HAT-GUESSING ON GRAPHS

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ABSTRACT. Hat-guessing games are combinatorial puzzles in which people try to guess the colors of their own hats. In the variant we study, each person is assigned 1 of q possible hat colors and is placed at the vertex on a graph G. Players can only see the hat colors of the people at adjacent vertices, but not their own. The players simultaneously guess the colors of their own hats. What is $HG(G)$, the largest q for which the players on G can devise a strategy that guarantees at least one person guesses correctly? We find the exact values of $HG(G)$ for the following graphs. $HG(K_{3,3}) = 3$. For windmill graphs, $HG(Wd(k,n)) = 2k-2$ for $n \geqslant \log_2(2k-2)$. For book graphs, $HG(B_{d,n}) = 1 + \sum_{i=1}^{d} i^i$ for sufficiently large n. Finally, we determine the hat-guessing number for all 5-vertex undirected graphs.

CONTENTS

1. INTRODUCTION

Hat-guessing games are combinatorial puzzles in which people try to guess the colors of their own hats. In the variant we study, presented by Butler, Hajiaghayi, Kleinberg, and Leighton [3], each person is assigned 1 of q possible hat colors and is placed at the vertex on a graph G. Players can only see the hat colors of the people at adjacent vertices, but not their own. The players can communicate to design their collective strategy only before the hats are assigned by their adversary. Then, the players simultaneously guess the colors of their own hats. The players collectively *win* if they can form a strategy that guarantees at least one person guesses correctly. Note that we identify a vertex with the player on that

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vertex. In this paper, we only consider games on undirected graphs, although other variants of the game allow for directed edges $v_i \to v_j$ corresponding to v_i seeing v_j, but not vice versa.

Definition 1.1. *The hat-guessing number of the graph G, denoted $HG(G)$, is the largest number of hat colors q for which the players on G can win.*

Consider the traditional example when n players can all see each other, popularized by Winkler [11]. Then, $HG(K_n) = n$. The strategy which wins on n colors is as follows: label the hat colors $\{0, 1, \cdots, n-1\}$ and the people $\{p_0, \cdots, p_{n-1}\}$. Then p_i guesses the hat color that makes the sum of everyone's hat colors $i \mod n$. Since the actual sum of everyone's hat colors must be some value $x \mod n$, exactly one person, p_x, will guess correctly. Probabilistic proofs show that the players on K_n cannot win with $n+1$ colors, and furthermore, the only graph with n vertices that can win with n colors is K_n.

Previous works on hat guessing have only explicitly classified the hat-guessing number of a small number of families. First, we have the folklore result that $HG(K_n) = n$. For any tree T with at least 2 vertices, $HG(T) = 2$ [3]. Finally, Szczechla showed that for all n, $HG(C_n) < 4$, with $HG(C_n) = 3$ if and only if $n = 4$ or $n \equiv 0 \mod 3$ [10]. Most previous results prove bounds on the hat-guessing number of various graph families. The most general result states that for any graph G, $HG(G) < e\Delta$ where Δ is the maximum degree of G. The probabilistic proof for this folklore result uses the Lovász Local Lemma. However, based on all known hat-guessing numbers, it is conjectured that $HG(G) \leqslant \Delta + 1$. It is also evident that if H is a subgraph of G, $HG(H) \leqslant HG(G)$.

The complete bipartite graph $K_{m,n}$ is a popular graph family to study for this problem. Initially, it was proved that for large n, $HG(K_{n,n}) = \Omega(\log\log n)$ [3]. Later, Gadouleau and Georgiou [5] proved that $HG(K_{m,n}) \leqslant \min(m,n) + 1$ and that $HG\left(K_{q-1,(q-1)^{q-1}}\right) \geqslant q$, implying $n + 1 \geqslant HG(K_{n,n}) = \Omega(\log n)$. Most recently, Alon, Ben-Eliezer, Shangguan, and Tamo explored complete multipartite graphs [1]. They proved that $HG(K_{n,n,\ldots,n}) \geqslant n^{\frac{r-1}{r} - o(1)}$ for the complete r-partite graph with all parts sized n, which implies that $HG(K_{n,n}) = \Omega\left(n^{\frac{1}{2} - o(1)}\right)$. However, $HG(K_{n,n})$ was only explicitly known in the cases $n = 1$ and $n = 2$; since $K_{1,1} = K_2$, $HG(K_{1,1}) = 2$; for $K_{2,2} = C_4$, a linear strategy proves that $HG(K_{2,2}) = 3$. Our first major result in this paper, discussed in Section 3, solves the problem for $n = 3$.

Theorem 1.2. $HG(K_{3,3}) = 3$.

This is the first n for which $HG(K_{n,n}) < n + 1$.

Next, in Section 4, we consider the family of windmill graphs $Wd(k, n)$, defined as n copies of K_k glued together at a central vertex. These graphs generalize the complete graph K_k. At first glance, it seems that only the center vertex in a windmill graph $Wd(k, n)$ has more information compared to the complete graph K_k, so $HG(Wd(k, n))$ cannot be very large compared to $HG(K_k)$. Surprisingly, the copies of K_k in a windmill graph can collaborate to win with almost twice as many colors as K_k alone, disproving a conjecture by Bosek, Dudek, Farnik, Grytczuk, and Mazur [2] that the hat-guessing number of a graph is at most the size of its largest clique minor.

Theorem 1.3. *For $n \geqslant \log_2(2k-2)$, $HG(Wd(k, n)) = 2k - 2$.*

We also prove the following general theorem about windmill graphs.

Theorem 1.4. *For any $n, d \in \mathbb{N}$, $HG(Wd(d^n - d^{n-1} + 1, n)) = d^n$.*

In Section 5, we explore the family of book graphs $B_{d,n}$, defined as a clique of size d with n vertices adjacent exactly to each vertex in the clique. The d-clique is called the *spine* of $B_{d,n}$ and the n vertices are called its *pages*. Gadouleau [4] proved that for any G, $HG(G) \leqslant 1+\sum_{i=1}^{\tau(G)} i^i$, where $\tau(G)$ is the size of G's minimum vertex cover. Since the d-clique is the minimum vertex cover of a book, Gadouleau's result implies $HG\left(B_{d,n}\right) \leqslant 1+\sum_{i=1}^{d} i^i$. Bosek, Dudek, Farnik, Grytczuk, and Mazur [2] reduce finding $HG\left(B_{d,n}\right)$ to an equivalent geometric problem, using it to show $HG(B_{d,n}) \geqslant 2^d$ for sufficiently large n. By extending their approach, He and Li [7] improve the lower bound to $HG\left(B_{d,n}\right) \geqslant (d+1)!$ for sufficiently large n. By solving the equivalent geometric problem exactly, we further improve the lower bound, matching Gadouleau's upper bound for sufficiently large n.

