MASS INFLATION IN GENERAL RELATIVITY

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Abstract. We hope to establish a mass inflation result for the Einstein–Maxwell–(real) scalar field system in spherical symmetry. For this model, a version of the strong cosmic censorship conjecture has been resolved in [39, 40], but it remains open whether the Hawking mass generically blows up in the black hole interior.

By [41], mass inflation follows from an improved upper bound for the scalar field. We follow and adapt the strategy of Luk–Oh in their upcoming work [38] on late-time tails to establish improved decay. In particular, we introduce a dynamically defined "scaling" vector field S (modelled after $t\partial_t + r\partial_r$ on Minkowski space). The difficulty in our coupled setting is to simultaneously control the scalar field and the geometry.

1. Introduction

1.1. **Overview.** Rotating Kerr black holes pose a challenge to determinism in general relativity: the fate of an observer unfortunate enough to fall into one is not determined by how they fall in. Roger Penrose [48] formulated the *strong cosmic censorship conjecture* that this phenomenon is a particular feature of the Kerr model and that observers falling into a realistic black hole meet their fate at a singularity. How strong are the *singularities* of Penrose's conjecture? Poisson and Israel [49, 50] put forth the scenario of *mass inflation*, which exhibits a curvature singularity in the model case of a spherically symmetric charged black hole. Our goal is to rigorously establish this scenario.

1.2. **Black holes in general relativity.** In a series of four papers in 1915, Einstein [24, 25] developed the theory of general relativity. This geometric theory of gravitation is governed by the Einstein equations, a system of nonlinear partial differential equations relating the geometry of spacetime to its matter and energy content. A solution to Einstein's equations is a spacetime, namely a $(1+3)$-dimensional Lorentzian[1] manifold $(\mathcal{M}, g)$ satisfying

$$\mathrm{Ric}(g) - \frac{1}{2} g R(g) = 2T, \tag{1}$$

where $\mathrm{Ric}(g)$ and $R(g)$ are respectively the Ricci curvature tensor and scalar curvature associated to the spacetime metric g and T is the energy-momentum tensor. The case $T = 0$ is known as the Einstein vacuum equations. In general, T can depend on matter fields (massless or massive scalar fields, an electromagnetic field, fluids, etc.), in which case one couples eq. (1) to equations governing the evolution of matter.

A striking prediction of general relativity that has been confirmed by physical observations is the existence of black holes, namely regions of spacetime from which even light cannot escape to far away observers.[2] Although this interpretation took decades to develop [26], Schwarzschild [59] wrote down the first example of what we now understand as a black hole spacetime just a few months after Einstein finalized his field equations. Schwarzschild wanted to model the gravitational field *outside* a static spherically symmetric body. In 1950, Synge [62] introduced the modern form of the Schwarzschild spacetime, the so-called *maximal analytic extension*, which in particular includes the inside of a black hole.[3] The Schwarzschild spacetimes are

Date: August 26, 2022.

[1]See [55] for an introduction to the terminology of Lorentzian geometry that we freely use here.

[2]The appropriate notion for asymptotically flat spacetimes is that the black hole region consists of points whose future is disjoint from null infinity.

[3]The maximal Schwarzschild spacetime is not a model of astrophysical black holes forming from gravitational collapse, since its Cauchy data is posed on a two-ended time slice (i.e. spacelike Cauchy hypersurface) $\mathbf{R} \times S^2$, rather than the physically relevant one-ended time slice $\mathbf{R}^3$.

parameterized by mass $M \geq 0$. When $M = 0$ one recovers the empty Minkowski spacetime $\mathbf{R}^{1+3}$, and when $M > 0$ these solutions are geodesically incomplete: all observers that enter the black hole reach a curvature singularity in finite proper time. In fact, the metric cannot be continuously extended past this singularity, so infinite tidal deformations rip such an observer apart; this classical expectation already noted in [29] is now a theorem of Sbierski [58]. Singular black hole solutions were once thought to be unphysical artifacts of symmetry assumptions, but in view of Penrose's incompleteness theorem [47] and Christodoulou's work on the dynamic formation of black holes in vacuum from the focusing of gravitational waves [10], black holes are an unavoidable feature of general relativity.

