Elliptic Regularity and the Hodge Decomposition Theorem

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Abstract

We present a proof of the Hodge Decomposition Theorem by way of Elliptic Regularity results for differential operators on manifolds. Along the way, we develop some aspects of the theory of vector bundles, distribution theory, and differential forms.

1 Introduction and Motivation

Hodge Theory is a way of studying the Cohomology groups of Riemannian manifolds using methods from the analysis of partial differential equations. A key result in Hodge Theory is the Hodge Decomposition theorem, which we will state and prove below. The Hodge Decomposition can be viewed as a generalization of the Helmholtz decomposition from $\mathbb{R}^3$ to Riemannian manifolds. Of course, this generalization is not quite a strict generalization as $\mathbb{R}^3$ is not compact.

2 Distribution Theory

Here we let U denote an open subset of $\mathbb{R}^n$.

2.1 Notation

An n-dimensional *multi-index* is an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$ we define

$$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n},\ x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n} \text{ and } D = \frac{1}{i}\partial \tag{2.1}$$

We also define the *norm* of a multi-index by

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \tag{2.2}$$

Then, for non-negative integers k we say that a function $\varphi : U \to \mathbb{C}$ is C^k provided that $\partial^\alpha \varphi$ exist and are continuous for all α with $|\alpha| \leq k$. The space of all C^k functions with domain U is denoted $C^k(U)$. We define the norm $\|\varphi\|_{C^k(U)} = \max_{|\alpha|\leq k} \sup |\partial^\alpha \varphi|$ on this space. Then the space of smooth functions is defined by $C^\infty(U) = \bigcap_{k\in\mathbb{N}} C^k(U)$ and the subspace of smooth functions with compact support is denoted by $C_c^\infty(U)$. The smooth functions with compact support will be the *test* functions we use to define distributions.

2.2 Distribution Preliminaries

We start by defining distributions and prove some results concerning their basic properties.

Definition 2.1. For an open set $U \subset \mathbb{R}^n$ we define a *distribution* on u as a linear map $u : C_c^\infty(U) \to \mathbb{C}$ such that for all $K \subset U$ compact, there exists $C \geq 0, n \in \mathbb{N}$ for which $|u(\varphi)| \leq C\|\varphi\|_{C^n(U)}$ for all $\varphi \in C_c^\infty(U)$ supported in K. We let $D'(U)$ denote the space of distributions on U. For $u \in D'(U)$ and $\phi \in C_c^\infty(U)$ we take $(u, \varphi) := u(\varphi)$. In this way, we can think of distributions as acting on test functions as well as test functions acting on distributions.

Example 2.2. For $f \in L^1_{\text{loc}}(U)$ we define $T_f(\varphi) = \int_U f(x)\varphi(x)dx$. We claim that $T_f \in D'(U)$. Indeed, for all $K \subset U$, we have that

$$|T_f(\varphi)| = \left| \int_U f(x)\varphi(x)dx \right| \leq \left(\int_K f(x)dx \right) \|\varphi\|_{C^0(U)} \tag{2.3}$$

In fact, using mollifiers it can be shown that if $T_f = 0$ then $f = 0$ so the map $f \to T_f$ is a linear injection of $L^1_{\text{loc}}(U)$ into $D'(U)$. In this way we can (and do) identify $L^1_{\text{loc}}(U)$ as a subspace of $D'(U)$.

Example 2.3. For $U \subset \mathbb{R}^n$ and $y \in U$ we define the distribution $\delta_y(\varphi) = \varphi(y)$. This is indeed a distribution as for all $\varphi \in C_c^\infty(U)$ we have

$$|\delta_y(\varphi)| = |\varphi(y)| \leq \|\varphi\|_{C_0(U)}. \tag{2.4}$$

We define a notion of *convergence* on $D'(U)$ by saying that $u_k \to u$ in $D'(U)$ if $(u_k, \varphi) \to (u, \varphi)$ for all $\varphi \in C_c^\infty(U)$.