Theorem 1.5. *For sufficiently large n, $HG\left(B_{d,n}\right) = 1+\sum_{i=1}^{d} i^i$.*

Previous results, together with our result that $HG(Wd(3,2)) = 4$, solve the hat-guessing problem for all but three 5-vertex undirected graphs. In Section 6, we determine the hat-guessing numbers exactly for these remaining graphs.

2. Preliminaries

In this section, we discuss how vertices choose their guessing strategies and define concepts necessary for later proofs.

2.1. How to Strategize.

The only specific guessing strategy described above is for K_n, involving modular arithmetic. Here, we motivate more general guessing strategies.

Recall that the players win if one of them guesses correctly, and that everyone knows everyone else's strategy. This means that each vertex can assume that the hat assignment makes all other vertices guess incorrectly, and guess accordingly.

As an example, we use the K_n game with n colors. Say that vertex p_0 has already decided to guess the hat color that makes the sum of everyone's hat colors $0 \mod n$. Then, vertex p_1 knows that if the actual hat assignment makes the sum $0 \mod n$, p_0 will already guess correctly, and everyone wins, regardless of what p_1 guesses. Thus, p_1 will want to take care of hat assignments in which p_0 does not already guess correctly. That is, p_1 will want to guess a hat color that makes the sum of everyone's hat colors not $0 \mod n$. Thus, without loss of generality, it guesses that the sum of everyone's hat colors is $1 \mod n$. Then, p_2 guesses that the sum is neither 0 nor $1 \mod n$, and so forth. For other classes of graphs, the strategies are more complicated, but the principle of assuming other players guess wrong to deduce info about one's own hat remains.

2.2. Definitions and Notations.

Notation 2.1. *We use $[q]$ to denote the set of colors $\{0, 1, \ldots, q-1\}$.*

Definition 2.2. *We say that G is partially q-solvable with a set S of hat assignments if the vertices can win with q colors as long as they know their hat assignment is in S.*

Note that a graph G with n vertices is partially q-solvable with $S = [q]^n$ if and only if $HG(G) \geqslant q$.

Definition 2.3. *If G is partially q-solvable with a set of hat assignments S, we say that S is a solvable set of G on q colors, or simply a solvable set if G and q are clear from context. We denote the maximum size of S as $H(G,q)$.*

Definition 2.4. *We say that a point contained in n sets is an n-intersection (e.g. a point contained in two sets is a two-intersection). We say a point is a multiple-intersection if it is an n-intersection for some $n \geqslant 2$.*

Definition 2.5. *Given a base set S, a maximal packing of n subsets in S is a choice of $S_1, \ldots, S_n \subset S$ that satisfies the following condition. It maximizes the number of elements in S which are contained in exactly k S_i, where k is taken from the following list ordered in decreasing priority: $\{1, n, n-1, \ldots, 3, 2\}$.*

Definition 2.6. *Given a base set S that can be represented as a matrix and subsets $S_1, \ldots, S_n$, an intersection matrix for S is defined as follows: the entry in a point is the number of sets S_i that contain it.*

Definition 2.7. *A big Hamming ball around point p, denoted $bHb(p)$, is the set of points which share at least one coordinate with p.*

3. The Bipartite Graph $K_{3,3}$

Here we prove Theorem 1.2, that $HG(K_{3,3}) = 3$.

Proof of Theorem 1.2. Since $HG(K_{2,2}) = 3$ and it is a subgraph of $K_{3,3}$, $HG(K_{3,3}) \geqslant 3$. Since $HG(K_{m,n}) \leqslant \min(m,n) + 1$ [5], we know that $HG(K_{3,3}) \leqslant 4$. It remains to show that $HG(K_{3,3}) \neq 4$.

Suppose for the sake of contradiction that $HG(K_{3,3}) = 4$. The big Hamming ball interpretation of hat guessing on complete bipartite graphs tells us the following [1]: $HG(K_{3,3}) = 4$ if and only if there are three partitions of a $4 \times 4 \times 4$ cube into four parts each, such that removing a part from each partition leaves behind a set contained in a big Hamming ball. This condition is equivalent to the complementary condition that the union of a part from each partition must always contain a $3 \times 3 \times 3$ combinatorial cube (i.e., a set of the form $\{p\}^c \times \{q\}^c \times \{r\}^c$, for $p, q, r \in [4]$).

We will denote the three partitions of $[4]^3$ as P, Q, and R, and denote their respective parts with subscripts from 1 to 4. First, we pick the smallest part from P and call it P_1, so $|P_1| \leqslant 16$. Then choose the part $Q_i \in Q$ that minimizes $|Q_i \backslash P_1|$ and call it Q_1. By the pigeonhole principle, $|Q_1 \backslash P_1| \leqslant 12$, so $|P_1 \cup Q_1| \leqslant 28$. It must be that $P_1 \cup Q_1 \cup R_i$ contains a cube for $1 \leqslant i \leqslant 4$. Therefore, there must be four cubes that multiple-intersect in at most 28 points (see Definition 2.4).

Claim 3.1. *The multiple-intersection of four cubes with at most 29 points can only have one of two forms. The structure of the multiple-intersection must either be a cube (27 points), or a cube missing a point (26 points).*

Proof. Suppose we have four cubes $C_1 = \overline{bHb(p_1)}$, $C_2 = \overline{bHb(p_2)}$, $C_3 = \overline{bHb(p_3)}$, and $C_4 = \overline{bHb(p_4)}$. We can assume without loss of generality that $p_1 = (0,0,0)$. Further, if $p_i = p_j$ with $i \neq j$, then the multiple-intersection is either a cube or contains at least 31 points (a cube with at least 4 extra multiple-intersections), which is in accordance with our claim. We now assume that no two of the four points are equal.

First, suppose two of the four points have a Hamming distance of 1 between them. Without loss of generality, this occurs when $p_2 = (1,0,0)$. If all the points have a Hamming distance of 1 from p_1, then the multiple-intersection is a cube missing a point, so we can then assume that p_3 has at least 2 non-zero coordinates. So without loss of generality, $p_3 \in \{(0,1,1),(1,1,0),(2,1,0),(1,1,1),(2,1,1)\}$. We consider the cases in turn. (Several of these cases require exhaustively considering the possibilities for p_4, although symmetries make the task easier.)

First, if $p_3 = (0,1,1)$, then we can create a multiple-intersection of 30 by setting $p_4 = (0,1,0)$, but one can see that no other choice of p_4 achieves a smaller multiple-intersection.

Second, if $p_3 = (1,1,0)$, then it is possible to achieve a multiple-intersection of 26 by setting $p_4 = (1,0,1)$, and this is the cube missing a point as desired. Further, we can get a multiple-intersection of 30 by setting $p_4 = (0,0,1)$ or $p_4 = (1,1,1)$, but no smaller multiple-intersections are possible besides the cube missing a point.

Third, if $p_3 = (2,1,0)$, this case can easily be dismissed because the multiple-intersection is already 30.

