HIGH ORDER DISPERSIVE ESTIMATES FOR THE SPHERICALLY SYMMETRIC EINSTEIN SCALAR FIELD

ERIC KILGORE

1. The Einstein Scalar Field

The Einstein Scalar Field system is one of the simplest, and thus most frequently studied, matter-coupled variations on the famous Einstein's equations upon which general relativity is based, expressed in coordinates as:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{GR}$$

for $g_{\mu\nu}$ the (Lorentzian) metric, $R_{\mu\nu}, R$ the Ricci and scalar curvature respectively, and $T_{\mu\nu}$ some function of g and its derivatives (for a thorough treatment of this equation, and explanation of $R_{\mu\nu}, R$ and g, see [12]). The principle difference between GR and the Einstein Scalar Field system is the presence of an extra quantity, ϕ, which one can think of as a massless, non-self-interacting matter field distributed throughout space, which is propagated by the geometry of the system, and we allow to determine $T_{\mu\nu}$ in the following manner.

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \\ T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{-1})^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ \Box_g\phi = 0 \end{cases} \tag{ESF}$$

Where $\Box_g$ denotes the wave operator associated to g:

$$\Box_g u = \frac{1}{\sqrt{-\det g}}\partial_\mu\left((g^{-1})^{\mu\nu}\sqrt{-\det g}\partial_\nu u\right)$$

The goal here is to solve for a triple $(\mathcal{M}, g, \phi)$, where $\mathcal{M}$ a smooth 3+1 dimensional manifold, g a Lorentzian metric, and ϕ the scalar field. You may note that I have not specified any form for the initial data on the problem, nor even said where it ought to be prescribed. As one might expect when part of the problem is the solve for the background space on which g and ϕ live, prescribing initial data to Einstein's equations in general is a rather technical subject, which is once again treated excellently in [12].

This coupling to some form of matter is the main physical motivation of studying this system, as it provides a model of matter couple general relativistic phenomena which can still be studied analytically. Of course, the statement that ESF can be studied analytically should be taken with a grain of salt. In full generality even this simple matter coupled system, like the vacuum Einstein's equations themselves, is quite hard to study. As such we look for a way to reduce the complexity of the problem by studying some special case possessing some additional nice properties. It turns out that a good choice (i.e. one that makes the problem manageable but is not so restrictive as to make it uninteresting) is to impose spherical symmetry. That is, we ask that our initial data, (g_0, ϕ_0), be posed on some 3-manifold Σ that admits a smooth, faithful $SO(3)$ action s.t. for any $h \in SO(3)$, $x \in \Sigma$ $\phi_0(h \cdot x) = \phi_0(x)$, and $(h^* g_0)(x) = g_0(x)$. So we require ϕ to be invariant under our $SO(3)$ action, and moreover we ask that each group element induce an isometry of Σ. One can see without too much trouble that ESF will also obey this condition, and thus propagates the spherical symmetry to the solution $(\mathcal{M}, g, \phi)$. This reduction by spherical symmetry allows us to actually consider the domain $\mathcal{M}/SO(3) := \mathcal{Q}$, a $1+1$ dimensional domain. For the purposes of all that follows, we will also assume that $\mathcal{Q}$ admits a global null coordinate system (u, v) in which the metric takes the form

$$g_{ab} dx^a dx^b = -\Omega^2 du dv$$

It turns out that this assumption is justified, so long as we take our data to be prescribed on some Σ diffeomorphic to $\mathbb{R}^3$. Finally, in order to state the reduction of ESF to spherical symmetry, we will define the quantities

$$r(x) := \sqrt{\frac{A(SO(3) \cdot x)}{4\pi}} \qquad \mu := \frac{2m}{r} := 1 + \frac{4 \partial_u r \partial_v r}{\Omega^2}$$

where $A(SO(3) \cdot x)$ is the area of the sphere given by the orbit of the point x under our group action. We will also define the set $\Gamma = \{(u, v) \,|\, r(u, v) = 0\}$, and require it be connected. It turns out that the equations in spherical symmetry can be phrased entirely in terms of r, m and ϕ. They become:

$$\text{(SSESF)} \qquad \begin{cases} \partial_u \partial_v r = \frac{2m \partial_u r \partial_v r}{r^2(1-\mu)} \\ \partial_u \partial_v (r\phi) = \frac{2m \partial_u r \partial_v r}{r^2(1-\mu)} \phi \\ \partial_u r \partial_u m = \frac{1}{2}(1-\mu) r^2 (\partial_u \phi)^2 \\ \partial_v r \partial_v m = \frac{1}{2}(1-\mu) r^2 (\partial_v \phi)^2 \end{cases}$$

So we see that we end up with non-linear wave equations for $r, r\phi$, and some first order equations governing m.

