# Polynomial Convexity of Simple Complex Shapes

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## Background

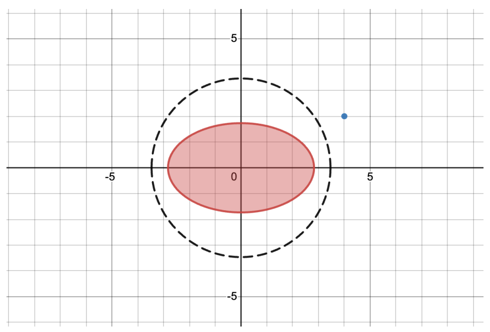
We consider polynomially convex sets, a generalization of convex sets.

**Definition.** For any compact $Y \subset \mathbb{C}^n$, we define the polynomial hull of $Y$ to be

$$Y^\wedge = \left\{ x \in \mathbb{C}^n : |p(x)| \leq \sup\{|p(y)| : y \in Y\} \text{ for all polynomials } p \right\}.$$

We say $Y$ is polynomially convex if $Y = Y^\wedge$.

In particular, a convex set is polynomially convex.




Our research focuses on the disjoint union the following objects:

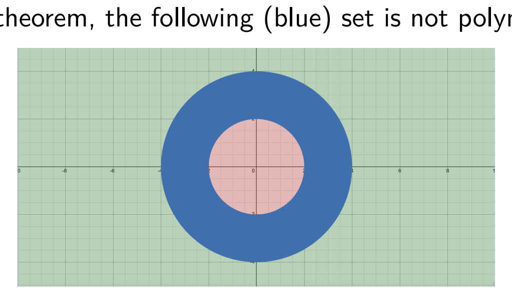
- sphere $= \{z \in \mathbb{C}^n : |z_1 - a_1|^2 + \ldots + |z_n - a_n|^2 \leq r^2\}$
- polydisk $= \{z \in \mathbb{C}^n : |z_1 - a_1| \leq r_1, \ldots, |z_n - a_n| \leq r_n\}$
- generalized super-ellipsoid (GSE) $= \{z \in \mathbb{C}^n : |z_1 - a_1|^k + \ldots + |z_n - a_n|^k \leq r^k\}$ for some exponent $k \geq 2$, which we call the degree of the GSE

## Characterizing Polynomial Convexity in C

**Theorem.** A compact set $Y \subset \mathbb{C}$ is polynomially convex if and only if the complement $\mathbb{C} \setminus Y$ is connected.

- The forward direction follows from the *maximum modulus principle*.
- The reverse direction follows from a clever application of *Runge's theorem*.

**Example.** By the theorem, the following (blue) set is not polynomially convex.



## Important Results from Kallin

Kallin's paper [1] forms the basis of our work, providing us with:

- A method for proving polynomial convexity of the disjoint union of several objects, called the separation lemma.
- A method for generating a counterexample to show the disjoint union of sevearl objects is not polynomially convex.

### Using the Separation Lemma to Prove Polynomial Convexity

**Separation Lemma.** If $X_1, X_2 \subset \mathbb{C}^n$ compact and $f$ is a polynomial such that $(f[X_1])^\wedge \cap (f[X_2])^\wedge = \emptyset$, then $(X_1 \cup X_2)^\wedge = X_1^\wedge \cup X_2^\wedge$.

The proof arises from *Runge's theorem*.

## Important Results from Kallin (continued)

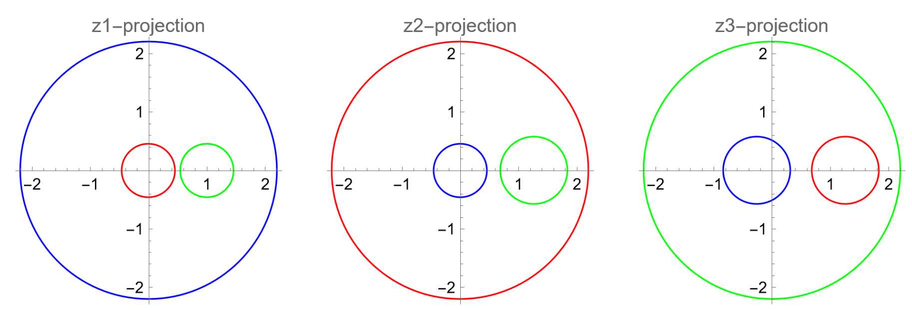
**Theorem (Kallin).** The disjoint union of any three balls $S_1, S_2, S_3$ are polynomially convex in $\mathbb{C}^n$.

- Two balls are polynomially convex so it suffices to separate $S_1$ from $S_2$ and $S_3$.
- Scale the balls such that the largest ball $S_1$ has radius 1.
- Choose coordinates and rotate the balls such that $S_1$ has center $(0,0)$, $S_2$ has center $(\gamma, 0)$ with $\gamma \in \mathbb{C}$ and $S_3$ has center $(\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R}$.
- The polynomial $f(z) = z_1^2 + z_2^2$ will separate $S_1$ from $S_2$ and $S_3$.

### Using the Maximum Modulus Principle to Find Counterexamples

**Theorem (Kallin).** There exists a collection of three disjoint polydisks that is not polynomially convex in $\mathbb{C}^3$.