Fourth, if $p_3 = (1,1,1)$, then a multiple-intersection of 30 is possible with $p_4 = (1,1,0)$, but no smaller multiple-intersection is possible.

Fifth, if $p_3 = (2,1,1)$, we can do a multiple-intersection of size 32 with $p_4 = (0,1,0)$, but no smaller multiple-intersection is possible, which is not difficult to see once one notes that the size of the multiple-intersection is already 26 after adding the third cube.

Thus, we have handled the case where two points have a Hamming distance of 1 between them. We now suppose that all pairs of points have a Hamming distance of at least 2 between them. We take the perspective of considering the four 4×4 layers that make up the entire cube. Each combinatorial cube is a combinatorial square repeated on three of the four layers (a combinatorial square is a set of the form $\{p\}^c \times \{q\}^c$, for $p, q \in [4]$). By the condition we impose on Hamming distances, all four cubes must be made up of distinct squares.

One can verify that within a layer, 2 squares must multiple-intersect in at least 4 points, and 3 squares must multiple-intersect in at least 8 points. Although not as obvious, one can also verify that 4 distinct squares must multiple-intersect in at least 12 points (the minimal multiple-intersection is achieved with the squares defined by the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$).

With four cubes each composed of a square repeated on three layers, there are five ways the squares can be distributed across the layers, without loss of generality: $(0,4,4,4)$, $(1,3,4,4)$, $(2,2,4,4)$, $(2,3,3,4)$, and $(3,3,3,3)$ (here, the ith coordinate within the tuple is the number of squares on the ith layer). Using the facts from the previous paragraph, this gives us a lower bound for the total number of multiple-intersection points for each of these arrangements.

Arrangement	Number of multiple-intersections
$(0,4,4,4)$	$\geqslant 3 \cdot 12 = 36$
$(1,3,4,4)$	$\geqslant 2 \cdot 12 + 8 = 32$
$(2,2,4,4)$	$\geqslant 2 \cdot 12 + 2 \cdot 4 = 32$
$(2,3,3,4)$	$\geqslant 12 + 2 \cdot 8 + 4 = 32$
$(3,3,3,3)$	$\geqslant 4 \cdot 8 = 32$

Since for all of these cases, the number of multiple-intersections is at least 30, none of these cases can be a counterexample to the claim, so the claim is proved. □

By the previous claim, $|P_1 \cup Q_1| \geqslant 26$, and so $|Q_1 \backslash P_1| \geqslant 10$. Then we can apply the pigeonhole principle again to pick a Q_i such that $|Q_i \backslash P_1| \leqslant 12$; call this part Q_2. If $|Q_1 \backslash P_1| \geqslant 11$ or $|Q_2 \backslash P_1| \geqslant 11$, we can apply the pigeonhole principle a third time to get a Q_i such that $|Q_i \backslash P_1| \leqslant 13$, which we will call Q_3. Thus, we have two cases to consider: the case where Q_1,

Q_2, and Q_3 each have at most 13 points outside of P, and the case where Q_1 and Q_2 have exactly 10 points outside of P. The first case implies that there are three distinct cubes or cubes missing a point that multiple-intersect in at most 16 points.

Claim 3.2. *Three $3 \times 3 \times 3$ cubes in $[4]^3$ must multiple-intersect at a minimum of 20 points.*

Proof. Call the cubes $C_1 = \overline{bHb(p_1)}$, $C_2 = \overline{bHb(p_2)}$, and $C_3 = \overline{bHb(p_3)}$, where $p_1, p_2, p_3 \in [4]^3$. Let x, y, and z represent the number of distinct x-, y-, and z-coordinates among p_1, p_2, and p_3, respectively. By inclusion-exclusion,

$$|\text{multiple-intersection of } C_1, C_2, C_3| = |C_1 \cap C_2| + |C_2 \cap C_3| + |C_1 \cap C_3| - 2|C_1 \cap C_2 \cap C_3|$$

We see that a point is in $C_1 \cap C_2 \cap C_3$ if and only if none of its coordinates is used by p_1, p_2, or p_3. Thus $|C_1 \cap C_2 \cap C_3| = (4-x)(4-y)(4-z)$.

Let $d_1 = d(p_1, p_2)$, $d_2 = d(p_2, p_3)$, and $d_3 = d(p_1, p_3)$ be the Hamming distances between each pair of points. Then, in an x-, y-, or z-coordinate where p_1 and p_2 agree, a point in $C_1 \cap C_2$ has 3 options since it must only avoid the one option p_1 and p_2 use. In a coordinate where p_1 and p_2 disagree, a point in $C_1 \cap C_2$ has 2 options since it must avoid both options that p_1 and p_2 use. Thus $|C_1 \cap C_2| = 3^{3-d_1}2^{d_1}$. By the same argument, $|C_2 \cap C_3| = 3^{3-d_2}2^{d_2}$ and $|C_1 \cap C_3| = 3^{3-d_3}2^{d_3}$.

$$\begin{aligned}
|\text{multiple-intersection of } C_1, C_2, C_3| &= \sum_{i=1}^{3} 3^{3-d_i}2^{d_i} - 2(4-x)(4-y)(4-z) \\
&= 27\sum_{i=1}^{3}\left(\frac{2}{3}\right)^{d_i} - 2(4-x)(4-y)(4-z) \\
&\geqslant 27 \cdot 3 \cdot \left(\frac{2}{3}\right)^{(d_1+d_2+d_3)/3} - 2(4-x)(4-y)(4-z) \text{ by AM-GM}
\end{aligned}$$

We now perform a change of variables so $a_1 = \#\{x, y, z = 1\}$, $a_2 = \#\{x, y, z = 2\}$, and $a_3 = \#\{x, y, z = 3\}$. Then, we see that $d_1 + d_2 + d_3 = 2a_2 + 3a_3$ and $(4-x)(4-y)(4-z) = 3^{a_1}2^{a_2}$. Substituting gives

$$|\text{multiple-intersection of } C_1, C_2, C_3| \geqslant 27 \cdot 3 \cdot \left(\frac{2}{3}\right)^{(2a_2+3a_3)/3} - 3^{a_1}2^{a_2+1}.$$

Conditioned on $a_i \in \mathbb{Z}$, $a_1 + a_2 + a_3 = 3$, and $0 \leqslant a_i \leqslant 3$, calculating all cases shows the minimum of the right hand side is 20, attained when $a_1 = 0$, $a_2 = 3$, and $a_3 = 0$. $\square$

We now know three cubes must multiple-intersect in at least 20 points. Removing one point can subtract at most 1 from the multiple-intersection. Thus it is impossible for three cubes missing a point to multiple-intersect at fewer than 17 points. Thus the first case is impossible, and $|Q_1 \backslash P_1| = |Q_2 \backslash P_1| = 10$. Furthermore, we use this claim about three cubes to show that $|P_1| = 16$, implying the partitions are balanced.

