Mass Inflation in General Relativity

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Introduction

Mathematical general relativity uses techniques from geometry and partial differential equations to study spacetimes, which are solutions to Einstein's equations.

General relativity predicts the existence of black holes, namely regions of spacetime from which even light cannot reach far away observers. The *strong cosmic censorship conjecture* is roughly the expectation that a black hole contains a singularity.

One scenario exhibiting a singularity is mass inflation (Goal 1). The blowup of mass inside a spherically symmetric black hole follows from a decay result outside the black hole (Goal 2).

The Formal Setup

A spacetime is a $(1+3)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ solving Einstein's equations. We study the Einstein–Maxwell–scalar field system in spherical symmetry. Spherical symmetry means that $\mathcal{M} = \mathcal{Q} \times S^2$ and $g = g_{\mathcal{Q}} \times r^2 \gamma_{\mathbf{S}^2}$ for $(\mathcal{Q}, g_{\mathcal{Q}})$ a $(1+1)$-dimensional Lorentzian manifold and $\gamma_{\mathbf{S}^2}$ the round metric on a unit sphere.

This model takes the following form, where φ is a real-valued massless scalar field on $\mathcal{M}$ and F is the Maxwell field (a 2-form), and $T(\varphi)$ and $T(F)$ are tensors constructed from them:

$$\begin{cases} \mathrm{Ric}(g) - \frac{1}{2} g R(g) = T(\varphi) + T(F) \\ \Box_g \varphi = 0 \qquad \mathrm{d}F = \star \mathrm{d} \star F = 0 \end{cases}$$

Strong cosmic censorship is the expectaion that general relativity is a deterministic theory. That is, a spacetime that exhausts the possibilities determined by initial data cannot be extended as a suitably regular Lorentzian manifold, perhaps due to a singularity.

Penrose Diagrams

We can draw all of spacetime and read which events are in the future of others (see figures).

"Time" and "space" are on the vertical and horizontal axes. Light travels along 45° lines.

Bootstrap Method

Q: How do you prove $E(t) \leq 10$ if you know E is continuous and $E(0) \leq 5$?

A: Show $E(t) \leq 20$ implies $E(t) \leq 10$!

This lets us make assumptions, as long as we can recover them later.

Reissner–Nordström

These spherically symmetric black hole spacetimes $(\mathcal{M}_{\mathrm{RN}}, g_{\mathrm{RN}})$ are parametrized by mass M and charge $\mathbf{e}$ $(0 \leq |\mathbf{e}| < M)$:

$$\mathcal{M}_{\mathrm{RN}} = \mathbf{R}_t \times (r_-, \infty)_r \times S^2$$
$$g_{\mathrm{RN}}|_{\{r > r_+\}} = -\Omega^2 \, \mathrm{d}t^2 + \Omega^{-2} \, \mathrm{d}r^2 + r^2 \gamma_{\mathbf{S}^2}$$
$$\Omega^2 = 1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2},$$

There is *no singularity* inside this black hole when the charge is nonzero. In fact, there are infinitely many smooth extensions (past the Cauchy horizon $\mathcal{CH}^+$), so determinism fails. This feature is unstable under perturbations, which are generically extendible in C^0 and inextendible in C^2.

Goal 1 (Mass inflation)

Mass inflation occurs for generic small perturbations of Reissner–Nordström data. That is, the Hawking mass is infinite along $\mathcal{CH}^+$, the lightlike future boundary of the black hole interior.

Goal 2 (Price's law decay)

The scalar field decays like $|\partial_v^k \varphi||_{r=R} \leq C_{\epsilon,R} t^{-3-k+\epsilon}$ on curves of constant r for *all* small perturbations of Reissner–Nordstrom.

Energy Method

Observation: If

$$0 = \Box_{\mathbf{R}^{1+3}} \varphi := -\partial_t^2 \varphi + \sum_{i=1}^{3} \partial_i^2 \varphi$$

then

$$\int_0^T \partial_t \varphi \Box_{\mathbf{R}^{1+3}} \varphi \, \mathrm{d}x \, \mathrm{d}t = 0. \qquad (*)$$

This implies that the energy

$$E[\varphi](t) := \int_{\mathbf{R}^3} (\partial_t \varphi)^2 + \sum_{i=1}^{3} (\partial_i \varphi)^2 \, \mathrm{d}x$$

is conserved. To go further, consider $(*)$ with $X\varphi$ in place of $\partial_t \varphi$ for another "multiplier" vector field X, and also consider $E[Y\varphi](t)$ for "commutator" vector fields Y.

Proof Outline

- Make *bootstrap assumptions* on geometric quantities that are small when φ is.
- Control all other geometric quantities.
- Prove weighted energy estimates.
- Use spacetime elliptic estimates to show some derivatives of φ decay better.
- Write wave equation as ODE for bad derivatives with source of good derivatives to show all derivatives of φ decay better.
- Use the fundamental theorem of calculus to show φ decays better.
- Recover initial assumptions.

