From Morse Theory to Heegaard Floer Homology

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Heegaard Floer Theory: An Overview

Heegaard Floer Theory is a form of Lagrangian Intersection Floer homology which is specially adapted to use in 3-manifold and 4-manifold topology.

The theory transforms the data provided by a Heegaard diagram for a 3-manifold into a graded module, called Heegaard Floer homology, which is a topological invariant of the 3-manifold.

It has been used with great success in knot theory and in investigating the structure of the homology cobordism group, in large part because of its high computational tractability relative to other Floer theories.

The Morse-Smale-Witten Complex

One of the most important developments in Morse's "doctrine of critical point theory" was the realization that the critical points of a Morse function $f : M \to \mathbb{R}$ can be used to compute the homology of M.

The *Morse-Smale-Witten* chain complex serves as an important model for the Heegaard Floer chain complex.

(i) The chain groups $C_*(M, f)$ are generated by the critical points of the Morse function. The complex is graded by critical point index.

(ii) The differential operator ∂_* is given by counting, with sign (according to certain orientation rules), the number of *gradient flow lines* of f from a given critical point to the critical points of index one below.

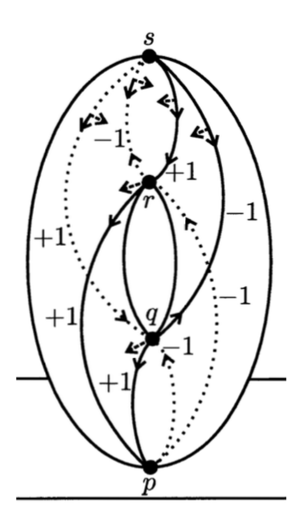

Fig. 1: The critical points and gradient flow lines on a tilted torus.

Roadblocks to Floer Theory

There are two primary challenges when trying to create the analogue of this chain complex in an infinite-dimensional context (the setting of Floer theory):

- **The notion of index is challenging to define** – critical points have "infinite index." The best one can hope for is a notion of "relative index."
- **The moduli space of gradient trajectories may not be compact** – gradient flow lines cannot be "counted."

Heegaard Floer Homology

Morse theory reveals that 3-manifolds admit *Heegaard splittings*: a decomposition into two handlebodies glued along their boundary. This can be seen by choosing a self-indexing Morse function $h : Y \to \mathbb{R}$ on the 3-manifold, and considering the two manifolds-with-boundary $h^{-1}([0, 3/2])$ and $h^{-1}([3/2, 3])$.

The information which informs exactly how this gluing takes place can be conveniently stored in the form of a *Heegaard diagram* (shown below).

