Elliptic Regularity and the Hodge Decomposition Theorem

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Introduction

- Hodge Theory is a means of studying the cohomology groups of manifolds using methods from the analysis of partial differential equations.
- A key result in Hodge Theory is the **Hodge Decomposition theorem**, which we will state and prove below.
- The Hodge Theorem provides an interested an unexpected connection between the study of distributions and pseudodifferential operators, and algebraic topology.

Background: Notation

- An n-dimensional multi-index is an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.
- For multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$ we define

$$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \dots \partial_n^{\alpha_n},\ x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} \text{ and } D = \frac{1}{i}\partial \qquad (1)$$

- For an open set $U \subset \mathbb{R}^n$ and non-negative integers k we let $C^k(U)$ denote the space of function φ from U into C such that $\partial^\alpha \varphi$ exist and are continuous for all α with $|\alpha| \leq k$. Define the norm $\|\varphi\|_{C^k(U)} = \max_{|\alpha|\leq k} \sup |\partial^\alpha \varphi|$ on this space.
- Define the space of smooth functions as $C^\infty(U) = \bigcap_{k\in\mathbb{N}} C^k(U)$ and then let $C_c^\infty(U)$ denote the subspace of $C^\infty(U)$ consisting of smooth functions with compact support.

Background: Distribution Theory

For an open set $U \subset \mathbb{R}^n$ we define a **distribution** on u as a linear map $u : C_c^\infty(U) \to \mathbb{C}$ such that for all $K \subset U$ compact, there exists $C \geq 0, n \in \mathbb{N}$ for which $|u(\varphi)| \leq C\|\varphi\|_{C^n(U)}$ for all $\varphi \in C_c^\infty(U)$. We let $D'(U)$ denote the space of distributions on U. For $u \in D'(U)$ and $\phi \in C_c^\infty(U)$ we take $(u, \varphi) := u(\varphi)$.

Example

For $f \in L^1_{\text{loc}}(U)$ we define $T_f(\varphi) = \int_U f(x)\varphi(x)dx$. We claim that $T_f \in D'(U)$. Indeed, for all $K \subset U$, we have that

$$|T_f(\varphi)| = \left|\int_U f(x)\varphi(x)dx\right| \leq \left(\int_K f(x)dx\right) \|\varphi\|_{C^0(U)} \qquad (2)$$

- We define a notion of convergence (but not a topology) on $D'(U)$ by saying that $u_k \to u$ in $D'(U)$ if $(u_k, \varphi) \to (u, \varphi)$ for all $\varphi \in C_c^\infty(U)$.
- We can also define (by duality), differentiation of distributions, multiplication of distributions by smooth functions, and convolution of distributions.

Example

Consider a sequence $f_k \in L^1_{\text{loc}}(U)$ converging pointwise (almost everywhere) to $f \in L^1_{\text{loc}}(U)$. We would like it $f_k \to f$ in $D'(U)$. Indeed, if the f_k are uniformly bounded above by some locally integrable function, then by the dominated convergence theorem we find that for all $\varphi \in C_c^\infty(U)$,

$$\lim_{k\to\infty}(f_k, \varphi) = \lim_{k\to\infty} \int_U f_k(x)\varphi(x)dx = \int_U f(x)\varphi(x) = (f, \varphi) \qquad (3)$$

and so f_k indeed converges to f.

Distribution Theory (Continued)

Example

Let $f_k(x) = k\mathbf{1}_{0\leq x\leq \frac{1}{k}}$. Then $f_k \to 0$ almost everywhere but $f_k \to \delta_0$ in the sense of distributions.

The above example illustrates how the notion of convergence in the sense of distributions is relatively weak.

Elliptic Regularity

We are motivated by questions concerning the regularity of solutions to PDEs. Intuitively, the singular support of a distribution $u \in D'(U)$ consists of the set of points of U where u is not locally a smooth function.

Theorem

Let P be a constant coefficient differential operator that is elliptic. Then, for all $U \subseteq \mathbb{R}^n$ and $u \in D'(U)$, sing supp $u \subseteq$ sing supp Pu.