Example 2.4. Consider a sequence $f_k \in L^1_{\text{loc}}(U)$ converging pointwise (almost everywhere) to $f \in L^1_{\text{loc}}(U)$. We would like it $f_k \to f$ in $D'(U)$. Indeed, if the f_k are uniformly bounded above by some locally integrable function, then by the dominated convergence theorem we find that for all $\varphi \in C_c^\infty(U)$,

$$\lim_{k\to\infty} (f_k, \varphi) = \lim_{k\to\infty} \int_U f_k(x)\varphi(x)dx = \int_U f(x)\varphi(x) = (f, \varphi) \tag{2.5}$$

and so f_k indeed converges to f.

The above example shows that our notion of convergence on the space of distributions is relatively weak.

Example 2.5. Let $f_k(x) = k1_{0\leq x \leq \frac{1}{k}}$. Then $f_k \to 0$ almost everywhere but $f_k \to \delta_0$ in the sense of distributions as for all $\varphi \in C_c^\infty(\mathbb{R})$,

$$\left| \lim_{k\to\infty} (f_k, \varphi) - \varphi(0) \right| = \left| \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)(\varphi(x) - \varphi(0)dx \right| \leq \lim_{k\to\infty} \sup_{0\leq x\leq \frac{1}{k}} |\varphi(x) - \varphi(0)| = 0. \tag{2.6}$$

Definition 2.6. For $u \in D'(U)$, we define the distributions $\partial_j u$ for appropriate $1 \leq j \leq n$ by

$$(\partial_j u, \varphi) := -(u, \partial_j \varphi). \tag{2.7}$$

This is indeed a valid definition. Let $K \subseteq U$. Then, there exists $C \geq 0$ and non-negative n such that $|(u, \varphi)| \leq C\|\varphi\|_{C^N(U)}$ for all $\varphi \in C_c^\infty(U)$ supported in K. Thus, because $\partial_j\varphi \in C_c^\infty(U)$ as well, we find that

$$|(\partial_j u, \varphi)| = |(u, \partial_j \varphi)| \leq C\|\partial_j \varphi\|_N \leq C\|\varphi\|_{n+1} \tag{2.8}$$

and so $\partial_j u$ is indeed a distribution. If $f \in C^1(U)$ then, using integration by parts it is easy to verify that $T_{\partial_j f} = \partial_j T_f$. Thus, the distributional and normal definitions of the partial derivative agree. Additionally, if $u_k \to u$ in $D'(U)$ then for all $\varphi \in C_c^\infty(U)$,

$$\lim_{k\to\infty} (\partial_j u_k, \varphi) = -\lim_{k\to\infty} (u_k, \partial_j \varphi) = -(u, \partial_j \varphi) = (\partial_j u, \varphi) \tag{2.9}$$

so $\partial_j u_k \to u$ in $D'(U)$ as well. In this way, for all multi-indices α, we can define the distribution $\partial^\alpha u$ by $(\partial^\alpha u, \varphi) = (-1)^{|\alpha|}(u, \partial^\alpha \varphi)$.

Example 2.7. Let $f(x) = 1_{[a,b]}$ for some $a \leq b \in \mathbb{R}$. Then for all $\varphi \in C_c^\infty(U)$ we find that

$$(\partial_j f, \varphi) = -(f, \partial_j, \varphi) = -\int_a^b \partial_j \varphi = \varphi(a) - \varphi(b) = (\delta_a - \delta_b)(\varphi) \tag{2.10}$$

so $\partial_j f = \delta_a - \delta_b$.

Example 2.8. For $a \in U$, $\varphi \in C_c^\infty(U)$ and multi-index α we find that

$$(\partial^\alpha \delta_a, \varphi) = (-1)^{|\alpha|}(\delta_a, \partial^\alpha \varphi) = (-1)^{|\alpha|}(\partial^\alpha \varphi)(a) \tag{2.11}$$

Similarly to the above construction of differentiation of distributions, we can define multiplication of distributions by smooth functions.

Definition 2.9. For $u \in D'(U)$, $a \in C^\infty(U)$ and $\varphi \in C_c^\infty(U)$ we let $(ua, \varphi) := (u, a\varphi)$.