Much of the gain here is actually in considering this problem independently from ESF itself, i.e. solving SSESF in the region $\mathcal{Q}$ with suitably defined boundary conditions. In this case, since we have sufficiently controlled the background manifold, we can state these explicitly. It turns out that the correct restraints to impose are the following.

For some $u_0 \in \mathbb{R}$ we require $m(u_0, u_0) = r(u_0, u_0) = 0$, and on the line $C_{u_0} := \{(u_0, v) \in \mathcal{Q} \,|\, v \geq u_0\}$ we impose the initial data $\partial_v (r\phi)(u_0, v) = \Phi(v)$ for some function Φ. For technical reasons we must also fix what gauge we work in by prescribing the value of $\partial_v r$ on C_{u_0}. We will take this to be $\frac{1}{2}$, though this condition will not be particularly important for the remaining contents of this note. When we prescribe initial data in this manner, there are well established local existence results for the solution. That is, for any such data with $\Phi \in C^1$, we know that a solution exists for at least a short time. For more detail on the set up of the system see (e.g.) [10].

2. Known Results

Of course, as mentioned above, SSESF has been studied quite a lot in the past thirty years, thus we in fact know much more than this simple local existence result. Many of the earliest results are due to Christodolou (see [6, 5, 3, 2, 1, 4], studying the singularity structure of solutions to SSESF. In particular, he established that generic (with exception of a class of data with codimension 2), asymptotically flat, BV initial data (i.e. data that tends toward Minkowski at long range, and has finite BV norm) gives rise to a solution that either disperses, satisfying favorable decay estimates, or collapses to form a black hole with a curvature singularity. These results, though quite strong, giving a resolution to the famous cosmic censorship conjecture for this class of solutions to SSESF, unfortunately do not provide a very detailed understanding of the long term behavior of solutions outside of the small initial data case (i.e. data with BV norm strictly bounded by some fixed constant).

Better understanding of this long term behavior has come only relatively recently in the past 10-15 years. Some decay estimates for the scalar field and its derivatives were established in the mid 2000's by Dafermos and Rodnianski ([7]) for the black hole case. More recently Luk and Oh established quantitative decay rates for the scalar field, as well as geometric factors (i.e. derivatives of r) in the dispersive case in [10], and it is this case which we will folllow from here on.

In the last few year there has been even more progress in understanding the large data case. In 2018 Luk, Oh and Yang published [11], in which they establish the existence of a class of initial data with arbitrarily large (in fact even infinite) BV norm, which gives rise to dispersive, causal geodescially complete solutions. In [11] they also announced a forthcoming paper [9], which gives criteria for the stability of spherically symmetric initial data to ESF against non-spherically symmetric perturbations, in the sense that one can perturb spherically symmetric data giving rise to a dispersive, causal geodesically complete solution and preserve these properties.

3. New Results

It is here that my work begins. In particular, the goal is to establish that a certain subset of the class of solutions found in [11] in fact satisfies the stability condition provided in [9]. There are two main obstacles to this. First, the conditions of [9] require strong control of at least the first 11 derivatives of ϕ and the metric, and second, these decay estimates must in fact be satisfied by derivatives w.r.t. coordinates in the larger manifold $\mathcal{M}$, rather than the spherical symmetry reduced $\mathcal{Q}$.

The first problem is interesting in its own right. To this point (so far as I am aware) quantitative decay estimates for solutions to SSESF have not been extended past differential order 2. In order to resolve this, I prove the following result:

Theorem 3.1. *For Φ, initial data to SSESF of class C^k prescribed on the curve C_{u_0} satisfying the estimates*

$$\sup_{C_{u_0}} (1+r)^{2+n} \left|\partial_v^n \Phi\right| \leq C$$

for all $n \leq k$, we obtain the following quantitative estimates on the solution $r, \phi, r\phi$.

$$|\partial^\alpha \partial_v r|, |\partial^\alpha \partial_u r| \lesssim \min\Big\{ r^{-(\alpha_v+1)}, r^{-(\alpha_v+1)} u^{-\alpha_u}, u^{-(|\alpha|+1)}, 1 \Big\} \tag{3.1}$$

$$|\partial^\alpha(r\phi)| \lesssim \min\Big\{r^{-(\alpha_v+1)\vartheta(\alpha_v)}, r^{-(\alpha_v+1)\vartheta(\alpha_v)}u^{-\alpha_u}, u^{-(|\alpha|+1)}, 1\Big\} \tag{3.2}$$

$$|\partial^\alpha\phi| \leq \min\Big\{r^{-(\alpha_v+1+\vartheta(\alpha_v))}, r^{-(\alpha_v+1+\vartheta(\alpha_v))}u^{-\alpha_u}, u^{-(|\alpha|+1)}, 1\Big\} \tag{3.3}$$

for all $|\alpha| \leq k$*, multi-indices in* u, v*, where* $\vartheta(n) = \begin{cases} 1 & n > 0 \\ 0 & n \leq 0 \end{cases}$.