Claim 3.3. *The partitions are balanced; that is, all of the parts are of size 16.*

Proof. Let P_1 be the smallest part in P and assume for sake of contradiction that $|P_1| \leqslant 15$. Then, since $|P_1 \cup Q_1| \geqslant 26$ and $|P_1 \cup Q_2| \geqslant 26$, we see that $|Q_1 \backslash P_1| \geqslant 11$ and $|Q_2 \backslash P_1| \geqslant 11$. Then, $|(Q_3 \cup Q_4) \backslash P_1| \leqslant 64 - 15 - 11 \cdot 2 = 27$, so $|Q_3 \backslash P_1| \leqslant 13$. Then, $|P_1 \cup Q_3| \leqslant 28$ so we have that $P_1 \cup Q_3$ must be either a cube or a cube missing a point. By the lower bound on the multiple-intersection of three sets that are either cubes or cubes missing a point, it is impossible for the sets Q_1, Q_2, and Q_3 to exist. Thus, $|P_1| = 16$. Since $|P_i| \geqslant |P_1|$ for $i \in \{2,3,4\}$ and $\sum_{i=1}^{4} |P_i| = 64$, $|P_i| = 16$ for all i and P is a balanced partition. This argument is symmetric for all partitions P, Q, and R. $\square$

We now know $|P_1| = 16$. Then, since $|P_1 \cup Q_1| = |P_1 \cup Q_2| = 26$, P_1 is a set of 16 points such that adding two disjoint sets of 10 points, $Q_1 \backslash P_1$ and $Q_2 \backslash P_1$, creates two distinct sets that are cubes missing a point.

Claim 3.4. *P_1 has the structure of a $2 \times 3 \times 3$ prism missing two points.*

Proof. $P_1 \cup Q_1 = C_1 \backslash p_1$ and $P_1 \cup Q_2 = C_2 \backslash p_2$ for some $3 \times 3 \times 3$ cubes C_1 and C_2 and points p_1 and p_2. They multiple-intersect precisely at P_1. To think about $(P_1 \cup Q_1) \cap (P_1 \cup Q_2) = (C_1 \backslash p_1) \cap (C_2 \backslash p_2) = P_1$, consider the possible structures of $C_1 \cap C_2$. This is either a $2 \times 2 \times 2$ cube, a $2 \times 2 \times 3$ prism, or a $2 \times 3 \times 3$ prism. Since $|P_1| = 16$, and removing p_1 and p_2 can only make the multiple-intersection smaller, we see that the $2 \times 2 \times 2$ and $2 \times 2 \times 3$ options are too small. The $2 \times 3 \times 3$ option is large enough as $18 \geqslant 16$, and we see that the p_1 and p_2 must be removing distinct points. $\square$

Furthermore, since the partitions are balanced, the argument is symmetric and we see that every $P_i \in P$ has the structure of a $2 \times 3 \times 3$ prism missing two points.

Claim 3.5. *It is impossible for four sets P_i of the form of a $2 \times 3 \times 3$ prism missing two points to partition $[4]^3$.*

Proof. Consider four sets of the form of a $2 \times 3 \times 3$ prism. We simplify this to four sets $T_i = S_i \times \{p\}^c$ where S_i is some set with 6 points. Then, we examine the four S_i's using an intersection matrix with maximal packing of four sets of size 6 (Figure 1). Note that in an intersection matrix, the value at (x, y) corresponds to how many sets contain it.

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 4 & 4 \end{matrix}$$

FIGURE 1. Max packing four sets of size 6

Claim 3.6. *Any point (x, y) that is marked with a 3 or 4 in an intersection matrix must be a multiple-intersection point of the T_i in at least 3 layers.*

Proof. Assume for sake of contradiction that (x, y) is marked with a 3 or 4 in the intersection matrix but is only a multiple-intersection point in at most 2 layers A and B. At least 3 sets (without loss of generality, S_1, S_2, and S_3) contain (x, y). This implies that T_1, T_2, and T_3 contain (x, y) across 3 layers. Say T_1 contains (x, y, A), (x, y, B), (x, y, C), and T_2 contains (x, y, A), (x, y, B), (x, y, D). Then T_3 can only contain (x, y, A) and (x, y, B) since the others are not multiple-intersection points, a contradiction. $\square$

Claim 3.7. *Any point (x, y) marked with a 2 in an intersection matrix must be a multiple-intersection point in at least 2 layers.*

Proof. Assume for sake of contradiction that (x, y) is marked 2 in the intersection matrix for layer A and 1 in the other 3 layers. Then, 2 sets (without loss of generality, S_1 and S_2) contain (x, y). This implies T_1 and T_2 contain (x, y) across 3 layers. Say T_1 contains (x, y, A), (x, y, B), and (x, y, C). Then T_2 can only contain (x, y, A) and (x, y, D), a contradiction. □

Thus, each of the (x, y) points containing 3's and 4's in the matrix imply multiple-intersections at (x, y) in at least 3 layers. By convexity, the maximal packing has the smallest number of multiple-intersections. Thus there are at least 9 multiple-intersections with four sets of the form of a $2 \times 3 \times 3$ prism. Since every P_i is actually a $2 \times 3 \times 3$ prism missing two points, and each missing point can subtract at most one from the number of multiple-intersections, we can subtract at most $2 \cdot 4 = 8$ but we still have at least $9 - 8 = 1$ multiple-intersection with four sets of the form of $2 \times 3 \times 3$ prisms missing two points. Thus it is impossible for four sets of the form of $2 \times 3 \times 3$ prisms missing two points to partition $[4]^3$, as parts in a partition must be disjoint. □

□

On the basis of this theorem, we conjecture the following generalization.

Conjecture 3.8. *For $n \geqslant 3$, $HG(K_{n,n}) \leqslant n$.*

4. Windmills

In this section, we discuss the windmill graph family, defined below. In particular, we prove Theorems 1.2 and 1.3.

Definition 4.1. *A graph G is the windmill graph $Wd(k, n)$ if it is formed by joining n copies of K_k at a single universal vertex. For example, $Wd(3, 2)$ is the bow tie.*

For some of the upper bounds, we require the following lemma.

Lemma 4.2. *If $q \geqslant n$, then $H(K_n, q) = nq^{n-1}$.*

Proof. First, we show that $H(K_n, q) \leqslant nq^{n-1}$. This follows from a counting argument. Let f_i be the guessing function of the ith person. The configurations in which at least one person is correct are of the form $(f_1(x_2, x_3, ..., x_{n-2}), x_2, x_3, ..., x_n)$, $(x_1, f_2(x_1, x_3, ..., x_n), x_3, ..., x_n)$, etc., where x_i has q possibilities. Thus, each element in the list encompasses q^{n-1} configurations, and there are n such elements in that list (corresponding to the fact that n different people can win). Thus, it is impossible for them to guarantee a win if there are more than nq^{n-1} configurations they can be in.

