Random Walks on Infinite Graphs

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Introduction to Random Walks

A discrete random walk is a stochastic process in which the vertices of a graph are traversed along its edges. The random walk begins at the origin, and during each time step, a single step is made along an edge in a graph $G = (V, E)$.

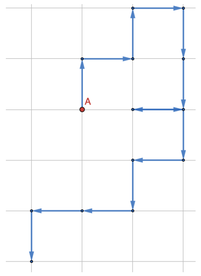


Figure 1. A random walk on $\mathbb{Z}^2$

The probability that a walk currently at x will next visit vertex y is called the *transition probability*. Each edge is assigned a weight $c(x, y)$, a measure of how likely the random walk is to take that path. The transition probability is

$$p(x, y) = P(X_{t+1} = y | X_t = x) = \frac{1_{x\sim y} c(x, y)}{\sum_{v\in V} 1_{x\sim v} c(x, v)}.$$

For a random walk with uniform weights, if x is connected to y, then $p(x, y) = \frac{1}{\deg(x)}$.

We call a random walk (and the graph on which it takes place) *recurrent* if every vertex in the graph is visited almost surely infinitely many times. We call a random walk (and the graph on which it takes place) *transient* if every vertex in the graph is visit almost surely finitely many times.

Connections to Electrical Networks

Representing random walks as electrical networks is a useful tool for solving certain problems on random walks. In this technique, the weight on each edge of a graph represents its conductance, a measure of how likely the random walk is to traverse that edge, the reciprocal of which is the resistance of that edge.

Using electrical networks significantly simplifies the computation of hitting probabilities. Suppose A and B are subsets of the set of vertices V. To compute the probability that a random walk beginning at x reaches A before B, fix a 1 volt potential difference between A and B (perhaps by connecting them with a 1 volt battery with B grounded). Then the voltage $v(x)$ at point x is equal to the probability that the random walk reaches A before B. This technique works because voltage and hitting probabilities are both harmonic functions, and by defining identical boundary conditions (in this case the voltages and probabilities at A and B), uniqueness ensures the functions are the same.

Figure 2. Computing the probability a random walk starting at X would reach A before B would require solving a system of 6 equations.

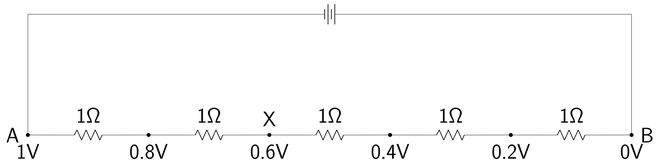


Figure 3. Converting to an electrical network and setting voltages allows a very simple computation to see that the probability of reaching A before B is 0.6.

Furthermore, since the resistance— the inverse of the conductance— represents the difficulty of getting from one point to another, the *effective resistance* between two points is a useful value. In fact, if $R_{\text{eff}}(a \leftrightarrow \infty) = \infty$, much like a closed edge which has resistance ∞, there is probability 0 the random walk reaches ∞, so it is recurrent. Similarly, $R_{\text{eff}}(a \leftrightarrow \infty) < \infty$ implies the random walk is transient, because the random walk can "easily" reach ∞.

Rayleigh's Monotonicity Principle *Increasing the resistances in an electrical network, results in $R_{\text{eff}}(a \leftrightarrow \infty)$ also increasing. Therefore, increasing resistances preserves recurrence and decreasing resistances preserves transitivity.*

This principle is useful in proving recurrence and transience. In particular, it's vital to the proof of Pólya's Theorem, a famous result in the study of random walks.

