Methods of functional analysis and geometry in transportation theory

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Introduction and Background

The study of optimal transport dates back to the 18th century, where Gaspard Monge considered minimizing the ‘cost’ of transporting mounds of dirt into holes, where maps from these subsets of Euclidean space defined the transport and cost of the map was determined by the distance moved. In the 20th century, Soviet economist/mathematician Kantorovich generalized the problem to a variety of spaces by reformalizing the source and the sink as probability measures on Polish spaces X and Y (i.e. normalized to aggregate mass 1).

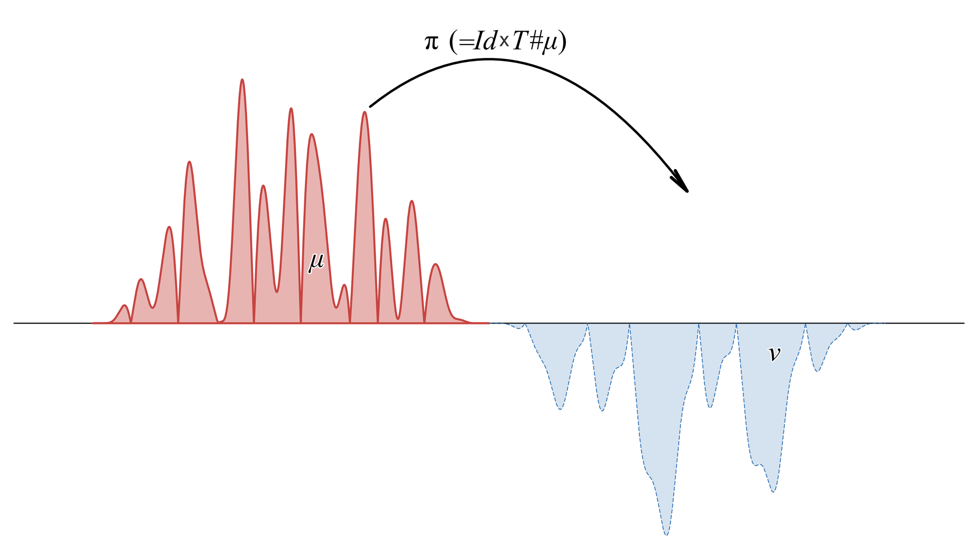

Figure 1. Example of mass transport where $X = Y = \mathbb{R}$

A *transference plan* π is then a probability measure on the space $X \times Y$, where for a rectangle $A \times B$, $\pi(A \times B)$ we say $\pi(A \times B)$ is the mass transported from A to B. To be admissible, $\pi(A \times Y) = \mu(A)$ and likewise $\pi(X \times B) = \nu(B)$; we say π *has marginals* μ and ν and denote $\Pi(\mu, \nu)$ the space of these admissible plans. Given a function determining the cost of moving mass between spaces $c : X \times Y \to \mathbb{R}^+ \cup \{+\infty\}$, the cost of a plan is given by

$$I[\pi] = \int_{X \times Y} c(x, y) d\pi(x, y)$$

The Kantorovich problem then becomes minimizing $I[\pi]$ over $\Pi(\mu, \nu)$. The Monge problem restricts us to plans which doesn't ‘split’ mass; i.e. there exists some map $T : X \to Y$ so that π is the pushforward of mass from μ to ν by T, denoted $\pi = (Id \times T)\sharp\mu$.

Kantorovich Duality

Theorem (Kantorovich Duality): Let X, Y be Polish spaces and $c : X \times Y \to \mathbb{R}$ be a lower semi-continuous cost function. Then,

$$\inf_{\Pi(\mu,\nu)} \int_{X \times Y} c(x, y) d\pi = \sup_{\Phi_c} \int_X \varphi(x) d\mu + \int_Y \psi(y) d\nu,$$

where the infimum on the left-hand side is achieved. For the general Kantorovich statement, we rely on the following minimax principle (as well as the Riesz Representation theorem) to go between the Kantorovich problem of measures and its dual functional problem.

Theorem (Fenchel-Rockafellar): Any pair of proper, convex functions $\varphi, \psi : X \to \mathbb{R}^+ \cup \{+\infty\}$ which share domain satisfy

$$\inf\{\varphi(x) + \psi(x)\} = \max_{f \in X^*}\{-\varphi^*(-f) - \psi^*(f)\},$$

where φ^* denotes the convex conjugate, defined by

$$\varphi^*(f) = \sup_{x \in X}\{\langle f, x\rangle - \varphi(x)\}.$$

Optimal Transport of Quadratic Cost Functions

In the more geometric setting where $X, Y \subset \mathbb{R}^n$ and our cost function is $c(x, y) = |x - y|^2$, we may recover some geometric facts.

Knott-Smith Optimality Criterion: A transference plan π is optimal if and only if there exists a convex lower semi-continuous function φ such that

$$\text{supp}(\pi) \subset \text{Graph } (\partial\varphi)$$

and the pair (φ, φ^*) minimizes

$$\int_{\mathbb{R}^n} \varphi(x) d\mu + \int_{\mathbb{R}^n} \varphi^*(y) d\nu$$

where $\varphi^*(y) := \sup\{x \cdot y - \varphi(x) : x \in \mathbb{R}^n\}$ in Euclidean space.