Next, we show that $H(K_n, q) \geqslant nq^{n-1}$. Let the set of configurations be all of those for which the sum of the hat colors is between 0 and $n - 1 \mod n$. If we index the players from 0 to $n - 1$, then each player guesses that its hat color is the one that will make the sum of all their hat colors equal to its index modulo q. □

4.1. Proving Theorem 1.3.

Now, we prove Theorem 1.3. Recall the statement: for $n \geqslant \log_2(2k - 2)$,

$$HG(Wd(k, n)) = 2k - 2.$$

This follows from the two lemmas immediately below.

Lemma 4.3. *For $n \geqslant \log_2(2k - 2)$, $HG(Wd(k, n)) \geqslant 2k - 2$.*

Proof. First consider the problem with $2k-2$ hat colors. We can think of $Wd(k,n)$ as being n copies of K_{k-1} that each see a single vertex.

Take the perspective of a particular K_{k-1}. Among the $(2k-2)^{k-1}$ configurations for K_{k-1}, there is a partition of this set into two parts such that each part is solvable by K_{k-1}. Namely, in the $k-1$ dimensional hypercube representing the possible color assignments, we take one part to be a set consisting of 2^{k-2} subhypercubes with side length $k-1$ stationed at non-adjacent corners of the hypercube. More explicity, for $v \in \{0,1\}^{k-1}$ we define the subhypercube $C_v \in \{0,...,2k-3\}^{k-1}$ to be $\{x \mid \text{ if } v_i = 0, x_i < k-1; \text{ if } v_i = 1, x_i \geqslant k-1\}$. Then let C be the union of each C_v such that v has an odd number of 1s, and $\overline{C}$ its complement. If K_{k-1} is restricted to either of these sets of configurations, it can guarantee a win. This is because because given $k-2$ coordinates, each vertex can determine which subhypercube of side length $k-1$ it is in by the aforementioned parity constraint, and any set of configurations in a hypercube of side length $k-1$ is solvable by K_{k-1}, because the problem is equivalent to the problem on K_{k-1} with $k-1$ colors. In other words, K_{k-1} is partially $2k-2$-solvable with respect to both C and $\overline{C}$.

Then, the middle vertex's strategy is a partition of the configurations of all the other vertices. After the removal of one of these parts, one of the copies of K_{k-1} needs to be able to guarantee a win. The middle vertex can make 2^n partitions as follows:

$$\begin{gathered} C \times C \times ... \times C \\ C \times C \times ... \times \overline{C} \\ \vdots \\ \overline{C} \times \overline{C} \times ... \times \overline{C}, \end{gathered}$$

where each line is the Cartesian product of n terms.

If $2^n > 2k-2$, then it can merge the $(2k-2)^{\text{th}}$ part with everything after it to make $2k-2$ parts in total. The result is a partition, because the parts are exhaustive and disjoint. Further, both C and $\overline{C}$ are complements of winning sets of configurations for K_{k-1}, so removing any of those Cartesian products leaves behind a set of configurations in which at least one of the copies of K_{k-1} is restricted to a set of winning configurations. Thus, removing any one of the parts allows one of the copies of K_{k-1} to guarantee a win. $\square$

Lemma 4.4. *For $n \geqslant \log_2(2k-2)$, $HG(Wd(k,n)) < 2k-1$.*

Proof. Suppose $HG(Wd(k,n)) \geqslant 2k-1$. for the sake of contradiction. Then, we would still need the middle vertex to partition the configurations of all other vertices such that removing any one part always left one of the copies of K_{k-1} with a set of winning configurations. In other words, we need each part to contain the Cartesian product of the complements of winning configurations for K_{k-1}.

With $2k-1$ colors, the largest winning sets of configurations for K_{k-1} are of size $(k-1)(2k-1)^{k-2}$ by Lemma 4.2, which is less than half the size of the total set of $(2k-1)^{k-1}$ configurations for K_{k-1}. Thus, the complement of a winning set of K_{k-1} is always larger than half the total set of configurations for K_{k-1}, so any two such complements must overlap. If A_i intersects B_i for some i, then $\prod A_i$ intersects $\prod B_i$. Thus, forming two disjoint parts for the center vertex's partition that each contain such a complement in a particular term in a Cartesian product is impossible. Therefore, there is no adequate set of partitions for the middle vertex that will suffice with more than $2k-2$ colors. ☐

Corollary 4.5. *Let $K_{n,\dots,n,1}$ be the r-partite graph with $r-1$ parts of size n and 1 part of size 1. Then if $n \geqslant \log_2(2r-2)$, $HG(K_{n,\dots,n,1}) \geqslant 2r-2$.*

Proof. Since $Wd(r,n)$ is a subgraph of $K_{n,\dots,n,1}$, the corollary follows from Theorem 1.3. ☐

4.2. **Proving Theorem 1.4.** Recall that the statement of Theorem 1.4 is as follows:

$$HG(Wd(d^n - d^{n-1} + 1, n)) = d^n.$$

To prove the theorem, we require this definition.

Definition 4.6. *Consider the sets of residues $A_1, \dots, A_n$. We say that these sets have no more than a single intersection under translation if for all $(c_1, \dots, c_n)$, $|\bigcap_{i=1}^n (A_i + c_i)| \leqslant 1$.*

Note that this condition is equivalent to $\bigcap_{i=1}^n (A_i - A_i) = \varnothing$, for there are multiple intersections under some translation if and only if there are elements with the same difference in each set.

Now, we prove the theorem.

Proof. First we show $HG(Wd(d^n - d^{n-1} + 1, n)) \leqslant d^n$. Suppose for the sake of contradiction that $HG(Wd(d^n - d^{n-1} + 1, n)) \geqslant d^n + 1$. Note that $Wd(d^n - d^{n-1} + 1, n)$ has a single point, the center, whose strategy partitions the hat configurations of n cliques of size $d^n - d^{n-1}$ into $d^n + 1$ parts. Removing a part leaves on average $\frac{d^n}{d^n+1}(d^n+1)^{(d^n-d^{n-1})n}$ configurations for the other vertices, so removing a particular part leaves at least this many configurations. Thus, we require that $\frac{d^n}{d^n+1}(d^n+1)^{(d^n-d^{n-1})n} \leqslant H(nK_{d^n-d^{n-1}}, d^n+1)$. We claim the following:

Lemma 4.7. *Let nK_{k-1} be n disconnected copies of K_{k-1}. Then $H(nK_{k-1}, q) = q^{(k-1)n} - (q^{k-2}(q-k+1))^n$.*

Assuming this lemma, which will be proved after completing the proof of the theorem, the requirement is not satisfied. (In fact, for d^n colors, the sides of the inequality are equal, so any strategy with d^n colors is "perfect.")