It is easy to verify that if $f \in L_1^{\text{loc}})(U)$ then $T_{af} = aT_f$ for smooth a and that if $u_k \to u$ in $D'(U)$ then $au_k \to au$ in $D'(U)$.

Example 2.10. Let $f \in C^\infty(U)$ and $a \in U$. Then, for all $\varphi \in C_c^\infty(U)$,

$$(f\delta_a, \varphi) = (\delta_a, f\varphi) = f(a)\varphi(a) = (f(a)\delta_a, \varphi) \tag{2.12}$$

so $f\delta_a = f(a)\delta_a$.

Proposition 2.11. *For $u \in D'(U)$ and $a \in C^\infty(U)$,*

$$\partial_j(au) = a(\partial_j u) + (\partial_j a)u \tag{2.13}$$

This is a generalized version of Leibniz' rule.

Proof. For $\varphi \in C_c^\infty(U)$, we compute that

$$\begin{aligned}
(\partial_j(au), \varphi) &= -(au, \partial_j \varphi) \\
&= -(u, a(\partial_j \varphi)) \\
&= -(u, \partial_j(a\varphi) - (\partial_j a)\varphi)) \\
&= (\partial_j u, a\varphi) + (u, (\partial_j a)\varphi) \\
&= (a(\partial_j u), \varphi) + ((\partial_j a)u, \varphi) \\
&= (a(\partial_j u) + (\partial_j a)u, \varphi)
\end{aligned}$$

where we have made use of the classical version of Leibniz' rule.

□

Now for $u \in D'(U)$ and $V \subseteq U$ open we define $u|_V$ by $(u|_V, \varphi) = (u, \varphi)$ for $\varphi \in C_c^\infty(V)$ where we note the inclusion of $C_c^\infty(V)$ into $C_c^\infty(U)$ by setting φ to be 0 on V/U. This is called *localization* of a distribution. We say that two distributions $u, u' \in D'(U)$ are equal on V if $u|_V = u'|_V$

Definition 2.12. For $u \in D'(U)$ we define the *support* of a distribution by

$$\text{supp } u = \{x \in U : u = 0 \text{ on a neighborhood of x}\}^C \tag{2.14}$$

It is easy to see from the above definition that the support of a distribution in $D'(U)$ is closed in U.

Proposition 2.13. *If $f \in C^0(U)$ then supp f = supp T_f.*

Proof. If $x \notin$ supp f then there exists $r > 0$ such that $f \equiv 0$ on $B_r(x)$. Thus, $T_f|_{B_r(x)} = 0$ so $x \notin$ supp T_f. If $x \notin$ supp f then for some $x \in V \subseteq U$, $T_f|_V = 0$ so for all $\varphi \in C_c^\infty(V)$, $\int_V f(x)\varphi(x) = 0$ and hence $f \equiv 0$ on V so $x \notin$ supp f. □

Example 2.14. We claim that for $a \in U$, supp $\delta_a = \{a\}$. Indeed, consider any neighborhood V of a. Let $r > 0$ such that $B_{2r}(a) \subseteq V$. Let f be a bump function which is 1 on $B_r(a)$ and 0 outside of $B_{2r}(a)$. Then $(\delta_a, f) = f(a) = 1$ so $\delta_a|_V \neq 0$. Thus $a \in$ supp δ_a. Now let $b \neq a \in V$ and $r > 0$ such that $a \notin B_r(b)$. Then, for all $\varphi \in C_c^\infty(B_r(b))$,

$$(\delta_a|_{B_r(b)}, \varphi) = \varphi'(a) = 0 \tag{2.15}$$

so $\delta_a|_{B_r(b)} = 0$. Thus $b \notin$ supp δ_a, completing the proof.