A larger class of axisymmetric, stationary black hole solutions to the Einstein vacuum equations was discovered by Kerr [35] in 1963, and Carter [6] later found the maximal extensions of the Kerr spacetimes. The (subextremal) Kerr family models rotating black holes parameterized by mass M and angular momentum a ($0 \leq |a| < M$).[4] When $a = 0$, the Kerr spacetime recovers the spherically symmetric, non-rotating Schwarzschild solution. We cannot do justice here to the rich history of work on the long-standing conjecture that the exterior of a Kerr black hole is stable to perturbations of initial data, but we mention the pioneering works [43, 52, 63, 64, 67]. We also highlight a few recent breakthroughs, such as the proof of full nonlinear stability of slowly spinning Kerr ($|a|/M \ll 1$) by Giorgi–Klainerman–Szeftel [28]. Earlier progress includes the work of Dafermos–Rodnianski–Shlapentokh-Rothman [22] on the linear stability of Kerr to scalar perturbations, the proof of linear gravitational stability of Schwarzschild ($a = 0$) by Dafermos–Holzegel–Rodnianski [18] and of Kerr in the full subextremal parameter range $|a| < M$ by Shlapentokh-Rothman and Texeira da Costa [60], and the full nonlinear stability of Schwarzschild ($a = 0$) due to Dafermos–Holzegel–Rodnianski–Taylor [23]. See also Hintz–Vasy [31] for nonlinear stability of Kerr–de Sitter in the case of positive cosmological constant ($|a|/M \ll 1, \Lambda > 0$).

1.3. **Rotating black holes and failure of determinism.** Rotating Kerr black holes ($a \neq 0$) admit infinitely many smooth extensions, which are in particular independent of initial data. If realistic black hole interiors look like Kerr, then general relativity is not a deterministic theory. This is because an observer who falls into a Kerr black hole will reach the *Cauchy horizon* (the boundary of the region determined by their initial conditions), and no singularity is there to stop them from crossing this boundary and leaving determinism behind. The physically troubling implications of spacetimes with smooth Cauchy horizons are resolved if such a phenomenon turns out to be unstable with respect to perturbations. In 1968, Penrose [48] formulated the *strong cosmic censorship conjecture* that this is in fact the case.

1.4. **Initial value problem in general relativity.** Before we continue our discussion of strong cosmic censorship, it is worth clarifying how to pose initial data for the Einstein equations. The mathematical formulation of eq. (1) as an initial-value problem began with the seminal work of Choquet-Bruhat in 1952 [27] on the local well-posedness of the Cauchy problem. She understood Einstein's vacuum equations as a system of quasilinear wave equations and established the following local existence result (in the smooth category): for a Riemannian 3-manifold $(\overline{M}, \overline{g})$ and symmetric 2-tensor $\overline{k}$ on $\overline{M}$ satisfying an underdetermined system of constraint equations, there exists a vacuum spacetime $(\mathcal{M}, g)$ and an embedding $\iota : \overline{M} \to \mathcal{M}$ inducing the metric $\overline{g}$ and second fundamental form $\overline{k}$ on $\iota(\overline{M}) \subset \mathcal{M}$ such that $\iota(\overline{M})$ is a Cauchy surface for $\mathcal{M}$. One calls $\mathcal{M}$ a *globally hyperbolic development* of the initial data set $(\overline{M}, \overline{g}, \overline{k})$. Choquet-Bruhat also showed a local uniqueness result, which she later extended with Geroch in 1969 [8] to establish that the class of globally hyperbolic developments of $(\overline{M}, \overline{g}, \overline{k})$ has a maximal element $\mathcal{M}$ that is moreover unique up to isometry. One can think of this so-called *maximal globally hyperbolic development* as the largest Lorentzian manifold that is determined by the initial data through Einstein's equations.

[4] We remark that a large population of observed black holes spin very quickly ($a/M \geq 0.9$) [54].

For a detailed introduction to the Cauchy problem in general relativity, we refer the reader to [55]. We also mention that Sbierski [57] has simplified the proof of the Choquet-Bruhat–Geroch result by avoiding the use of Zorn's lemma. These results can also be generalized to Einstein–matter models.