Brenier's Theorem There is a unique optimal π such that

$$\pi = [Id \times \nabla\varphi]\sharp\mu$$

and moreoever,

$$Supp(\nu) = \overline{\nabla Supp(\mu)}.$$

The pushforward maps φ are called Brenier maps.

Together, the Knott-Smith Criterion and Brenier's Theorem imply the *existence and uniqueness* of optimal maps for the Monge problem in the case where μ does not give mass to sets with Hausdorff co-dimension ≥ 1.

Cyclic Monotonicity

A subset $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *cyclically monotone* if for all $m \geq 1$, and for any $(x_1, y_1)...(x_m, y_m) \in \Gamma$, writing $x_{m+1} = x_1$,

$$\sum_{i=1}^{m} y_i \cdot (x_{i+1} - x_i) \leq 0.$$

Cyclic monotonicity provides yet another reason why optimal transference plans must be supported in the subdifferential of a lower semicontinuous function, and can be applied to general Hilbert spaces.

Theorem: All optimal transportation plans π for quadratic cost functions are supported by cyclically monotone subsets of $\mathbb{R}^n$. It is conjectured that the converse is true, however this remains an open problem.

The figure below illustrates why this should be true: if π gave mass to a subset not cyclically monotone, then we could cyclically permute the mass given by π to balls on the subset. Clearly the μ marginal is preserved; with some care, an argument preserving the ν marginal can be achieved. But then this new measure would have lower cost.

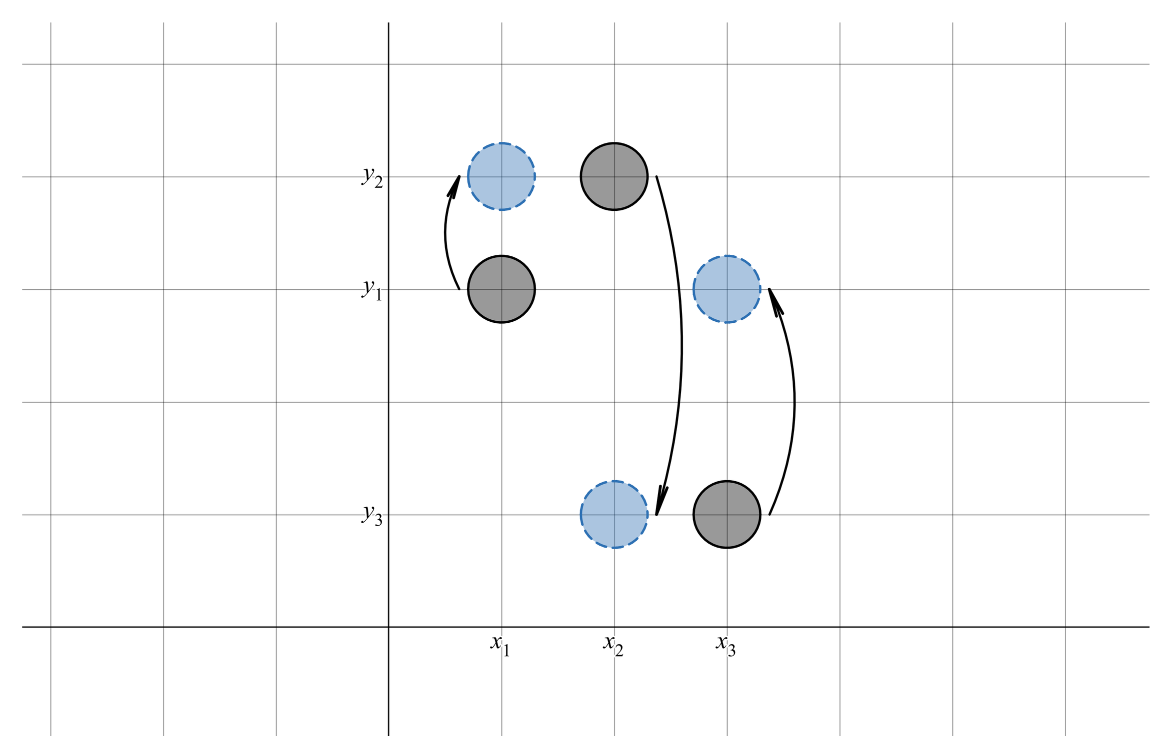

Figure 2. Example of an alternate cost function, under cyclic permutation

Cyclic Monotonicity (cont.)

It can be shown with little trouble that the cyclically monotone subsets of $\mathbb{R}^n$ are precisely the subsets of the sudifferentials of lower semi-continuous convex functions. These methods help us conclude with a stronger version of Brenier's theorem above that doesn't require a quadratic cost function (by exchanging the dot product with the scalar product for the duality).

Theorem (Brenier's, refined): When μ does not give mass to sets with codimension ≥ 1, then there is exactly one measurable map T such that $T\sharp\mu = \nu$ and $T = \nabla\varphi$ almost everywhere.

Applications

The Isoperimetric Inequality:

Problem: Which compact sets of $\mathbb{R}^n$ of given volume have minimal surface area?

Conclusion: It is well known that spheres are the solution to the above problem. This can be done using the existence of Brenier maps to confirm the Brunn-Minkowski inequality, which gives bounds for the Lebesgue measure of Minkowski sums of compact sets.

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