Next, we show $HG(Wd(d^n - d^{n-1} + 1, n)) \geqslant d^n$. Every player except the vertex in the middle will guess that for the clique on $d^n - d^{n-1} + 1$ vertices it belongs in, the sum of the hat colors on that clique is some value modulo d^n, distinct from all guesses of the sum by other vertices in the same clique. Thus, we can represent their strategies by n sets of residues of $\mathbb{Z}/d^n\mathbb{Z}$, where the i'th set consists of elements *not* guessed by anyone in the i'th $d^n - d^{n+1}$ sized clique (excluding the person in the middle); thus each set contains d^{n-1} elements.

By assuming every other player guesses wrong, the center vertex will deduce for each clique that its hat color is one of d^{n-1} colors, a translation of one of the sets of residues. If the intersection of all of these translated sets contains at most one element, then the center vertex can deduce its hat color. In other words, it has a strategy if these sets have no more than a single intersection under translation. We now assume the following lemma, proved later in this section.

Lemma 4.8. *If our residues are elements of $\mathbb{Z}/d^n\mathbb{Z}$ for some $d \in \mathbb{N}$, then there are n sets of size d^{n-1} residues that have no more than a single intersection under translation.*

Then, the theorem follows. ☐

We now prove Lemma 4.7, whose statement is as follows: let nK_{k-1} be n disconnected copies of K_{k-1}. Then $H(nK_{k-1}, q) = q^{(k-1)n} - (q^{k-2}(q-k+1))^n$.

Proof of Lemma 4.7. Recall that by Lemma 4.2, $H(K_{k-1}, q) = (k-1)q^{k-2}$. Then, nK_{k-1} wins on a set of configurations in exactly one of two ways: one set of $n-1$ cliques wins, or that set loses and the last clique wins. The first case provides $q^{k-1}H((n-1)K_{k-1}, q)$ configurations and the second case provides $(q^{(n-1)(k-1)} - H((n-1)K_{k-1}, q))H(K_{k-1}, q)$ configurations. Thus, we have the recurrence

$$H(nK_{k-1}, q) = q^{k-1}H((n-1)K_{k-1}, q) + (q^{(n-1)(k-1)} - H((n-1)K_{k-1}, q))(k-1)q^{k-2}$$

We omit the computations, but the solution to the recurrence is $H(nK_{k-1}, q) = q^{(k-1)n} - (q^{k-2}(q-k+1))^n$. □

Before proving Lemma 4.8, we must introduce some definitions.

Definition 4.9. *We define $\mathrm{d}_d(x, i)$ to be the ith digit of x in base d, where $\mathrm{d}_d(x, 0)$ is the least significant digit; that is $\mathrm{d}_d(x, i) = \lfloor \frac{x}{d^i} \rfloor$.*

Definition 4.10. *We define $\mathrm{r}_d(x)$ to be the number of trailing 0s in the base d representation of x.*

We now prove Lemma 4.8, which states that if our residues are elements of $\mathbb{Z}/d^n\mathbb{Z}$ for some $d \in \mathbb{N}$, then there are n sets of size d^{n-1} residues that have no more than a single intersection under translation.

Proof of Lemma 4.8. We construct our sets as follows, indexing them from $0 \leqslant i \leqslant n-1$: $A_i = \{x \mid \mathrm{d}_d(x, i) = 0\}$. We claim that for each $1 \leqslant a \leqslant d^n - 1$, $a \notin \left(A_{\mathrm{r}_d(a)} - A_{\mathrm{r}_d(a)}\right)$. Indeed, for any $x, y \in A_{\mathrm{r}_d(a)}$, if $\mathrm{d}_d(x, \mathrm{r}_d(a)) = \mathrm{d}_d(y, \mathrm{r}_d(a)) = 0$, it is impossible for $\mathrm{d}_d(x-y, \mathrm{r}_d(a)) = 0$ unless $\mathrm{d}_d(x-y, m) \neq 0$ for some $m < \mathrm{r}_d(a)$, in which case $a \neq x - y$. Thus, the sets A_i have no more than a single intersection under translation. □

The windmill graph $Wd(4,3)$ has hat guessing number 6, disproving the conjecture that all planar graphs have hat guessing number at most 4 from [2]. He and Li [7] previously gave another planar graph with a hat guessing number of 6, namely $B_{2,n}$ for sufficiently large n. On the basis of these examples, we put forward the following revised conjecture.

Conjecture 4.11. *If G is planar, $HG(G) \leqslant 6$.*

5. Books

We call a graph G a book graph $B_{d,n}$ if the graph has n copies of K_{d+1} with a shared K_d. We say the book has n pages with the shared d-clique as the spine. In this section, we prove Theorem 1.5, which states that for sufficiently large n, $HG(B_{d,n}) = 1 + \sum_{i=1}^{d} i^i$.

Finding $HG(B_{d,n})$ for sufficiently large n reduces to a geometric problem about $h(\mathbb{N}^d)$, which is the largest t such that every subset of size t of $\mathbb{N}^d$ can be covered by picking at most one point from every axis-aligned line [2]. He and Li [7] showed that for any $d \geqslant 1$ and n sufficiently large, $HG(B_{d,n}) = h(\mathbb{N}^d) + 1$. We prove below that $h(\mathbb{N}^d) = \sum_{i=1}^{d} i^i$, implying Theorem 1.5.

Theorem 1.5 follows immediately from Lemmas 5.1 and 5.3 below.

Lemma 5.1. $h(\mathbb{N}^d) \leqslant \sum_{i=1}^{d} i^i$

Proof. First, it is evident that $h(\mathbb{N}) = 1$.

Next, we show the theorem inducting on $d \geqslant 2$ with a particular recursive construction S_d of size $d^d + |S_{d-1}| = \sum_{i=1}^{d} i^i + 1$ that is not coverable in $\mathbb{N}^d$. The construction is as follows: consider the hypercube $[d]^d$, which has a covering picking d^{d-1} points in every hyperplane-aligned direction. Then, we place S_{d-1} "on top" of the hypercube. Suppose $x_1, x_2, \ldots, x_d$ are the d directions to cover in the hypercube $[d]^d$. Without loss of generality, we say that the resulting set S_d should be a subset of $[d]^{d-1} \times [d+1]$ where the $(d+1)$ component is in the x_d direction. Formally, $S_d = [d]^d \cup (S_{d-1} \times \{d+1\})$.

Now, let $\pi_i(T)$ be the projection of the set T in the x_i direction. For the base case, consider the 2×3 set S_2, or a 2×2 square with S_1, which we define to be a 2×1 set, on top. S_2 is not coverable in two dimensions because we can pick at most $|\pi_1(S_2)| + |\pi_2(S_2)| = 2 + 3 = 5$ points from it.