1.5. **Strong cosmic censorship conjecture in spherical symmetry.** The strong cosmic censorship conjecture for the Einstein vacuum equations remains difficult (but see [19]), so we restrict ourselves to the simplified setting of spherical symmetry.[5] The natural candidate for a spherically symmetric model case of Kerr is Schwarzschild, the spherically symmetric member of the Kerr family. However, any spherically symmetric vacuum spacetime is a member of the Schwarzschild family. This is a theorem commonly attributed to Birkhoff [3], though actually proven earlier by Jebsen [33]. The upshot is that to study dynamical problems in spherical symmetry one must couple the Einstein equations to a matter model.

The simplest matter distribution is a homogeneous cloud of massless "null dust," famously considered by Oppenhymer and Snyder in 1939 [45] as a model of gravitational collapse. Christodoulou [12] showed that this Einstein–null dust model is unstable and develops naked singularities upon perturbation. He later considered the next simplest model, which involves a massless scalar field, and showed that generic solutions are indeed inextendible, with singularities like the one in Schwarzschild [9, 11].

The spherically symmetric Einstein–scalar field system that Christodoulou studied unfortunately cannot contend with the effects of angular momentum, which in some sense measures the departure from spherical symmetry of the metric (see the discussion in [15]). Thankfully, there is an analogy between angular momentum and the repulsive effects of *charge* that persists in spherical symmetry. One manifestation of this is that the charged members of the Reissner–Nordström [44, 53] family of spherically symmetric static *electrovacuum* black hole spacetimes has the same Penrose diagram as rotating Kerr, including the smooth Cauchy horizons that pose a challenge to determinism. We emphasize that the charged spherically symmetric model is of interest as an analogue for Kerr and that it is not itself physically relevant.[6]

The (subextremal) Reissner–Nordström spacetimes $(\mathcal{M}_{M,\mathbf{e}}, g_{M,\mathbf{e}})$, parameterized by mass M and charge $\mathbf{e}$ satisfying $0 \le |e| < M$ are given in coordinates by

$$\begin{aligned} \mathcal{M}_{M,\mathbf{e}} &= \mathbf{R}_t \times (r_-, \infty)_r \times S^2 \\ g_{M,\mathbf{e}}|_{\{r\neq r_+\}} &= -\Omega^2\,\mathrm{d}t^2 + \Omega^{-2}\,\mathrm{d}r^2 + r^2\gamma_{\mathbf{S}^2}, \qquad \Omega^2 = 1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2}. \end{aligned} \tag{2}$$

Here $\gamma_{\mathbf{S}^2}$ is the round metric of radius 1 on S^2, and we write $r_\pm = M \pm \sqrt{M^2 - \mathbf{e}^2}$ when $\mathbf{e} \neq 0$ and mean $(r_-, r_+) = (0, 2M)$ when $\mathbf{e} = 0$. Charged Reissner–Nordström ($\mathbf{e} \neq 0$) has a Cauchy horizon "at $\{r = r_-\}$" across which it admits infinitely many smooth extensions. When $\mathbf{e} = 0$, the Reissner–Nordström spacetime recovers the Schwarzschild solution to the Einstein vacuum equations, which is C^0-inextendible [58]. The relevant model is the Einstein–Maxwell–(real) scalar field system, whose solution consists of a $(1 + 3)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$, a massless uncharged real-valued scalar field $\varphi : \mathcal{M} \to \mathbf{R}$, and a 2-form F called the Maxwell field F. The scalar field φ solves the wave equation, the Maxwell field F

[5]Spherical symmetry is the assumption that SO(3) acts on the spacetime by isometry with spacelike spherical orbits. This induces a round metric on each (spherical) group orbit as well as a $(1 + 1)$-dimensional Lorentzian metric on the quotient manifold (with boundary) $\mathcal{Q} := \mathcal{M}/\,\mathrm{SO}(3)$. We can therefore define an *area-radius function* $r : \mathcal{M} \to \mathbf{R}_{>0}$ by $g = g_{\mathcal{Q}} + r^2\gamma_{\mathbf{S}^2}$, where we identify $\mathcal{M} = \mathcal{Q} \times S^2$ and $\gamma_{\mathbf{S}^2}$ is the round metric on the unit sphere. In this setting, we also assume that all matter fields are spherically symmetric, i.e. constant on each group orbit.

