}{v}Z^f$
(3) $\mathcal{D}_i\mathcal{D}_i(Z) = Z$ for any partition function Z
(4) $\mathcal{D}_{i+1}\mathcal{D}_i\mathcal{D}_{i+1}(Z) = \mathcal{D}_i\mathcal{D}_{i+1}\mathcal{D}_i(Z)$ for any partition function Z

where v is some constant.

In order to satisfy property (1), we begin by evaluating the action of the Demazure operator on two empty rows. The Demazure operator acting on two empty rows should only transpose the rows such that $\mathcal{D}_i(Z^e) = s_iZ^e$. However, by property (1), this then implies that $\mathcal{D}_i(Z^e) = vZ^e = s_iZ^e$. Looking at $\mathcal{D}_i(Z^e) = vZ^e$, we notice that

$$\left(\frac{R_{i+1,i}(a_1)s_i - R_{i+1,i}(c_1)}{R_{i+1,i}(b_2)}\right) Z^e = vZ^e$$
$$\left(\frac{R_{i+1,i}(a_1) - R_{i+1,i}(c_1)}{R_{i+1,i}(b_2)}\right) Z^e = vZ^e$$
$$\frac{R_{i+1,i}(a_1) - R_{i+1,i}(c_1)}{R_{i+1,i}(b_2)} = v$$
$$\implies vR_{i+1,i}(b_2) + R_{i+1,i}(c_1) = R_{i+1,i}(a_1).$$

To satisfy property (2), we evaluate the action of the Demazure operator on two full rows. Using a similar method to what was previously shown, we arrive at the condition that

$$R_{i+1,i}(a_1) = R_{i+1,i}(c_2) + \frac{1}{v}R_{i+1,i}(b_1).$$

Similarly, for property (3), we arrive at the condition that

$$R_{i+1,i}(a_1)R_{i,i+1}(a_1) = R_{i+1,i}(c_1)R_{i+1,i}(c_2) - R_{i+1,i}(b_1)R_{i+1,i}(b_2).$$

These three derivations allow us to now define the three conditions that enable a system of integrable weights to fulfill the above properties.

Definition 5.5. A system of weights is *Demazure integrable* if its horizontal cross weights satisfy the following conditions:

(1) $R_{i+1,i}(a_1)R_{i,i+1}(a_1) = R_{i+1,i}(c_1)R_{i+1,i}(c_2) - R_{i+1,i}(b_1)R_{i+1,i}(b_2)$
(2) $R_{i+1,i}(a_1) - R_{i+1,i}(c_2) = \frac{R_{i+1,i}(b_1)}{v}$
(3) $R_{i+1,i}(a_1) - R_{i+1,i}(c_1) = vR_{i+1,i}(b_2)$.

Theorem 5.6. *The Demazure operators acting on a Demazure integrable system, when parameterized, satisfy Matsumono's relation. That is,* $\mathcal{D}_i\mathcal{D}_{i+1}\mathcal{D}_i(Z) = \mathcal{D}_{i+1}\mathcal{D}_i\mathcal{D}_{i+1}(Z)$.

Proof. Let us parameterize our weights as follows:

$$R_{i,j}(a_1) = z_j - vz_i$$
$$R_{i,j}(b_2) = z_i - z_j$$
$$R_{i,j}(c_1) = (1-v)z_j.$$

Then our Demazure operator becomes the Demazure-Lusztig operator found in [BBBG19], which satisfies the braid relation.

□

Note that this also shows how our operators are more general than existing operators defined in literature.

6. Conclusion

Through our application of Lagrange interpolation for constant field weights, we have seen that the method is effective in computing the partition functions of graphs with domain wall boundary conditions. Since Lagrange interpolation requires data about a large number of points, which are specific to the choice of weights, it can be difficult to generalize to an arbitrary class of weights. Other methods, that do not require such data, may be easier to generalize, namely the Demazure operators and the functional method. The former is particularly effective at finding the partition function for general boundary conditions, which allow us to extend our results in Sections 3 and 4. However, in order to properly use Demazure operators, we must restrict to the class of Demazure integrable weights.

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References

[BBBG19] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Henrik P. A. Gustafsson. Colored five-vertex models and demazure atoms, 2019. URL: https://arxiv.org/abs/1902.01795, doi:10.48550/ARXIV.1902.01795.

[BBF09] Ben Brubaker, Daniel Bump, and Solomon Friedberg. Schur polynomials and the yang-baxter equation, 2009. URL: https://arxiv.org/abs/0912.0911, doi:10.48550/ARXIV.0912.0911.

[BMN14] Daniel Bump, Peter J. McNamara, and Maki Nakasuji. Factorial Schur functions and the Yang-Baxter equation. *Comment. Math. Univ. St. Pauli*, 63(1-2):23–45, 2014.

[Kup97] G. Kuperberg. Another proof of the alternating sign matrix conjecture, 1997. doi:10.48550/ARXIV.MATH/9712207.