We now do the inductive step. Assume S_i is not coverable in $\mathbb{N}^i$ for $i \leqslant d-1$. From the above construction of S_i, we know that $S_{d-1} \subset [d-1]^{d-2} \times [d] \subset [d]^{d-1}$. Thus it is indeed possible to put S_{d-1} "on top" of the hypercube $[d]^d$. Therefore, $S_d \subset [d]^{d-1} \times [d+1]$. We know that S_{d-1} is not coverable by the first $d-1$ dimensions, so after covering the maximum set possible in directions $x_1, \ldots, x_{d-1}$ in $S_{d-1} \times \{d+1\}$, there is at least one point p left uncovered. To cover all of S_d, we must cover this point p in $S_{d-1} \times \{d+1\}$, and we must pick it using the x_d direction. However, since $|\pi_d(S_d)| = d^{d-1}$, we can now only cover $d^{d-1} - 1$ points in the x_d direction in the hypercube $[d]^d$. On the other hand, it still remains true that $|\pi_i([d]^d)| = d^{d-1}$ for $i \in \{1, \ldots, d-1\}$, so we can pick at most d^{d-1} points of the hypercube in each of the first $d-1$ dimensions. Thus we can cover at most $d^d - 1$ points in the hypercube $[d]^d$, so we cannot cover all of S_d. ☐

Remark 5.2. *The above lemma has been proven before independently by Gadouleau [4]. However, we include this alternative proof for completeness as Gadouleau uses very different notation.*

Lemma 5.3. $h(\mathbb{N}^d) \geqslant \sum_{i=1}^{d} i^i$

Proof. We say that a set S is numerically coverable if $\sum_{i=1}^{d} |\pi_i(S)| \geqslant |S|$, where $\pi_i(S)$ is the projection of S orthogonal to the ith axis. It is clear that every coverable set is numerically coverable, and so are all its subsets. We claim the converse is also true.

Claim 5.4. *If every subset of S is numerically coverable, then S is coverable.*

Proof of Claim 5.4. We use Hall's Marriage Theorem [6], which states that a bipartite graph G on sets U and V contains a perfect matching from U to V if every subset U' of U has at least $|U'|$ total neighbors in V. Let U be the points in S and V be the axis-parallel lines intersecting S, with an edge (u, v) on the graph if and only if line v intersects point u. If every subset of S is numerically coverable, then the conditions of Hall's Marriage Theorem are satisfied. Thus, a perfect matching between points and axis-parallel lines exists, which is to say that S is coverable. ☐

Thus, to prove Lemma 5.3, it suffices to show that every set of size at most $\sum_{i=1}^{d} i^i$ is numerically coverable in $\mathbb{N}^d$.

First, we need to show the following weaker claim.

Claim 5.5. *All sets of size at most d^d are numerically coverable. That is, if $|S| \leqslant d^d$, then $\sum_{i=1}^{d} |\pi_i(S)| \geqslant |S|$.*

Proof of Claim 5.5. We use the Loomis-Whitney inequality [9]:

$$|S|^{d-1} \leqslant \prod_{i=1}^{d} |\pi_i(S)|.$$

Apply the AM-GM inequality and the inequality above implies

$$|S|^{d-1} \leqslant (\frac{1}{d}\sum_{i=1}^{d} |\pi_i(S)|)^d$$

$$d|S|^{\frac{d-1}{d}} \leqslant \sum_{i=1}^{d} |\pi_i(S)|, \tag{1}$$

If we have $|S| \leqslant d^d$, then it is equivalent to say $|S| \leqslant d|S|^{\frac{d-1}{d}}$. By plugging the equivalent condition in (1), we have $|S| \leqslant \sum_{i=1}^{d} |\pi_i(S)|$ as desired. □

Now, it remains to show the following claim.

Claim 5.6. *For all S with $|S| \leqslant \sum_{i=1}^{d} i^i$, it is true that $|S| \leqslant \sum_{i=1}^{d} |\pi_i(S)|$.*

Proof of Claim 5.6. We already know the claim holds for $|S| \leqslant d^d$. Thus, we now focus on all S with $d^d < |S| \leqslant \sum_{i=1}^{d} i^i$.

For $d = 1$, S must be the set with one point, and it holds that $|S| \leqslant |\pi_1(S)|$.

Assume for sake of contradiction that the claim does not hold for all $d \in \mathbb{N}$. Then, there must be some minimal $d > 1$ for which it does not hold. That is, there is some S with $d^d < |S| \leqslant \sum_{i=1}^{d} i^i$, such that $|S| > \sum_{i=1}^{d} |\pi_i(S)|$. It is sufficient to assume that S minimizes $\sum_{i=1}^{d} |\pi_i(S)|$ among all sets of the same size $|S|$. Lev and Rudnev [8] show that in that case, we may assume that S contains the hypercube $[d]^d$ and that $S\backslash[d]^d$ lies in one hyperface adjacent to the hypercube. Without loss of generality, say that it lies in the hyperface with $x_d = d + 1$. This implies that $\pi_i(S) = \pi_i\left([d]^d\right) + \pi_i\left(S\backslash[d]^d\right)$ for $i \in \{1, \ldots, d-1\}$ and that $\pi_d(S) = \pi_d\left([d]^d\right)$.

Then,

$$|S| > \sum_{i=1}^{d} |\pi_i(S)|$$

$$\left|[d]^d\right| = \sum_{i=1}^{d} |\pi_i\left([d]^d\right)|$$

$$|S| - \left|[d]^d\right| = |S\backslash[d]^d| > \sum_{i=1}^{d} |\pi_i(S)| - \sum_{i=1}^{d} |\pi_i\left([d]^d\right)|$$

$$|S\backslash[d]^d| > \sum_{i=1}^{d} \left(|\pi_i(S)| - |\pi_i\left([d]^d\right)|\right)$$

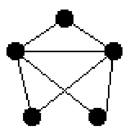
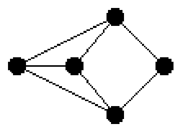
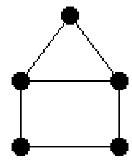
$$|S\backslash[d]^d| > \sum_{i=1}^{d-1} \pi_i\left(S\backslash[d]^d\right)$$

Since $|S| \leqslant \sum_{i=1}^{d} i^i$, we see $|S \backslash [d]^d| \leqslant \sum_{i=1}^{d-1} i^i$. This shows that the claim does not hold for $d-1$, contradicting the minimality of d. $\square$

This finishes the proof of Theorem 1.5. $\square$

6. 5-Vertex Graphs

There are three 5-vertex undirected graphs for which the hat-guessing number is not trivial given results from previous literature or theorems above (all undirected graphs with fewer vertices are already trivial). Each contains a triangle and is not complete, so its hat-guessing number is either 3 or 4. We determine the hat-guessing numbers of all three graphs explicitly.

$B_{2,3}$ Broken Wheel House

Proposition 6.1. *$HG(B_{2,3}) = 4$.*

Proof. We illustrate the strategies of the three pages with three 4×4 grids, where each axis is the color of one of the two vertices in the spine, and the number is the guess a given page will make.