Pólya's Theorem *A simple random walk on the $\mathbb{Z}^d$ square lattice is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Random Walks in Changing Environments

A *Random Walk in a Changing Environment* (RWCE) on a graph $G = (V, E)$ refers to a stochastic process $\{(X_t, G_t)\}_{t=0}^{\infty}$, where each graph $G_t = (V, E, C_t)$ is defined by a conductance function $C_t : E \to [0, \infty)$ applied over a fixed set of vertices V and edges E. For each time step t, the vertex X_t is in V, and the probability of transitioning to a vertex v at the next step is given by:

$$P(X_{t+1} = v \mid \{(X_0, G_0), \ldots, (X_t, G_t)\}) = \frac{C_t(X_t, v)}{\sum_{e\in E: X_t \in e} C_t(e)}.$$

While an ordinary random walk is either recurrent or transient, RWCE can be of *mixed-type*: random walks which are neither recurrent nor transient— i.e., some points are visited finitely many times, and some are visited infinitely many times.

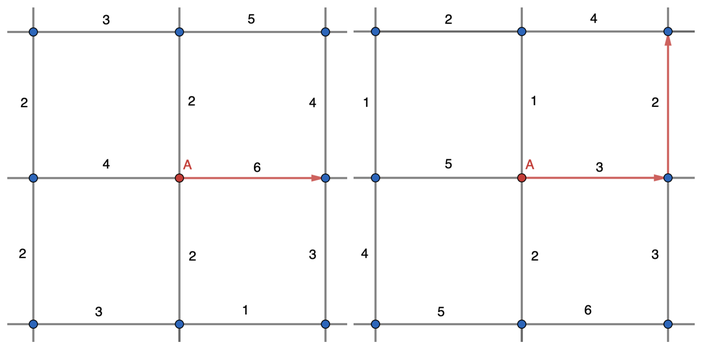


Figure 4. An example of a RWCE

Non-Backtracking Random Walk

A non-backtracking random walk is not allowed to "backtrack", or leave a vertex through the same edge it entered through.

Theorem. *If G is a regular infinite graph with degree $k \geq 3$, then a Non-Backtracking Random Walk (NBRW) on G is recurrent $\iff$ a Simple Random Walk (SRW) on G is recurrent.* (Jung, Markowsky 2020)

Random Walks in General Changing Environments

There are several related theorems for monotone decreasing RWCE bounded by a recurrent/transient graph— below is an example.

Theorem. *If $\{(X_t, G_t)\}$ is a monotone increasing RWCE on $\mathbb{N}$ with environment bdd. above by a graph $G_\infty = (\mathbb{N}, C_\infty)$, the walk is recurrent.* (Amir, Benjamini, Gurevich, Kozma 2020)

Random Walks in Slowly Changing Environments

Let $G = (V, E)$ be any simple, undirected, locally finite graph. Let $\{X_t, G_t\}$ be any proper and bounded from above RWCE on G.

Theorem. *If $\tau = \sum_{t\geq 0, e\in E} |R_t(e) - R_{t+1}(e)| < \infty$ almost surely, then the RWCE $\{X_t, G_t\}$ is recurrent if the random walk on (G, C_0) is recurrent, and transient if the random walk on (G, C_0) is transient.* (Park, Ray 2024)

If C_t is monotone increasing/decreasing, the condition $\sum_{e\in E} |R_0(e) - R_\infty(e)| < \infty$ is enough to prove that (V, E, C_0) and (V, E, C_t) have the same recurrence/transience properties, since the sequence $R_t(e)$ converges and $|R_\infty(e) - R_t(e)| \to 0$ as $t \to \infty$ for all $e \in E$.

Random Walk Collisions on G and Comb(G)

Let $G = (V, E)$ be a graph. We say G has the infinite (resp. finite) collision property, if 2 independent random walks X_n, Y_n starting at the same point meet infinitely (resp. finitely) times almost surely.

Theorem. *Two random walks $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ on a recurrent reversible random rooted graph G will almost surely collide infinitely many times.* (Hutchcroft, Peres 2015)

For a graph G, define Comb(G) as a graph with vertex set $V(G) \times \mathbb{Z}$ and edge set $\{(x, n), (x, m) : |m - n| = 1\} \cup \{(x, 0), (y, 0) : (x, y) \in E(G)\}$. Essentially, Comb(G) is G with a z-axis through every vertex.