- Define a surface cut out by $z_1 z_2 = 1, z_3(1 - z_1) = 1$. On the surface, take the curves $|z_1| = M$, $|z_2| = M$, $|z_3| = M$ for some $M > 2$.
- **Basic idea:** the polynomial hull of the three curves contains the part of the surface bounded by the three curves. If we can fit three (disjoint) polydisks over each of the three curves, then their polynomial hull will also contain this section of the surface.



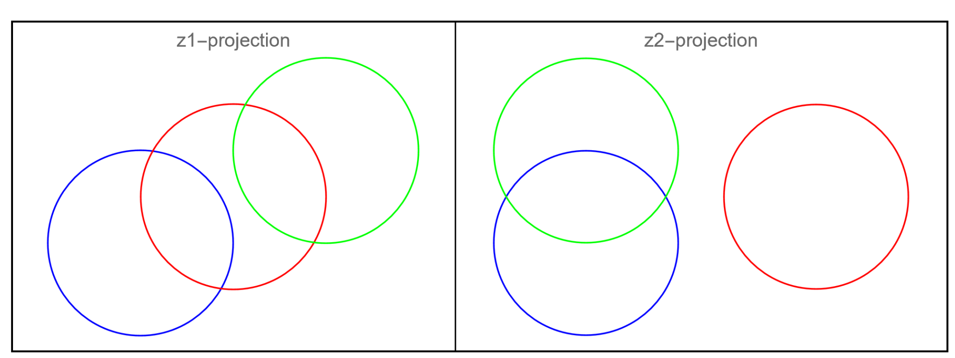

Projections of the three curves when $M = 2.2$.

- The polydisks of radius $M$, centered at $(-M + \frac{1}{M}, 0, M + \frac{M}{M+1})$ and $(M + 1 - \frac{1}{M}, M + \frac{M}{M+1}, 0)$ and $(0, -M + \frac{1}{M}, -M + \frac{1}{M+1})$ satisfy this condition.

## Three Polydisks in $\mathbb{C}^2$

**Theorem.** The disjoint union of any three polydisks $P_1, P_2, P_3$ are polynomially convex in $\mathbb{C}^2$.

- By the same argument as Kallin's three spheres, it suffices to separate one polydisk from the other two.
- Any two disjoint polydisks can intersect in at most one coordinate projection. (If they intersect in both projections, they are no longer disjoint.)



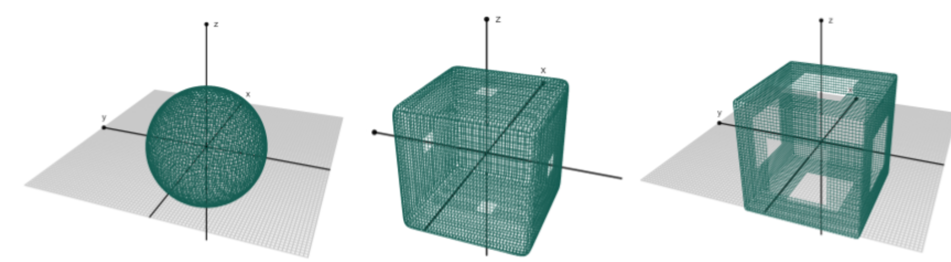

## Three GSEs in $\mathbb{C}^n$

**Question.** If we could slowly stretch a sphere into a polydisk, at what point in this process would the intermediate shape become not polynomially convex?

To answer this, we came up with the idea of GSEs, noting that a GSE with $k = 2$ is simply a sphere, and as $k \to \infty$, the GSE approaches a polydisk.

**Theorem.** When the degree $k \geq 18.121$, the disjoint union of three GSEs $E_1, E_2, E_3$ is not polynomially convex in $\mathbb{C}^3$.

- We use Kallin's surface cut out by $z_1 z_2 = 1, z_3(1 - z_1) = 1$, and on the surface, take the curves $|z_1| = M, |z_2| = M, |z_3| = M$ for some $M > 2$.
- We center $E_1, E_2, E_3$ at $(-M + \frac{1}{M}, 0, M + \frac{M}{M+1})$ and $(M + 1 - \frac{1}{M}, M + \frac{M}{M+1}, 0)$ and $(0, -M + \frac{1}{M}, -M + \frac{1}{M+1})$ respectively, with radii $r_1, r_2, r_3$.
- Two GSEs cannot intersect, allowing us to bound the radii from above.
- $E_1, E_2$ and $E_3$ must contain the curves $|z_1| = M, |z_2| = M, |z_3| = M$, allowing us to bound the radii from below.
- The difference between our two bounds (which are in terms of $M$ and $k$) must be positive in the worst case. Choose $M \approx 4$, then the condition holds $k \geq 18.121$.



A 3-real-dimensional analog of a GSE with $k = 2$, $k = 19$ and $k = \infty$.

## Conclusion and Future Work

Despite being a natural generalization of convexity, much less is understood about the idea of polynomial convexity. As our research demonstrates, even simple shapes fail to admit intuitive solutions. In the future, some problems we would like to explore include:

- Finding a separating polynomial to prove that three GSEs with degree $2 < k < 18.121$ are polynomially convex
- Using the Fubini-Study metric to show that $k \geq 18.121$ cannot be improved by raising the dimension
- Improving the bound of $k \geq 18.121$ by finding a more optimal surface and curves
- Determining if four polydisks are polynomially convex in $\mathbb{C}^2$ using surfaces in projective space
- Finding new techniques for proving and disproving polynomial convexity to tackle questions like the four spheres problem

## Acknowledgements

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## References

[1] Eva Kallin.
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