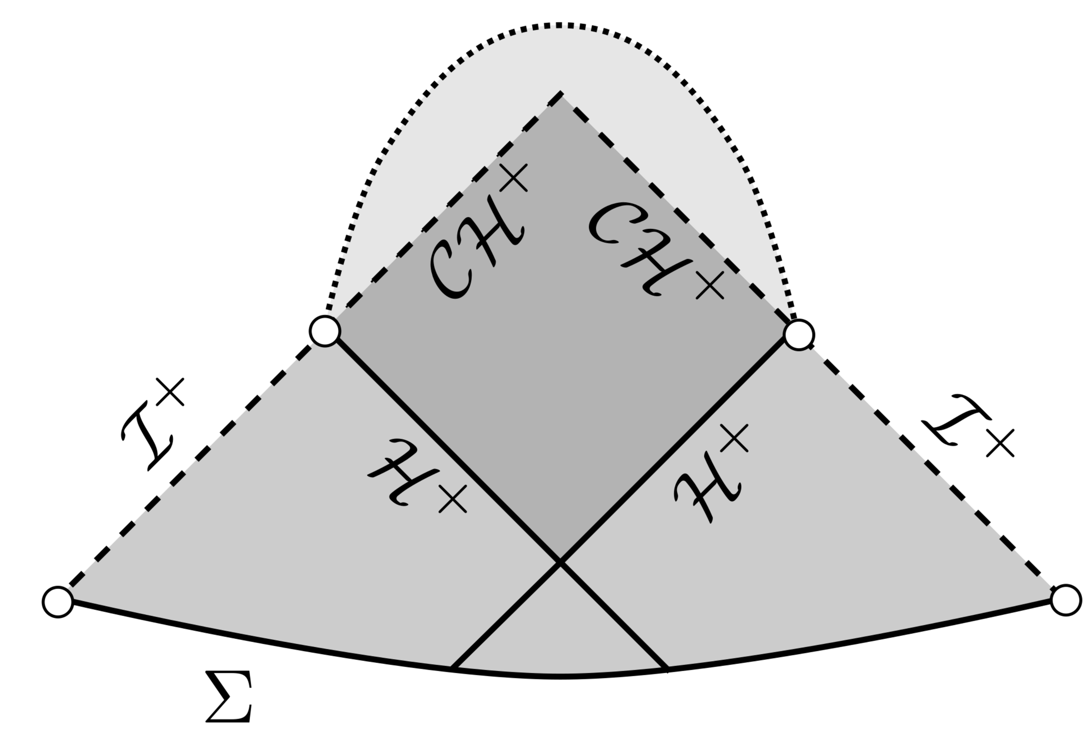

Figure: Reissner–Nordström for $\mathbf{e} \neq 0$. This spacetime can be extended in many ways beyond the Cauchy horizon $\mathcal{CH}^+$, so the future of observers is not determined by their past as prescribed on the surface Σ.

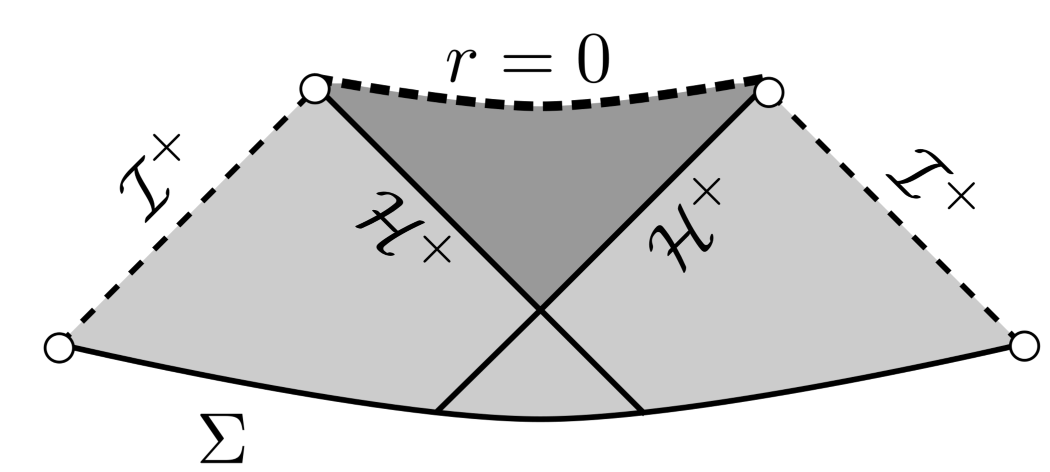

Figure: Schwarzschild ($\mathbf{e} = 0$). There is a curvature singularity at $\{r = 0\}$ beyond which this spacetime cannot be extended.

Further Directions

We obtain almost sharp upper bounds for φ and its derivatives. One can hope to obtain sharp decay rates (upper and lower bounds).

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References

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