[6]A quick heuristic argument considering the charge to mass ratio of elementary charged particles and the ratio of electromagnetic to gravitational force exerted on a test particle suggests that a body with charge to mass ratio larger than about 10^{-18} would attract particles of the opposite charge. Thus only the case of exceedingly small charge is physically relevant for the study of black holes. Moreover, the massless uncharged matter model that we study admits Cauchy data on a two-ended hypersurface $\mathbf{R} \times S^2$, rather than the physically relevant *one-ended* hypersurface $\mathbf{R}^3$.

solves Maxwell's equations, and the evolution of the metric g is coupled to φ and F as follows:

$$\begin{cases} \mathrm{Ric}(g) - \frac{1}{2}gR(g) = 2(T^{(\mathrm{sf})} + T^{(\mathrm{em})}), \\ T^{(\mathrm{sf})}_{\alpha\beta} = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}g_{\alpha\beta}\partial^\mu\varphi\partial_\mu\varphi, \\ T^{(\mathrm{em})}_{\alpha\beta} = F_\alpha{}^\nu F_{\beta\nu} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}, \\ \Box_g\varphi = 0 \quad \mathrm{d}F = \star\mathrm{d}\star F = 0. \end{cases} \tag{3}$$

Here $\star$ and $\Box_g$ are the Hodge star and Laplace–Beltrami operators with respect to the metric g, and all indices are raised with respect to the metric g. Reissner–Nordström is a solution to this system with $\varphi = 0$ and $F = \frac{1}{2}r^{-2}\mathbf{e}\Omega^2\,\mathrm{d}u \wedge \mathrm{d}v$, where Ω^2 is as in eq. (2). In fact, in spherical symmetry the Maxwell equations decouple from eq. (3) (see [16, §2]), and we can compute its contribution to the energy-momentum tensor in terms of the area-radius function r and a *constant* $\mathbf{e}$, which we call charge. For further information about eq. (3) in spherical symmetry, see [39, §2]. We only mention that eq. (3) can be expressed as a system of wave equations for the area-radius r, the scalar field φ, and the metric component Ω^2, and it can be further reformulated in terms of the Hawking mass

$$m = \frac{r}{2}\Big(1 - \nabla^\alpha r\nabla_\alpha r\Big) = \frac{r}{2}\Big(1 + \frac{4\partial_u r\partial_v r}{\Omega^2}\Big). \tag{4}$$

One may conjecture that, just as for rotating Kerr, the smooth Cauchy horizon of charged Reissner–Nordström is non-generic.

Conjecture 1.1 (Strong cosmic censorship in spherical symmetry)**.** *The maximal globally hyperbolic development arising from generic asymptotically flat initial data to the Einstein–Maxwell–scalar field system in spherical symmetry is future inextendible as a suitably regular Lorentzian manifold.*

We will say, for example, "strong cosmic censorship in C^2" as a shorthand for Conjecture 1.1 in the class of C^2 Lorentzian manifolds.[7]

Remark 1.2 (Genericity). The "generic" collection of initial data is to contain a set that is open and dense in the space of all data (with respect to topologies that depend on the particular result). For the sake of exposition, we will not revisit this technical detail.

We now state some main results on this conjecture.

Theorem 1.3 (Results on strong cosmic censorship in spherical symmetry)**.**
(1) (Christodoulou [9, 11]). *Strong cosmic censorship in C^0 is true for the Einstein–scalar field system (that is, $\mathbf{e} = 0$) for 1-ended and 2-ended asymptotically flat initial data on $\mathbf{R}^3$ and $\mathbf{R} \times S^2$, respectively.*
(2) (Dafermos [16, 17], Dafermos–Rodnianski [20]). *Strong cosmic censorship in C^0 is false for the Einstein–Maxwell–scalar field system with $\mathbf{e} \neq 0$ for 2-ended asymptotically flat initial data on $\mathbf{R} \times S^2$.*
(3) (Luk–Oh [39, 40], Sbierski [56]). *Strong cosmic censorship in C^2 (and in fact in $C^{0,1}_{\mathrm{loc}}$ for small data) is true for the Einstein–Maxwell–scalar field system for 2-ended asymptotically flat initial data on $\mathbf{R} \times S^2$.*

[7]Namely, we are referring to the statement that there is no isometric embedding $\iota : (\mathcal{M}, g) \to (\widetilde{\mathcal{M}}, \widetilde{g})$ of the maximal globally hyperbolic development $\mathcal{M}$ into a manifold $\widetilde{\mathcal{M}}$ with C^2 Lorentzian metric $\widetilde{g}$ such that there is $p \in \widetilde{\mathcal{M}} - \iota(\mathcal{M})$ whose future in $\widetilde{\mathcal{M}}$ is disjoint from $\iota(\mathcal{M})$.