Theorem. *For any recurrent infinite graph G, with constant degree d, comb(G) is recurrent, but has the finite collision property.* (Hutchcroft, Peres 2004)

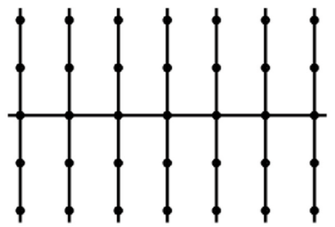
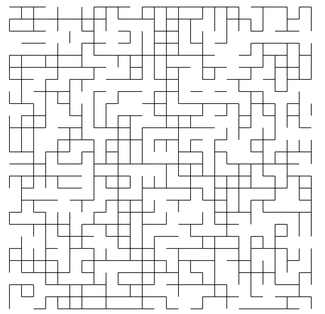


Figure 5. Comb($\mathbb{Z}$)

Random Walks on the Percolation Model

The percolation model is a stochastic model that represents natural diffusion through porous materials. Given a probability p, a *percolation configuration* is created from a graph by including each edge with probability p, independently of all the other edges.



A connected component of a configuration is called a *cluster*. One characteristic of interest is a potential infinite cluster. For any infinite graph, there exists a phase transition in p: there exists some $p_c \in [0, 1]$ such that if $p > p_c$ there is an infinite cluster with probability 1 and if $p < p_c$ there is no infinite cluster.

On $\mathbb{Z}^d$ this infinite cluster, called C_∞, is almost surely unique. This leads to the question of whether a random walk on C_∞ is transient. Clearly for $d \leq 2$, C_∞ is a subgraph of $\mathbb{Z}^2$, so by Rayleigh's Monotonicity Principle it must be recurrent.

Theorem: *The infinite cluster C_∞ on $\mathbb{Z}^d$ is transient for $d \geq 3$.* (Grimmett, Kesten, Zhang 1993)

The proof procedes by constructing a tree in C_∞. Each layer of the tree contains 4 times as many nodes as the previous layer and is 3 times as far from the origin. Linear paths from parent to child are defined, then neighborhoods are defined around them as seen in figure 6.

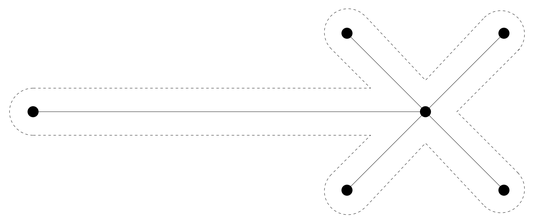


Figure 6. A diagram of a single node of the tree, its children, and the neighborhood around them.

Using inequalities about the likelihood of paths through cubes existing, it's shown that, by growing this tree far enough, the probability that there will be a path in C_∞ along the tree becomes 1. All that remains is to use electrical networks to show the effective resistance from the root of the tree to infinity is finite, then conclude with the monotonicity principle that C_∞ is transient.

We attempted to extend this proof to the triangular lattice. While we were able to follow a similar procedure in building a tree that would imply transitivity for $d \geq 3$, we were unable to prove it exists. While there is strong intuition that this tree should exist almost surely, the proof requires bounds on the crossing probabilities of certain large boxes which haven't yet been proved, and which we were unable to show.

Conclusion and Future Work

During the ten week research program, we learned about various types of random walks and their applications. We explored numerous problems related to random walks, from their recurrence and transience to the number of times two random walks collide. We utilized powerful and flexible techniques like electrical networks to answer some of these questions. After this work, we propose the following areas of further research

- The finiteness of collisions of transient branching random walks.
- The recurrence or transience of non-adaptive monotone random walks in changing environments bounded by recurrent and transient random walks
- The escape rate of transient random walks in changing environments.
- Recurrence and transience of the infinite cluster of the percolation model on non-square regular lattices.

Random walks are a broad subject of research, and can be intimidating. We invite you to find solace in the remark of mathematician Shizuo Kakutani on Pólya's Theorem: "A drunk man will find his way home, but a drunk bird may get lost forever". Keep studying random walks, and be grateful that you are not a bird.

Acknowledgements

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