In fact, we also gain control in the case where initial data is prescribed on past null infinity, under some additional mild conditions on the decay of ϕ towards the initial data. I also have estimates for m and its derivatives, but I have not yet verified the exact order of decay, so I will omit them for the time being so as to not claim something false.

These decay estimates are essentially proved by induction on $|\alpha|$, using the results of [10], and exploiting the null structure of the wave equations for $r\phi$ and r, with the help of a few tricks to gain control of non-mixed derivatives, and near the axis, Γ.

The second problem turns out to be rather tricky, since the (u, v) coordinate derivatives (as well as r) do not lift to smooth vector fields on $\mathcal{M}$ thanks to the singularity at the axis from our reduction by the $SO(3)$. Thus we actually have two questions:

(1) Does our solution to SSESF lift to a smooth solution to ESF?
(2) How do our decay estimates above translate near the axis?

As it turns out the answers are "yes," and "favorably," with the answer to the second following from the methods used to resolve the first. In order to answer the first question we need to show that all our quantities remain finite at the axis whne we apply the operator $\partial_{r^2} = \frac{1}{r}\partial_r$ an arbitrary (up to required order of smoothness) times. Once again we find ourselves required to work via induction on the number of derivatives taken, showing that each of $\phi, \partial_u r, \partial_v r$ and m are smooth at a given order before proceeding to the next. We also once again use the wave equations for ϕ, r however this time the goal is instead to show that our functions admit an even extension across the axis which remains smooth in derivatives w.r.t. r. These two conditions turn out to be equivalent to the desired differentiability, and follow from analysis of the wave equations independently. The answer to the second question then follows from the bounds used to obtain finiteness of the derivatives when resolving the first.

Togethere these results all give the required estimates to apply [9], and thus give an open class of non-linearly stable dispersive, causal geodesically complete solutions to ESF.

As of now, only a portion of the proof for the dispersive bounds has been written down in full detail, but I am in the process of writing up the remainder of the arguments outlined above. The full proof [8] will hopefully be available within the next 2-3 months.

References

[1] Demetrios Christodoulou, *The problem of a self-gravitating scalar field*, Comm. Math. Phys. **105** (1986), no. 3, 337–361.
[2] ______, *A mathematical theory of gravitational collapse*, Comm. Math. Phys. **109** (1987), no. 4, 613–647.
[3] ______, *The formation of black holes and singularities in spherically symmetric gravitational collapse*, Communications on Pure and Applied Mathematics **44** (1991), no. 3, 339–373.
[4] ______, *Bounded variation solutions of the spherically symmetric einstein-scalar field equations*, Communications on Pure and Applied Mathematics **46** (1993), no. 8, 1131–1220.

[5] Demetrios Christodoulou, *Examples of naked singularity formation in the gravitational collapse of a scalar field*, Annals of Mathematics **140** (1994), no. 3, 607–653.
[6] ———, *The instability of naked singularities in the gravitational collapse of a scalar field*, Annals of Mathematics **149** (1999), no. 1, 183–217.
[7] Mihalis Dafermos and Igor Rodnianski, *A Proof of Price's law for the collapse of a selfgravitating scalar field*, Invent. Math. **162** (2005), 381–457.
[8] Eric Kilgore, *Dispersive Stability For the Spherically Symmetric Einstein Scalar Field System*, In progress.
[9] Jonathan Luk and Sung Jin Oh, *Global Nonlinear Stability of Large Dispersive Solutions to The Einstein Equations*, Currently Unpublished, "In preparation".
[10] ———, *Quantitative decay rates for dispersive solutions to the einstein-scalar field system in spherical symmetry*, Analysis and PDE **8** (2015), no. 7, 1603–1674.
[11] Jonathan Luk, Sung-Jin Oh, and Shiwu Yang, *Solutions to the Einstein-Scalar-Field System in Spherical Symmetry with Large Bounded Variation Norms*, Annals of PDE **4** (2018), no. 1.
[12] Hans Ringstrom, *The cauchy problem in general relativity*, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2009.