# Knot Heegaard Floer Homology, Bordered Heegaard Floer Homology, and the Surgery Exact Triangle

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## Contents



# 1 Introduction

In this note, we survey some results about knot Heegaard Floer homology in the context of knot Heegaard Floer homology and the application of bordered Heegaard Floer homology as a general conceptual framework to understand how Heegaard Floer homology is affected under cutting and pasting, where the precise result is given by the pairing theorem, which leads to the proof of the surgery exact triangle as a corollary.

# 2 Heegaard Floer Homology

## 2.1 Knot Heegaard Floer Homology (General form)

Refer to [5] for more details. Let $K \subset \mathbb{S}^3$ be an oriented knot. The hat version $\widehat{HFK}(K)$ is the simplest form of knot Floer homology of $K$, which takes the form of a bi-graded, finitely generated Abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HFK}_i(K, s).$$

Here, $i$ is called the *Maslov* (or *homological*) grading, and $s$ is called the Alexander grading. The graded Euler characteristic of $\widehat{HFK}$ is the Alexander-Conway polynomial:

$$\sum_{s,i \in \mathbb{Z}} (-1)^i q^s \cdot \mathrm{rank}_{\mathbb{Z}}(\widehat{HFK}_i(K, s)) = \Delta_K(q)$$

Another version is called *minus* knot Floer homology $HFK^-$ and has the form of a bi-graded module over the polynomial ring $\mathbb{Z}[U]$ and contains more information. The most complete version is a doubly-filtered chain complex denoted by $CFK^\infty$ called the *full knot Floer complex.*

## 2.2 Motivating the definition of $\widehat{HFK}$

Use the same definition of a (multi-pointed) *Heegaard diagram* as defined in [5]. Let tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and $\mathrm{Sym}(\Sigma)$ be the ones associated with the Heegaard diagram.

The intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ generate the complex $\widetilde{CFK}(\mathcal{H})$ which admits a bigrading $(M, A)$. The *Maslov* grading is characterized by the following relative index:

$$M(\boldsymbol{x}) - M(\boldsymbol{y}) = \mu(\phi) - 2\sum_{i=1}^{k} n_{w_i}(\phi),$$

where $\phi$ is any relative homotopy class in $\pi_2(\boldsymbol{x}, \boldsymbol{y})$. Here, $n_{w_i}(\phi)$ is defined as the intersection number between $\phi$ and the mnanifold

$$R_v = v \times \mathrm{Sym}^{d-1}(\Sigma)$$

inside $\mathrm{Sym}^d(\Sigma)$, where $d = g + k - 1$ and $k$ is the number of marked points in the Heegaard diagram.

It can be shown that the right hand side in the relative index formula is independent of the choice of $\phi$. There also exists a way of fixing $M$ as an absolute grading on $\mathbb{Z}$.

The Alexander grading $A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ is uniquely determined by the following two properties:

- For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we have

$$A(\boldsymbol{x}) - A(\boldsymbol{y}) = \sum_{i=1}^{k} n_{z_i}(\phi) - \sum_{i=1}^{k} n_{w_i}(\phi).$$

- We have

$$\sum_{\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(\boldsymbol{x})} q^{A(\boldsymbol{x})} = (1 - q^{-1})^{k-1} \cdot \Delta_K(q),$$

where $\Delta_K$ is the Alexander-Conway polynomial of the knot $K$.

For more information on relating Heegaard Floer homology to Kauffman links and the Alexander polynomial, refer to [6].

Define the differential

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1; n_{z_i}(\phi)=n_{w_i}(\phi)=0, \forall i}} (\#\widehat{M}(\phi)) \cdot \boldsymbol{y}.$$

The fact $\partial\partial = 0$ follows from Gromov compactness. The reason that we restrict to $n_{z_i} = n_{w_i} = 0$ is that this reduces the dimension of the moduli space of the pseudo-holomorphic representatives of $\phi$ (Irving gives an example for $\mathrm{Sym}^2(\Sigma) = \mathbb{T}^4 \# \overline{\mathbb{CP}^2}$ where $n_{z_i}$ counts the intersection number with a holomorphic ball). Let $\widetilde{HFK}(\mathcal{H})$ be the homology associated with $\widetilde{gCFK}(\mathcal{H})$ and $\partial$. For $k = 1$, define $\widehat{HFK}(\mathcal{H})$ as $\widetilde{HFK}(\mathcal{H})$. It can be shown that $\widehat{HFK}(\mathcal{H})$ depends only on $K$, so $\widehat{HFK}(K)$ is well defined.