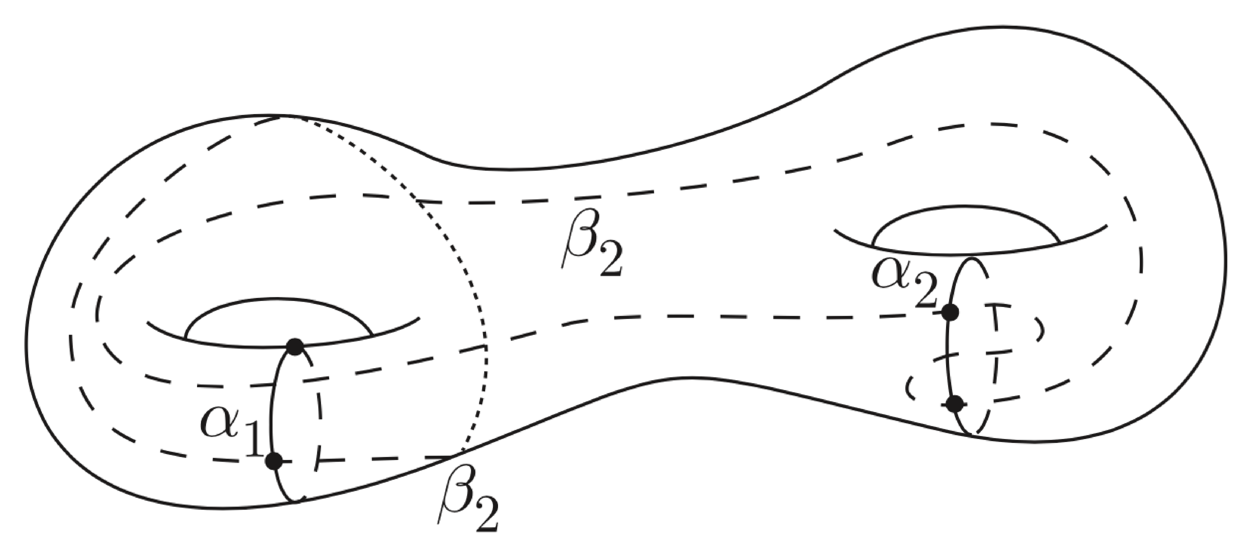

Fig. 2: A genus 2 Heegaard Splitting of S^3

The α and β curves shown provide all the information needed to determine the 3-manifold up to homeomorphism. One can view all the curves "simultaneously" by looking at the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ given by the products of the curves inside of the symmetric product $\text{Sym}^g(\Sigma_g)$ over the genus g surface Σ_g on which they reside.

The framework of the Morse-Smale-Witten complex can be applied to the space of paths $\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ between the two tori.

(i) The resulting chain complex $CF(Y)$ has the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ as generators.

(ii) The differential operator counts holomorphic disks between the intersection points.

The homology of this complex is the *Heegaard Floer homology* of the 3-manifold with which we started.

Additional structure may be added to the complex to obtain richer invariants, including grading by Spin^c structures, and tracking the intersection number of disks with a particular codimension 2 submanifold.

The aforementioned problems with adapting the Morse-Smale-Witten framework to the infinite dimensional space of paths $\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ were resolved as follows:

- **The Maslov index** provides an "expected dimension" of the space of embedded disks between intersection points, and thus resolves the issue of a "relative index."
- **Appropriate perturbations** could be applied to make the moduli space of holomorphic disks compact, in the case of a Maslov index of 1 [2].

Involutive Heegaard Floer Homology and Homology Cobordism

Involutive Heegaard Floer Homology incorporates information about the chain homotopy involution on $CF(Y)$ induced by switching the Heegaard splitting Morse function from h to $-h$. The involution arises from a change in the Heegaard data; namely, a "conjugation"

$$(\Sigma_g, \alpha, \beta) \mapsto (-\Sigma_g, \beta, \alpha).$$

This construction gives rise to some important homology cobordism invariants

The involutive theory has been used to give a reproof of Furuta's Theorem on the existence of a $\mathbb{Z}^\infty$ subgroup of the homology cobordism group $\Theta^3_\mathbb{Z}$, as well as an explicit generating set.

In fact, it is the only known approach which can be used to prove the stronger fact that $\Theta^3_\mathbb{Z}$ has a $\mathbb{Z}^\infty$ summand.

The Problem of Naturality

One drawback of Heegaard Floer theory is that its "higher order" naturality properties remain unsettled. This issue arises from the fact that the space of Heegaard data (i.e. the space of "choices" made when constructing the Heegaard Floer chain complex) is homotopically nontrivial.

Juhász, Thurston, and Zemke established first order naturality of the Heegaard Floer chain complex, up to a canonical chain homotopy class of chain homotopy equivalences [1]. This allowed for the construction of the involutive variant.

It remains to be shown whether there exists a canonical chain homotopy between any two representatives of the aforementioned chain homotopy class. Such a higher order result could lead to more powerful invariants.

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References

[1] András Juhász, Dylan P. Thurston, and Ian Zemke. *Naturality and mapping class groups in Heegaard Floer homology.* 2012. DOI: 10.48550/ARXIV.1210.4996. URL: https://arxiv.org/abs/1210.4996.

[2] Peter Ozsvath and Zoltan Szabo. *Holomorphic disks and topological invariants for closed three-manifolds.* 2001. DOI: 10.48550/ARXIV.MATH/0101206. URL: https://arxiv.org/abs/math/0101206.