This result can be generalized to the manifold setting which is critical for the proof of the Hodge Theorem.

Theorem

Let P be an elliptic operator in $\text{Diff}^m(M)$ for some non-negative integer m and manifold M. Then for all $u \in D'(M)$, we have sing supp $u \subseteq$ sing supp Pu.

- The proof of elliptic regularity relies on the development of the theory of pseudodifferential operators and the accompanying symbol calculus.
- The study of the singularities of distributions is important and interesting in its own right and is a large component of the field of microlocal analysis.

Vector Bundles

We define a (real) vector bundle as a triple (M, E, π) such that

- M is an n-dimensional manifold, E is an $n + m$ dimensional manifold
- $\pi : E \to M$ is surjective and $\pi^{-1}(x)$ has the structure of an m-dimensional vector space for all $x \in M$
- For all $x \in M$ there exists a neighborhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^m$ such that $\phi|_{\pi^{-1}(x)}$ is an isomorphism between $\pi^{-1}(x)$ and $\{x\} \times \mathbb{C}^m$

Example

The trivial bundle consists of a manifold M, $E = M \times \mathbb{R}^m$ and $\pi(x, v) = x$.

Example

The tangent bundle TM and the cotangent bundle T^*M are both vector bundles over M with π again simply being the projection map.

Given a vector bundle (M, E, π), a smooth map $\sigma : M \to E$ is a (smooth) section of the bundle provided $\pi \circ \sigma =$ Id. The collection of smooth sections of a vector bundle (M, E, π) is denoted $C^\infty(M, E)$.

Differential Forms

- Fix a compact, oriented, n-dimensional manifold M. We define the vector bundle of k-forms on M as $\Omega^k = \bigwedge^k(T^*M)$ and the vector bundle of all differential forms on M as $\Omega^o = \otimes_{k=0}^n \Omega^k$.
- Elements of $C^\infty(M, \Omega^k)$ are called differential k-forms.
- Let $d_k : C^\infty(M, \Omega^k) \to C^\infty(M, \Omega^{k+1})$ denote the exterior derivative. It is well known that $d_{k+1} \circ d_k = 0$ for all k.
- This allows us to define the deRham cohomology groups

$$H^k_{dR}(M) = \{\omega \in C^\infty(M, \Omega^k) : d_k\omega = 0\}/\{d_{k-1}\omega : \omega \in C^\infty(M, \omega^{k-1})\}. \qquad (4)$$

- Suppose we have a Riemannian Metric g on our manifold M. Via the Riesz representation theorem, g induces an inner product on $C^\infty(M, \Omega^1) = T^*M$. We can then use this define a canonical inner product on $C^\infty(M, \Omega^k)$, the space of differential differential k-forms.
- Define δ_{k+1} to be the adjoint of d_k for all $0 \leq k < n$. Then, define $\mathcal{H}^k = \{\omega \in C^\infty(M, \Omega^1) : (d + \delta)^2\omega = 0\}$. This is the space of Harmonic k-forms.

Hodge Decomposition Theorem

The Hodge Decomposition Theorem is important because it provides a gateway between de Rham cohomology (an algebraic topology concept), and the space of Harmonic Forms (a Riemannian Geometry concept).

Theorem

Let M be a n-dimensional oriented compact Riemannian Manifold. Then

$$C^\infty(M, \Omega^k) = \mathcal{H}^k \otimes \text{im } \delta_{k+1} \otimes \text{im } d_{k-1} \qquad (5)$$

and

$$\ker d_k = \mathcal{H}^k \otimes \text{im } d_{k-1} \qquad (6)$$

so in particular

$$H^k_{dR}(M) \cong \mathcal{H}^k \qquad (7)$$

- Proving that the sum is direct boils down to a simple linear algebra computation. The challenge of the proof is showing that any form can be decomposed in the desired manner, which makes use of some aspects of Fredholm theory for elliptic operators, which itself makes critical use of elliptic regularity.
- The Decomposition Theorem can be generalized to other settings, including the deRham cohomology of M over $\mathbb{C}$ and to the setting of Kahler manifolds.

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