In other words, in the uncharged case $\mathbf{e} = 0$, the C^0-inextendibility of Reissner–Nordström (i.e. Schwarzschild) is generic, while in the strictly charged case $\mathbf{e} \neq 0$, the C^2 Cauchy horizon of Reissner–Nordström is non-generic, even though continuous extensions are resistant to perturbations. In particular, generic spacetimes do not admit extensions with enough regularity to classically pose the Einstein equations.[8]

We also mention the results of Costa–Girão–Natário–Silva [14] for the present model in the case of positive cosmological constant and of van de Moortel [65, 66] and van de Moortel–Kehle [34] for strong cosmic censorship and mass inflation in the presence of a massive charged scalar field.

1.6. **Mass inflation.** An observer passing through the Cauchy horizon of a Reissner–Nordström black hole observes the outside universe shifted infinitely to the blue, as noted by Penrose [48, p. 222] when formulated the strong cosmic censorship conjecture. The idea is that an observer who enters the black hole exits through the Cauchy horizon in finite proper time, while an observer who stays outside the black hole exists for all time. As a result, if the "outside observer" sends signals into the black hole at what they measure as regular intervals, then the "inside observer" receives these intervals more and more frequently, until at the Cauchy horizon they meet a wall of radiation shifted infinitely to the blue. At the level of linear perturbations, [7, 61] argued that the blueshift effect should manifest by exponentially amplify gravitational disturbances from the black hole exterior, and so the Cauchy horizon should be unstable.

In [32] Hiscock made the first attempt to understand this instability with a nonlinear model, using an explicit solution to the Einstein–null dust system with one incoming dust. He showed that the metric remains continuous at the Cauchy horizon, but its Christoffel symbols blow up in a parallelly propagated frame. The full nonlinear phenomenon was uncovered in the seminal works of Poisson–Israel [49, 50], who considered a null dust model with an additional, outgoing null dust. They showed that the Hawking mass generically becomes infinite at the Cauchy horizon, and they named this scenario *mass inflation.* Because the Hawking mass bounds the Kretschmann scalar (namely the full trace $\mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}^{\alpha\beta\gamma\delta}$ of the Riemann curvature tensor $\mathcal{R}$), mass inflation immediately implies the C^2 formulation of strong cosmic censorship.[9] The phenomenon of mass inflation was corroborated in numerous analytic and numerical studies such as [4, 5, 46].

The first mathematical result verifying mass inflation is due to Dafermos in his pioneering work [16], which *assumes* pointwise upper and lower bounds for the generic decay of the scalar field along the event horizon. Although the required upper bound is established for *all* initial data by Dafermos–Rodnianski in their remarkable work [20], the lower bound (which must involve a genericity condition since Reissner–Nordström is a solution with $\varphi = 0$) remains open. Luk–Oh–Shlapentokh-Rothman [41] have recently shown a mass inflation result using the generic L^2 lower bound established in [39, 40] to prove strong cosmic censorship in C^2. Their result is again *conditional* on an improved L^2 upper bound. We now formally state their results in the setting of compactly supported Cauchy data to the Einstein–Maxwell–scalar field system.

Theorem 1.4 (Mass inflation from pointwise lower bound; Dafermos–Rodnianski [20], Dafermos [4])**.** *The following pointwise upper bound holds (for all $\epsilon > 0$):*

$$|\partial_v\varphi||_{\mathcal{H}^+} \leq C_\epsilon v^{-3+\epsilon}. \tag{5}$$

Mass inflation occurs if the following pointwise lower bound also holds:

$$|\partial_v\varphi||_{\mathcal{H}^+} \geq c v^{-9+\epsilon}. \tag{6}$$

[8]This suggests a formulation of strong cosmic censorship at a level of regularity between C^0 and C^2, namely the square integrability of Christoffel symbols (see [10, p. 13] and [13]). If this conjecture were true, then the continuous extensions in (2) of theorem 1.3 would be too singular to be considered solutions to Einstein's equations.