Proposition 2.15. *For all $u \in D'(U)$, $u|_{U/\text{supp } u} = 0$.*

Proof. Let $\varphi \in C_c^\infty(U/\text{supp } u)$. Then, for all $x \in$ supp φ there exists a neighborhood V_x of x such that $u|_{V_x} = 0$. Suppose $V_{x_1}, \ldots, V_{x_m}$ covers supp U (which is compact). Let $\chi_1, \ldots, \chi_m$ be a partition of unity subordinate to this cover (with $\chi_i \in C_c^\infty(V_{x_i})$ for all i). Then,

$$(u|_{U/\text{supp}(U)}, \varphi) = (u|_{U/\text{supp}(U)}, \varphi\chi_1 + \ldots + \varphi\chi_m) = (u|_{V_{x_1}}, \varphi\chi_1) + \ldots + (u|_{V_{x_m}}, \varphi\chi_m) = 0 \tag{2.16}$$

□

Now, we would like a way to characterize the points where a distribution is smooth, to do so we make the following definition.

Definition 2.16. For $u \in D'(U)$ we say that $x \in U$ is not in sing supp u (the *singular support* of u) if there exists a neighborhood $x \in V \subseteq U$ such that $u|_V \in C^\infty(V)$.

Proposition 2.17. *For $u \in D'(U)$, sing supp $u = \emptyset \iff u \in C^\infty(U)$.*

Proof. Suppose first that $u \in D'(U)$ and sing supp $u = \emptyset$. For $x \in U$ let V_x be a neighborhood around x such that $U|_{V_x} \in C^\infty(V_x)$. let $f_x = U|_{V_x}$. Suppose $V_x \cap V_{x'} \neq \emptyset$. Then by definition

$$f_x|_{V_x \cap V_{x'}} = (U|_{V_x})|_{V_x \cap V_{X'}} = U_{V_x \cap V_{x'}} = (U|_{V_{x'}})|_{V_x \cap V_{X'}} = f_{x'}|_{V_x \cap V_{x'}} \tag{2.17}$$

Thus, we can define $f \in C^\infty(U)$ by $f|_{V_x} = f_x$ for all x. Then, using a similar partition of unity argument to the one in the proof of proposition 2.14, we conclude that $(u, \varphi) = (f, \varphi)$ for all $\varphi \in C_c^\infty(U)$. The other direction is immediate.

□

The singular support also has a number of other properties, namely, sing supp $\partial_j u \subseteq$ sing supp u, sing supp $au \subseteq$ sing supp u and sing supp $u \subseteq$ supp u for all $u \in D'(U)$, $a \in C^\infty(U)$.

2.3 Distributions on Manifolds

In this section we will define distributions on manifolds and see some basic results about their properties. As we will see, distributions on manifolds have many of the nice properties we observed for distributions on open subsets of $\mathbb{R}^n$. We first need to define a notion of convergence for test functions on our manifold.

Definition 2.18. For $f_k, f \in C_c^\infty(U)$, we say that $f_k \to f$ provided that

- There exists a compact subset K of U such that supp $f_k \subseteq K$ for all k
- $\|f_k - f\|_{C^N(U)} \to 0$ for all non-negative integers N.

Definition 2.19. For $f_k, f \in C_c^\infty(M)$ we say that $f_k \to f$ provided that for all coordinate systems $(U_0, \varphi : U_0 \to V_0)$, $f_k \circ \varphi^{-1} \to f \circ \varphi^{-1}$ in $C_c^\infty(V_0)$.

We can then make our definition of distributions on manifolds.

Definition 2.20. A *distribution* on a manifold M is a linear map $u : C_c^\infty(M) \to \mathbb{C}$ such that if $f_k \to f$ in $C_c^\infty(M)$, $u(f_k) \to u(f)$.

Similarly to the case of open subsets of $\mathbb{R}^n$ we can define localization of distributions on manifolds and from there, the support and singular support of distributions.

3 Elliptic Regularity

We would like to understand the smoothness of distributional solutions to PDEs. In particular, given a differential operator P and a partial differential equation of the form $Pu = f$, we would like to understand what we can say about the smoothness of the solutions u of such an equation.