## 2.3 Knot Floer homology as the categorification of the Alexander polynomial

In this section, we show that the Euler characterstic of knot Floer homology is the symmetric version of the Alexander polynomial, following [6].

The key idea is that the generators for a chain complex associated to a knot Heegaard diagram corresponds to the set of Kauffman states. See below and [6] for details.

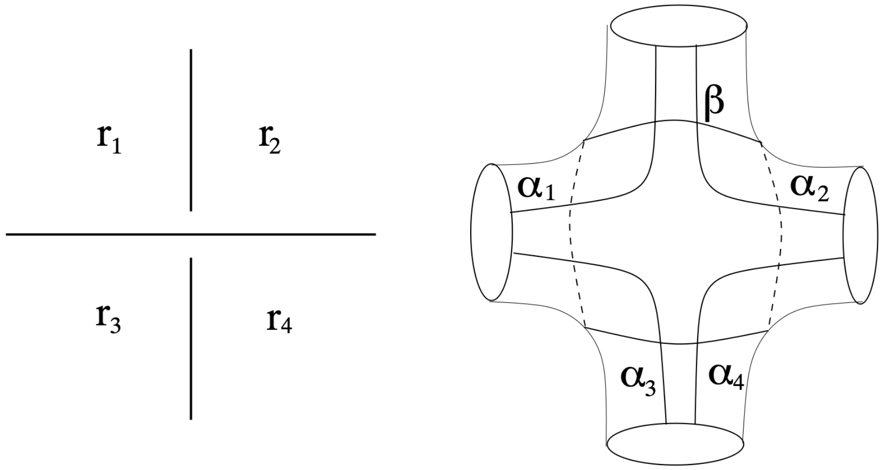



FIGURE 10. **Special Heegaard diagram for knot crossings.** At each crossing as pictured on the left, we construct a piece of the Heegaard surface on the right (which is topologically a four-punctured sphere). The curve $\beta$ is the one corresponding to the crossing on the left; the four arcs $\alpha_1, ..., \alpha_4$ will close up.

For a knot with $n$ double crossings and fix a vertex $c_i$ , each Hauffman state contributes a partial Alexander or Maslov grading (defined as functions $a_i, b_i : \mathcal{S} \to \frac{1}{2}\mathbb{Z}_n$, respectively, where $\mathcal{S}$ is the set of Kauffman states.) as shown in the following picture.

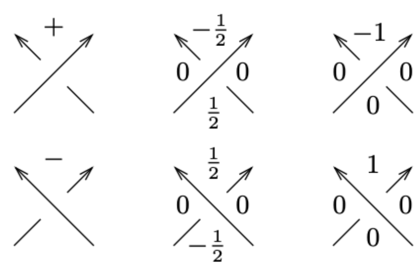



FIGURE 2. **Sign conventions for crossings, and local Alexander and Maslov contributions.** The first column illustrates the chirality of a crossing; the second the Alexander contribution of each quadrant; the third the Maslov contribution.

As shown in [6] and [1], the Kauffman states and $\Delta_K$ the symmetric version of the Alexander polynomial is related by the following formula:

$$\Delta_K(t) = \prod_{c\in\mathcal{K}} \sum_{i=1}^{n} (-1)^{b_i(c)} t^{a_i(c)}. \tag{1}$$

Moreover, it can be shown [6] that the Maslov and Alexander gradings have the formulas

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} a_i, \quad M(\boldsymbol{x}) = \sum_{i=1}^{n} b_i. \tag{2}$$

Therefore, it follows that the Euler characterstic of $\widehat{HFK}(K)$ is the symmetric Alexander polynomial $\Delta_K$.

# 3 Bordered Floer Homology

In general, the Heegaard Floer homology of a 3-mnanifold is difficult to compute because the definition involves counting pseudo-holomorphic curves, and it may not seem obvious how the Heegaard Floer homology is affected under operations such as surgering along a knot or taking connected sums. This motivates the development of bordered Heegaard Floer homology as a general conceptual framework that describes how Heegaard Floer homology behaves under cutting and pasting. In particular, we have a pairing theorem that recoves as special cases results proven formerly with Heegaard Floer homology, such as the surgery exact triangle.

## 3.1 Bordered Heegaard diagram

The definition of bordered Floer homology involves the following definition of bordered Heegaard diagrams which generalize Heegaard diagrams [4].