0	0	1	1
0	0	1	1
2	2	3	3
2	2	3	3

0	1	1	0
0	1	1	0
2	3	3	2
2	3	3	2

0	1	0	1
2	3	2	3
0	1	0	1
2	3	2	3

If one of the pages guesses correctly, then we are done. Otherwise, both of the vertices in the spine will assume that all three pages guess wrong. Thus, they can remove one part from each of the three partitions according to the hat colors of the pages. The vertices in the spine have a winning strategy if the set of remaining configurations is coverable in $\mathbb{N}^2$ for all choices of the three parts.

Without loss of generality, we can remove part 0 from the first partition. No matter how the last two parts are chosen, there will be no more than 7 points remaining in the grid. Any set of size 7 has an axis-parallel line that only intersects one point, so this point can be removed without affecting the coverability of the set. Thus, the only sets of size at most 7 that are not coverable are those containing a set of the form $\{a, b\} \times \{c, d, e\}$. One can easily verify that removing any part from the third partition excludes this possibility. $\square$

Proposition 6.2. *$HG($broken wheel$) = 3$.*

Proof. Assume for the sake of contradiction that HG(broken wheel) = 4. The broken wheel has two independent vertices each partitioning the possibilities of $K_1 \cup K_2$ into 4 parts.

Then, the remaining configurations after removing one part from each partition must be either solvable by K_2 or by K_1. The solvable set of K_1 is $\{p\}$ and a solvable set of K_2 is a set C which has at most one point with each x-coordinate and at most one point with each y-coordinate. This implies C has at most 8 points in $[4]^2$. Thus, the solvable set of $K_1 \cup K_2$ is $(C \times [4]) \cup ([4]^2 \times \{p\})$. This means the union of one part from each partition must always contain the complement of a solvable set of $K_1 \cup K_2$.

Lemma 6.3. *The union of one part from each partition must always contain $C^c \times \{p\}^c$, the complement of $(C \times [4]) \cup ([4]^2 \times \{p\})$.*

We will first focus on the smallest part P_1 of v_1's partition, which contains no more than 16 elements. If we call the four parts of v_2's partition Q_1, Q_2, Q_3, and Q_4 then we need $P_1 \cup Q_1$, $P_1 \cup Q_2$, $P_1 \cup Q_3$, and $P_1 \cup Q_4$ to each contain some set of the form $C^c \times \{p\}^c$. Furthermore, since Q_i is disjoint from Q_j for $i \neq j$, there will be at most 16 points (the size of P_1) that are shared by any two or more of the four sets $P_1 \cup Q_1$, $P_1 \cup Q_2$, $P_1 \cup Q_3$, and $P_1 \cup Q_4$ (which we call multiple-intersections, per Definition 2.4).

Since C^c has some structure to it that complicates the problem, we will weaken it to S, any set with 8 points. (Since $|C| \leqslant 8$ points and the total number of K_2 configurations is $4^2 = 16$, $|C^c| \geqslant 8$.)

Lemma 6.4. *Four sets that each contain a set of the form $S \times \{p\}^c$ (for any choice of S and p) must have at least 17 points of multiple-intersection.*

Having a larger set S can only increase the size of the multiple-intersection. Thus, given the lemma, it is impossible for four sets that contain $C^c \times \{p\}^c$ to have at most 16 points of multiple-intersection, and thus the sets $P_1 \cup Q_1$, $P_1 \cup Q_2$, $P_1 \cup Q_3$, and $P_1 \cup Q_4$ cannot exist. □

We now prove Lemma 6.4.

Proof of Lemma 6.4. It suffices to consider four sets $T_1 = S_1 \times \{p_1\}^c$, $T_2 = S_2 \times \{p_2\}^c$, $T_3 = S_3 \times \{p_3\}^c$, and $T_4 = S_4 \times \{p_4\}^c$, which equal rather than contain $S_i \times \{p_i\}^c$, because having more points cannot result in fewer multiple-intersections.

We refer to each 4×4 square corresponding to one color of K_1 as a "layer." The x and y

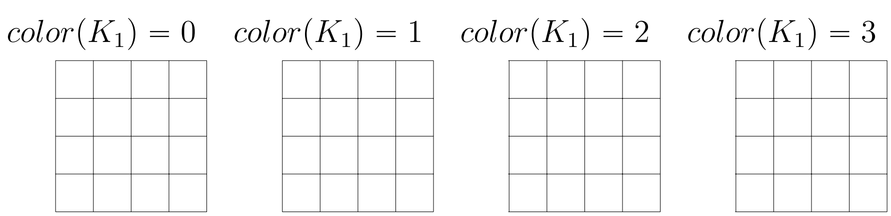


FIGURE 2. Layers

directions within the 4×4 square correspond to the colors of the first and second vertices of K_2 respectively.

We ignore the color of K_1 for now and focus on one 4×4 square (called the "K_2 square"), observing how the sets S_i fit inside. When considering one 4×4 square that contains four sets of size 8 (the S_i), since there are $8 \cdot 4 = 32$ elements and only 16 points, the sets must multiple-intersect in some points (x, y). With maximal packing (having all four sets multiple-intersect in one (x, y) point as many times as possible), we see that we must have at least 6 (x, y) points of multiple-intersection. This is shown as follows: the entries in the matrix correspond to how many sets S_i contain that point (x, y). We refer to such matrices as "intersection matrices" per Definition 2.6.

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 1 & 4 & 4 \end{matrix}$$

FIGURE 3. Max packing, 4 sets

By Claims 3.6 and 3.7, each point (x, y) marked 2 in the K_2 square's intersection matrix contributes at least 2 points of multiple-intersection (x, y, z) across all four layers, and each point (x, y) marked 3 or 4 contributes at least 3 points of multiple-intersection (x, y, z). That is, given an intersection matrix for the K_2 square,

$$\#(\text{total multiple-intersection points}) \geqslant 2 \cdot (\#2) + 3 \cdot (\#3 + \#4).$$

Thus, if we have one intersection matrix with maximal packing (Figure 3), we will have at least $5 \cdot 3 + 2 \cdot 1 = 17 > 16$ points of multiple-intersection total.

We can reformulate the inequality as follows: we have $a_1, \ldots, a_{16} \in \{0, 1, 2, 3, 4\}$ where $\sum a_i = 32$. We wish to find the minimum of $\sum f(a_i)$ where $f(0) = f(1) = 0$, $f(2) = 2$, and $f(3) = f(4) = 3$. We assume that $a_i \neq 0$ for all i, because having a 0 is worse than having a 1 when our goal is to minimize $\sum f(a_i)$. Then, by convexity, we see that the minimum is achieved by having the highest number of 1's. This tells us that the maximal packing is optimal, and others do worse. Thus it is impossible for T_1, T_2, T_3, and T_4 to multiple-intersect in at most 16 points. $\square$

Corollary 6.5. $HG(house) = 3$.

Proof. Since the house is a subgraph of the broken wheel, the former also has hat guessing number 3. $\square$

This completes the classification of 5-vertex undirected graphs.

7. Acknowledgments

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