Definition 3.1. Let $P = \sum_{|\alpha| \leq n} c_\alpha D^\alpha$ for some constants $c_\alpha \in \mathbb{C}$. We say that P is elliptic provided its principal symbol $\sigma(P)(\xi) = \sum_{|\alpha| \leq n} c_\alpha \xi^\alpha$ is invertible (ie. non-zero) away from 0.

For constant coefficient differential operators on $\mathbb{R}^n$ we have the following theorem we present here without proof. The proof involves constructing a parametrix to a fundamental solution of the given PDE.

Theorem 3.2. *Let P be a constant coefficient differential operator that is elliptic. Then, for all $U \subseteq \mathbb{R}^n$ and $u \in D'(U)$, sing supp $u \subseteq$ sing supp Pu.*

Now, we would like to define differential operators on manifolds.

Definition 3.3. An operator $P : C^\infty(M) \to C^\infty(M)$ is a differential operator provided

- It is local (ie. supp $Pu \subseteq$ supp u for all $u \in C^\infty(M)$)
- It is a differential operator in charts.

With some technicalities, we can also define a notion of ellipticity for differential operators on manifolds. Doing so, we have the following theorem.

Theorem 3.4. *Let P be an elliptic operator in $\mathrm{Diff}^m(M)$ for some non-negative integer m and manifold m. Then for all $u \in D'(M)$, we have sing supp $u \subseteq$ sing supp Pu.*

These theorems have important applications to the Fredholm theory of elliptic operators which itself has an application in the proof of the Hodge Decomposition Theorem

4 Vector Bundles and Differential Forms

4.1 Vector Bundles

Definition 4.1. We define a vector bundle as a triple (M, E, π) such that

- M is an n-dimensional manifold, E is an $n + m$ dimensional manifold
- $\pi : E \to M$ is surjective and $\pi^{-1}(x)$ has the structure of an m-dimensional vector space for all $x \in M$
- For all $x \in M$ there exists a neighborhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^m$ such that $\phi|_{\pi^{-1}(x)}$ is an isomorphism between $\pi^{-1}(x)$ and $\{x\} \times \mathbb{C}^m$

Example 4.2. The trivial bundle consisting of a manifold M, $E = M \times \mathbb{R}^m$ and $\pi(x, v) = x$.

Example 4.3. Other examples of vetor bundles include the tangent and cotangent bundles TM and T^*M.

Definition 4.4. Given a vector bundle (M, E, π), a smooth map $\sigma : M \to E$ is a smooth section of the bundle provided $\pi \circ \sigma = \text{Id}$. The collection of smooth sections of a vector bundle (M, E, π) is denoted $C^\infty(M, E)$.

Using the local trivializations of our vector bundles, similarly to how we defined differential operators on manifolds, we can define differential operators on sections of vector bundles.

4.2 Differential Forms

Fix a compact, oriented, n-dimensional manifold M.

Definition 4.5. We define the vector bundle of k-forms on M as $\Omega^k = \bigwedge^k(T^*M)$ and the vector bundle of all differential forms on M as $\Omega^\circ = \otimes_{k=0}^n \Omega^k$.

Elements of $C^\infty(M, \Omega^k)$ are called differential k-forms.

Definition 4.6. Now for $0 \leq k \leq n-1$ we define the exterior derivative $d_k : C^\infty(M, \Omega^k) \to C^\infty(M, \Omega^{k+1})$ in local coordinates. In particular,

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) := df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \tag{4.1}$$

for $1 \leq i_1 < \ldots < i_k \leq n$.

It is easy to verify that $d_k \circ d_{k-1} = 0$. This allows us to define the deRham cohomology groups

$$H^k_{dR}(M) = \{\omega \in C^\infty(M, \Omega^k) : d_k\omega = 0\}/\{d_{k-1}\omega : \omega \in C^\infty(M, \omega^{k-1})\} = \ker(d_k)/\text{im}(d_{k-1}). \tag{4.2}$$

We note that this definition does not depend on any particular choice of Riemannian metric on M.