**Definition 3.1.** *A bordered Heegaard diagram is a quadruple $\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \beta, z)$ consisting of*

1. *a compact, oriented surface $\overline{\Sigma}$ with one boundary component, of some genus $g$;*
2. *a $g$-tuple of pairwise-disjoint circles $\beta = \{\beta_1, \ldots, \beta_g\}$ in the interior of $\Sigma$;*
3. *a $(g+k)$-tuple of pairwise-disjoint curves $\overline{\alpha}$ in $\overline{\Sigma}$, split into $g-k$ circles $\alpha^{\mathbf{c}} = (\alpha_1^c, \ldots, \alpha_{g-k}^c)$ in the interior of $\overline{\Sigma}$ and $2k$ arcs $\overline{\alpha^{\mathbf{a}}} = (\overline{\alpha_1^a}, \ldots, \overline{\alpha_{2k}^a})$ in $\overline{\Sigma}$ with boundary on $\partial\overline{\Sigma}$ (and transverse to $\partial\overline{\Sigma}$); and*
4. *a point $z$ in $(\partial\overline{\Sigma}) \setminus (\overline{\alpha} \cap \partial\overline{\Sigma})$,*

*such that the intersections are transverse and $\overline{\Sigma} \setminus \overline{\alpha}$ and $\overline{\Sigma} \setminus \beta$ are connected.*

In [4], lemma 4.4 defines a matching $M$ and a pointed matched circle quadruple $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ associated with a bordered Heegaard diagram $(\overline{\Sigma}, \overline{\alpha}, \beta, z)$, such that $M(\alpha_i^a \cap \partial\overline{\Sigma}) = i$. One can define the strand algebra $A(\mathcal{Z})$ associated to the matched circle which is used in the definition of bordered Heegaard Floer homology to specify how Heegaard floer homology is affected when two bordered 3-manifolds are glued together by their homeomorphic boundaries (for details, see [4]).

We then construct an oriented surface $F(\mathcal{Z})$ to a matched circle $\mathcal{Z}$ by first attaching a 2-dimensional 1-handle to a disk with boundary $Z$ for each $\alpha_i^a$ such that the 1-handle is pasted in an orientation-preserving way specified by the matching $M(\alpha_i^a \cap \partial\overline{\Sigma}) = i$. It follows from properties in the definition of the bordered Heegaard diagram that the resulting compact, oriented surface with has boundary homeomorphic to $\mathbb{S}^1$, and we construct $F(\mathcal{Z})$ by pasting a $\mathbb{D}^2$ along the boundary $\mathbb{S}^1$ in a way that preserves orientation.

Given a matched circle $\mathcal{Z} = (Z, \mathbf{a}, M, z)$, we can form a bordered 3-manifold $Y$ such that $\partial Y = F(Z)$. For a complete algorithmic construction, we refer the reader to [4]. The authors in [4] also suggest an equivalent but more concrete way to construct such a bordered 3-manifold; in this approach, we first thicken $F(\mathcal{Z})$ to $F(\mathcal{Z}) \times [0,1]$ and then attach 3-dimensional 2-handles along the curves $\alpha_i^c$ and $\beta_j$, so the resulting bordered manifold has boundary $F(\mathcal{Z})$.

There is also a Morse-theoretic way to describe the bordered 3-manifold formed from a matched circle $\mathcal{Z}$ which leads to the following theorem (Lemma 4.9 in [4]).

**Theorem 3.1.** *Any bordered 3-manifold $(Y, \mathcal{Z}, \phi)$ is represented by some bordered Heegaard diagram $\mathcal{H}$.*

Before proceeding to the proof, we need to have the definitions of what it means for a Riemannian metric and self-indexing Morse functiono to be compatible with matched circle $\mathcal{Z}$ and a bordered 3-manifold $(Y, \mathcal{Z}, \phi)$ (see [4] and [3]).

**Definition 3.2.** *Given a closed, orientatable surface $F$ of genus $k$, let $f : F \to \mathbb{R}$ be a Morse function with a unique index 0 critical point and a unique index 2 critical point, and such that $f(p_i) = 1$ for all index 1 critical points $p_1, \ldots, p_{2k}$. Suppose that $f^{-1}(\frac{3}{2})$ is non-empty. Fix also a Riemannian metric $g$ on $F$. Let $Z = f^{-1}(\frac{3}{2})$, and let $\mathbf{a} \in Z$ denote the ascending spheres of the index 1 critical points. Define $M(a_j) = i$ if $a_j$ is in the ascending sphere of $p_I$. Choose also a point $z \in Z \setminus \ni$. Then $\mathcal{Z} = (Z, \ni, M, z)$ is a pointed matched circle, and $F(\mathcal{Z}) \cong F$. We say that $(f, g)$ is compatible with the pointed matched circle $\mathcal{Z}$.*

It is obvious that the existance of a Riemmanian metric and Morse function pair compatible with a matched circle follows from standard Morse theory. The following definition gives a similar compatibility criterion for bordered manifolds.