[9]From the form of the Hawking mass (see eq. (4)), mass inflation also implies that the spacetime is inextendible in the class of C^1 spherically symmetric Lorentzian manifolds.

Theorem 1.5 (Mass inflation from L^2 upper bound; Luk–Oh [39, 40], Luk–Oh–Shlapentokh-Rothman [41])**.** *The following L^2 lower bound generically holds:*

$$\int_{\mathcal{H}^+} v^8(\partial_v\varphi)^2 \,\mathrm{d}v = \infty. \tag{7}$$

Mass inflation occurs if eq. (7) *and the following L^2 upper bounds hold:*

$$\int_{\mathcal{H}^+} v^6(\partial_v\varphi)^2 \,\mathrm{d}v < \infty, \qquad \int_{\mathcal{H}^+} v^8(\partial_v^2\varphi)^2 \,\mathrm{d}v < \infty. \tag{8}$$

These results demonstrate a competition between instability in the interior and decay in the exterior. If a collapsing rotating star conspires to leave no trace of its existence, then the resulting black hole would have a smooth Cauchy horizon. The interpretation in our model is that if late-time decay is so strong as to give $\partial_v\varphi|_{\mathcal{H}^+\cap\{v\geq V\}} \equiv 0$ for large V, then the interior of the black hole coincides with Reissner–Nordström in a neighbourhood of the Cauchy horizon. On the other hand, the blueshift effect threatens to amplify sufficiently slow decay, as captured by a lower bound on $|\partial_v\varphi|$, into a singularity.

1.7. **Goals of the project.** We hope to establish the following improved decay results in view of theorems 1.4 and 1.5.

Goal 1.6 (Generic mass inflation)**.** *Solutions arising from* all *small, smooth, compactly supported, perturbations of (subextremal) Reissner–Nordström data obey the following estimate for $k = 1, 2$:*

$$\int_{\mathcal{H}^+\cap\{v\geq 1\}} v^{4+2k}(\partial_v^k\varphi)^2(v) \,\mathrm{d}v < \infty. \tag{9}$$

By [41, Thm. 7.5], generic such spacetimes exhibit mass inflation when $\mathbf{e} \neq 0$.

Goal 1.7 (Price's law decay)**.** *Let $0 \leq |\mathbf{e}| < M$ and fix $\epsilon > 0$. There is a family of norms $D^k_{M,\mathbf{e}}$ measuring the difference of the first k derivatives of the Cauchy data from Reissner–Nordström data with parameters $(M, \mathbf{e})$, constants $C_\epsilon, \delta_0 > 0$ (depending on M and $\mathbf{e}$), and an integer $N_0 \geq 0$ such that for an integer $N \geq 0$ and data satisfying $D_{N+N_0} \leq \delta_0$, the following estimates hold in a future-normalized null coordinate system (u, v) for $0 \leq j + k \leq N$:*

$$\begin{cases} |\partial_v^k\varphi| \leq C_\epsilon D_{N+N_0} v^{-3-k+\epsilon} & \textit{on } \mathcal{H}^+, \\ |\partial^k\varphi| \leq C_\epsilon D_{N+N_0} v^{-3+\epsilon} & \textit{in } \{r \leq 30 r_{\mathcal{H}^+}\}, \\ |\partial_u^j\partial_v^k\varphi| \leq C_\epsilon D_{N+N_0} u^{-2+\epsilon} r^{-1} \max(r^{-j}, u^{-j}) r^{-k} & \textit{in } \{r \geq 10 r_{\mathcal{H}^+}\}. \end{cases} \tag{10}$$

Here $r_{\mathcal{H}^+} := \sup_{\mathcal{H}^+} r$.