**Definition 3.3.** *Fix a bordered 3-manifold $(Y, \mathcal{Z}, \phi)$. We say that a pair consisting of a Riemannian metric $g$ on $Y$ and a self-indexing Morse function $f$ on $Y$ are compatible with $(Y, \mathcal{Z}, \phi)$ if*

1. *the boundary of* $Y$ *is geodesic*

2. *the gradient vector field* $\nabla f|_{\partial Y}$ *is tangent to* $\partial Y$.

3. $f$ *has a unique index* $0$ *and a unique index* $3$ *critical point, both of which lie on* $\partial Y$, *and are the unique index* $0$ *and* $2$ *critical points of* $f|_{\partial Y}$, *respectively.*

4. *the index* $1$ *critical points of* $f|_{\partial Y}$ *are also index* $1$ *critical points of* $f$, *and*

5. *the pair of* $(f \circ \phi, \phi^* g)$ *on* $F(\mathcal{Z})$ *are compatible with the pointed circle* $\mathcal{Z}$.

We now give a proof that any bordered 3-manifold $(Y, \mathcal{Z}, \phi)$ is represented by some bordered Heegaard diagram $\mathcal{H}$.

*Proof.* We construct a bordered 3-manifold from the following steps:

1. Choose a Morse function $f_F$ and metric $g_F$ on $F(\mathcal{Z})$ compatible with $\mathcal{Z}$. Let $f_F$ be self-indexing except that $f_F$ takes 3 on the index 2 critical point.

2. Extend $f_F \circ \phi^{-1}$ and $\phi_* g_F$ to $f$ and $g$ on a collar neighbourhood of $\partial Y$ satisfying conditions (1), (2), and (4), so that the index 0 and 2 critical points of $f_F \circ \phi^{-1}$ are index 0 and 3 critical points of $f$.

3. Extend the Morse function $f$ and Riemannian metric $g$ to the rest of $Y$.

4. Consider the graph formed by flows between index 0 and 1 critical points. This is connected, otherwise consider the manifold $Y' = -Y \cup_{F(\mathcal{Z})} Y$, where $-Y$ denotes a copy of $Y$ with the reversed orientation. By condition (2), we can extend the Morse function $f$ in an obvious way to $Y'$. If the flows between index 0 and 1 critical points of $Y$ are disconnected, then the flows between index 0 and 1 critical points of $Y'$ are also disconnected, a contradiction since that means $H_0(Y')$ has at least two summands, a contradiction since $Y'$ is connected.

5. If there exists an interior index 0 critical point, then its flow must connect with a unique interior 1 critical point by a dimensionality argument, since $f$ and $g$ are compatible with $F(\mathcal{Z})$. Therrefore, by standard Morse theory, we can cancel the interior index 0 critical points. Similarly, we cancel all interior index 3 critical points.

6. Modify $f$ in the interior so that the result is self-indexing.

7. Construct the bordered Heegaard diagram $(\overline{\Sigma}, \overline{\alpha}, \beta, z)$ by taking $\overline{\Sigma}$ to be $f^{-1}(\frac{3}{2})$, the curves $\overline{\alpha}$ the intersection of the ascending disks of the index 1 critical points of $f$ with $\overline{\Sigma}$, and the curves $\beta$ the intersection of the descending disks of the index 2 critical points of $f$ with $\overline{\Sigma}$.

□

## 3.2 Bordered Heegaard Floer homology and the pairing theorem

For a matched circle $\mathcal{Z}$, we can associate an $A_\infty$-algebra $\mathcal{A}(\mathcal{Z})$. (Refer to [4] and [2] for definitions of $A_\infty$ algebras and modules.) Bordered Heegaard Floer homology associates to each bordered Heegaard diagram a type $D$ module $\widehat{CFD}$ and a type $A$ module $\widehat{CFA}$ which are both $A_\infty$ modules (see [4] for their definitions.).

The following pairing theorem expresses how Heegaard Floer homology of a 3-manifold is determined by the bordered Heegaard Floer homology of its pieces of bordered manifolds.