Note that Goal 1.7 implies Goal 1.6. The type of decay result in Goal 1.7 is known is as Price's law, due to heuristics by Price [51]. The best known Price's law result for the Einstein–Maxwell–scalar field system is due to Dafermos–Rodnianski [20], who show $|\varphi| + |\partial_v\varphi| \leq C_\epsilon v^{-3+\epsilon}$ along the horizon. In particular, even a $v^{-3+\epsilon}$ decay rate is not known for more than two derivatives. Our goal is to obtain bounds for *all derivatives* and show that each additional ∂_v derivative gains an extra power of v-decay along the horizon. The expectation is that $|\partial_v^k\varphi||_{\mathcal{H}^+} \sim v^{-3-k}$, which has been established for linear wave equations on spacetimes including subextremal Reissner–Nordström [1] and Kerr [30].

Acknowledgements

I am grateful to my advisor Jonathan Luk for teaching me about mathematical general relativity, introducing me to the problem of mass inflation, and for his patience and guidance in many enlightening discussions. I also thank the SURIM program for supporting my work on this project in the summer of 2022.

2. Proof strategy

For a detailed introduction to the Einstein–Maxwell–scalar field system in spherical symmetry, including the notation used below, we refer the reader to [40, §2]. We use three families of null coordinate systems:

- (Cauchy) data-normalized (U,V): These coordinates are normalized by $\frac{\mathrm{d}U}{\mathrm{d}\rho}|_\Sigma = -1$ and $\frac{\mathrm{d}V}{\mathrm{d}\rho}|_\Sigma = 1$, where Σ is the Cauchy surface on which the data is prescribed (in spherical symmetry Σ becomes a curve in $\mathcal{Q}$), and ρ parametrizes Σ. Norms of the initial data are measured in these coordinates, and
- (Characteristic-)future-normalized (u,v): These coordinates are adapted to a neighbourhood of timelike infinity in the form of a characteristic rectangle $\mathcal{R} = \{(U,V) : U \geq U_0, V \geq V_0\}$. Then the null hypersurfaces $\underline{C}_{\mathrm{in}} = \{(U,V) : U \geq U_0, V = V_0\}$ and $C_{\mathrm{out}} = \{(U,V) : U = U_0, V \geq V_0\}$ make up the past boundary of $\mathcal{R}$, and the future boundary of $\mathcal{R}$ (in the topology of the Penrose diagram) contains segments of $\mathcal{I}^+$ and $\mathcal{H}^+$. We normalize the (u,v) coordinates by the conditions $\frac{\partial_u r}{1-\mu}|_{\mathcal{I}^+} = -1$ and $\frac{\partial_v r}{1-\mu}|_{\mathcal{H}^+} = 1$ and fix the freedom of translation by $\underline{C}_{\mathrm{in}} = \mathcal{R} \cap \{(u,v) : v = 1\}$ and $C_{\mathrm{out}} = \mathcal{R} \cap \{(u,v) : u = 1\}$. Then in (u,v)-coordinates we have $\mathcal{R} = \{(u,v) : u \geq 1, v \geq 1\}$.
- Bootstrap (future-normalized) $(u_{(t)}, v)$: For analysis in the bootstrap regions $\mathcal{R}_{(t)} = \{(u,v) : u \in [1,\infty), v \in [1,t]\}$ $(t \geq 1)$, we use the future-normalized outgoing null coordinate v and an ingoing null coordinate $u_{(t)}$ adapted to $\mathcal{R}_{(t)}$ in the sense that $\frac{\partial_{u_{(t)}} r}{1-\mu}(\cdot, v = t) = -1$ and $\underline{C}_{\mathrm{in}} = \{(u_{(t)}, v) : u_{(t)} = 1\}$.

We conclude by roughly outlining the proof. The energy estimates we use (step 6) involve vector field multipliers such as those in [40, §8.8–8.9] (see [21] for a general introduction to energy estimates on black hole spacetimes, including the use of the redshift vector field multiplier). We commute with a dynamically defined "scaling" vector field S which in the static case of the Reissner–Nordström exterior takes the form $v\partial_t$ near the horizon and $t\partial_t + r^*\partial_{r^*} = u\partial_u + v\partial_v$ away from the horizon. The use of the analogous vector field $S = t\partial_t + r\partial_r$ as a commutator on Minkowski space goes back to Klainerman's seminal work introducing the vector field commutator method [36]. The method of obtaining decay by commuting with S and performing elliptic estimates on spacelike hypersurfaces can be found in Klainerman–Sideris [37], but the idea of *spacetime elliptic estimates* is from forthcoming work of Luk–Oh [38]. Finally, the ODE argument on ingoing null hypersurfaces that propagates decay from good derivatives of $S^k\varphi$ to all derivatives $\partial S^k\varphi$ (and to $S^k\varphi$ itself) is inspired by the work of Metcalfe–Tataru–Tohaneanu on Price's law decay for nonstationary spacetimes [42] (see also the degenerate elliptic estimate in [2, §7.1]). Here are the main steps of the proof:

(1) (Choosing the characteristic rectangle) First, we choose $\mathcal{R}$ "close enough to timelike infinity" that $\sup_{\mathcal{R}}|\varpi - \varpi_f|$ and $\sup_{\mathcal{R}\cap\mathcal{H}^+}|r/r_{\mathcal{H}} - 1|$ are smaller than an absolute constant.
(2) (Reducing to the characteristic rectangle) Use a Cauchy stability argument and an argument near spacelike infinity to show that small Cauchy data on Σ implies small characteristic data on $\underline{C}_{\mathrm{in}} \cup C_{\mathrm{out}}$.
(3) (Bootstrap assumptions on the geometry) We now begin a bootstrap argument in the regions $\mathcal{R}_{(t)}$ in $(u_{(t)}, v)$-coordinates. We make assumptions on the geometric quantities that are small when the scalar field is small. Namely, we assume that $S^k \log\kappa$ and $\mathbf{1}_{r\geq R} S^k \log(-\gamma)$ for $0 \leq k < N$ are $O(v^{-1})$, and we make the same assumption for $\lambda|_{\mathcal{H}^+}$ (see [39, App. B]). We also assume that $S^k\varpi$ is $O(1)$ for $1 \leq k \leq N$.
(4) (Obtain full geometric control) Use the bootstrap assumptions to show that the following quantities $|S^k v - v|, |S^k u - u|, |S^k r|, |S^k\mu|, |S^k\lambda|, |S^k\nu|$ are $O(1)$.
(5) (Control commutators) Use this geometric control to show that $[\Box, S^k]\varphi = \sum_{j=1}^{k-1} O(r^{-2}\log r)\partial S^j\varphi$.
(6) (Energy estimates) Establish energy estimates and commute with S^k for $1 \leq k \leq N$.
(7) (Control of data in $(u_{(t)}, v)$ coordinates) Use the control of commutators and geometric quantities to show that the energy of $\partial_u S^k\varphi$ on $\underline{C}_{\mathrm{in}}$ and $\partial_v S^k\varphi$ on C_{out} in $(u_{(t)}, v)$-coordinates is bounded by the energy quantities known to be small by step 2. This involves commuting with $\frac{1}{(-\nu)}\partial_u$ (and making

a suitable bootstrap argument for geometric quantities differentiated by this vector field). Now we know that energy quantities involving $S^k\varphi$ are small, since they are bounded by the data.

(8) (Decay argument) Begin by obtaining L^∞-bounds (with r-decay) on $S^k\varphi$ for $k \leq N/2$ using the energy quantities.

(a) (Spacetime elliptic estimates) Use elliptic estimates in spacetime regions to show that each derivative ∂ hitting on φ gains an extra power of decay (of r in the region $r \lesssim u$ and of u in the region $u \lesssim r$). Use the form of S to show that certain derivatives hitting on φ have v-decay, namely T in the region $r \lesssim u$ and ∂_v in the region $u \lesssim r$.

(b) (ODE argument) In an induction on k, write the wave equation as an ODE for $\partial_u S^k\varphi$ whose source term consists of terms that decay better, such as $TS^j\varphi$ and $\partial TS^j\varphi$, and obtain one power of v-decay for $\partial S^k\varphi$ ($k \leq N/2$).

(9) (Recovering the bootstrap assumptions) Use the equations for the geometric quantities in step 3, the normalization conditions in the definition of $\mathcal{R}_{(t)}$, and the decay for lower order $\partial S^k\varphi$ obtained in step 8 to recover the bootstrap assumptions of step 3.

(10) (Improving decay rate) The argument of step 8 (along with an additional argument improving decay in the region $u \lesssim r$) can be iterated as many times as needed to get the desired decay rates. These improved conclusions are not needed to close the bootstrap argument, so this step can be done outside a bootstrap setting.

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