**Theorem.** *Let $Y_1$ and $Y_2$ be two 3-manifolds with parameterized boundary $\partial Y_1 = F = -\partial Y_2$, where $F$ is specified by the pointed matched circle $\mathcal{Z}$. Fix corresponding bordered Heegaard diagrams for $Y_1$ and $Y_2$. Let $Y$ be the closed 3-manifold obtained by gluing $Y_1$ and $Y_2$ along $F$. Then $\widehat{CF}(Y)$ is homotopy equivalent to the $A_\infty$ tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$. In particular,*

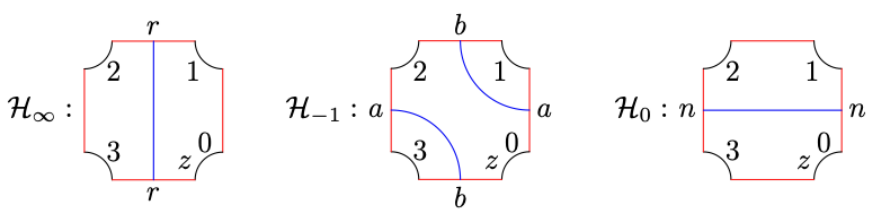
$$\widehat{HF}(Y) \cong H_*(\widehat{CFA}(Y_1)\widetilde{\oplus}_{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(Y_2)).$$

As a direct consequence of the pairing theorem, we recover the surgery exact triangle.

**Corollary.** *There is a long exact sequence relating the manifolds $M_\infty$, $M_{-1}$, and $M_0$, which corrspond to the results of applying $\infty$, $-1$, and $0$ surgeries to the 3-manifold $M$, respectively.*

$$\cdots \widehat{HF}_n(M_\infty) \to \widehat{HF}_n(M_{-1}) \to \widehat{HF}_n(M_0) \to \widehat{HF}_{n-1}(M_\infty)\cdots$$

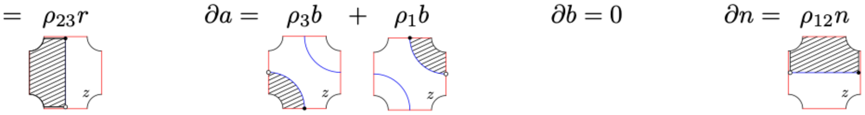
We follow the proof given in [4], in which the authors construct three bordered Heegaard diagrams corresponding to surgeries with coefficients $\infty$, $-1$, and 0, respectively.



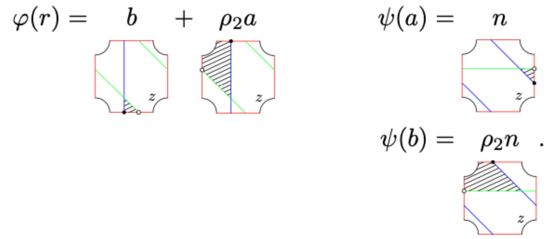

The generalors for each bordered Heegaard diagram are represented by the intersection of the blue $\beta$ curve and two red $\alpha$ arcs.

The boundary maps are given by the following.

$\partial r = \rho_{23} r$ $\quad$ $\partial a = \rho_3 b + \rho_1 b$ $\quad$ $\partial b = 0$ $\quad$ $\partial n = \rho_{12} n$



There are maps $\phi : \widetilde{CFD}(\mathcal{H}_\infty) \to \widetilde{CFD}(\mathcal{H}_{-1})$ and $\psi : \widetilde{CFD}(\mathcal{H}_{-1}) \to \widetilde{CFD}(\mathcal{H}_0)$ defined by




This induces the following short exact sequence

$$0 \to \widetilde{CFD}(\mathcal{H}_\infty) \xrightarrow{\phi} \widetilde{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widetilde{CFD}(\mathcal{H}_0) \to 0$$

Let $N$ the bordered 3-manifold formed from removing a tubular neighbourhood $Nb(K)$ from $M$. It then follows from Proposition 2.36 in [4] that we have the following long exact sequence

$$\cdots H_n(\widetilde{CFA}(N) \boxtimes \widetilde{CFD}(\mathcal{H}_\infty)) \to H_n(\widetilde{CFA}(N) \boxtimes \widetilde{CFD}(\mathcal{H}_{-1}))$$
$$\to H_n(\widetilde{CFA}(N) \boxtimes \widetilde{CFD}(\mathcal{H}_0)) \to H_{n-1}(\widetilde{CFA}(N) \boxtimes \widetilde{CFD}(\mathcal{H}_\infty)) \cdots,$$

where $H_*()$ denotes taking homology with respect to the chain complex.

We can then apply the pairing theorem to see that

$$H_n(\widetilde{CFA}(N) \boxtimes \widetilde{CFD}(\mathcal{H}_\circ)) \cong \widetilde{HF}(M_\circ),$$

where $\circ \in \{\infty, -1, 0\}$. This gives the surgery exact triangle formula

$$\cdots \widetilde{HF}_n(M_\infty) \to \widetilde{HF}_n(M_{-1}) \to \widetilde{HF}_n(M_0) \to \widetilde{HF}_{n-1}(M_\infty) \cdots .$$

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