4.3 Inner Product on Form Bundle

Now, suppose we have a Riemannian Metric g on our manifold M. Via the Riesz representation theorem, g induces an inner product on $C^\infty(M, \Omega^1) = T^*M$. Now, for each $x \in M$, and $k > 1$, we can define an inner product on Ω^k via $\langle \bigwedge \alpha_i, \bigwedge \beta_j \rangle = \det([\langle \alpha_i, \beta_j \rangle]_{i,j})$ for 1-forms α, β. We can then extend this an inner product on all of Ω^k by linearity and this inner product ends up being smoothly varying. We can then use this define an inner product on $C^\infty(M, \Omega^k)$, the space of differential differential k-forms via $\langle \omega, \omega' \rangle = \int_M \langle \omega(x), \omega'(x) \rangle dV_g x$ where dV_g is the volume form on M induced by g. Using this, we define $\delta_{k+1} : C^\infty(M, \Omega^{k+1}) \to C^\infty(M, \Omega^k($ to be the adjoint of d_k for all k.

5 Hodge Decomposition Theorem

This section is devoted to proving the Hodge Decomposition Theorem. We first need to make the following definitions.

Definition 5.1. We define the operator $d : C^\infty(M, \Omega^\circ) \to C^\infty(M, \Omega^\circ)$ via $d|_{C^\infty(M,\Omega^k)} = d_k$.

Definition 5.2. We define the *Laplace-Beltrami operator* $\Delta_g : C^\infty(M, \Omega^\circ) \to C^\infty(M, \Omega^\circ)$ by

$$\Delta_g = (d + \delta)^2 = d\delta + \delta d. \tag{5.1}$$

It is immediately clear from this definition that Δ_g is a self-adjoint operator and is elliptic provided $d+\delta$ is elliptic. Additionally, for all k we have that Δ_g restricts to an operator $C^\infty(M, \Omega^k) \to C^\infty(M, \Omega^k)$.

Definition 5.3. We define the space of *harmonic* k-forms as $\mathcal{H}^k = \{\omega \in C^\infty(M, \Omega^k)\}$

Let us now prove the following lemmas.

Lemma 5.4. *For all $0 \leq k \leq n$ we have that*

$$\mathcal{H}^k = \{\omega \in C^\infty(M, \Omega^k) : \Delta_g\omega = 0\} = \{\omega \in C^\infty(M, \Omega^k) : d\omega = \delta\omega = 0\} \tag{5.2}$$

.

Proof. First suppose that $\Delta_g\omega = 0$. Then, we find that

$$0 = \langle \delta_g\omega, \omega\rangle_g = \langle d\delta\omega, \omega\rangle_g + \langle \delta d\omega, \omega\rangle_g = \|\delta\omega\|_g^2 + \|d\omega\|_g^2 \tag{5.3}$$

and so $\delta\omega = d\omega = 0$. The other direction is immediate. □

Lemma 5.5. *As an operator $C^\infty(M, \Omega^0) \to C^\infty(M, \Omega^0)$, $d + \delta$ is elliptic (and hence Δ_g as well).*

As a consequence of

Theorem 5.6 (Hodge Decomposition Theorem)**.** *Let M be a n-dimensional oriented compact Riemannian Manifold. Then*

$$C^\infty(M, \Omega^k) = \mathcal{H}^k \otimes im\ \delta_{k+1} \otimes im\ d_{k-1}. \tag{5.4}$$

Proof. Let us first prove that the sum is direct. Suppose there exists $\alpha \in \mathcal{H}^k, \beta \in C^\infty(M, \Omega^{k+1}), \gamma \in C^\infty(M, \Omega^{k-1})$ such that $\alpha + \delta\beta + d\gamma = 0$. Then applying d to both sides we immediately find that $d\delta\beta = 0$ and hence $0 = \langle d\delta\beta, \beta\rangle_g = \langle \delta\beta, \delta\beta\rangle_g = \|\delta\beta\|_g^2$ so $\delta\beta = 0$. Similarly, applying δ we find that $d\gamma = 0$ and hence $\alpha = 0$ as well. Thus, the sum is indeed direct.

The rest of the proof follows rather simply from the Fredholm property of elliptic differential operators on sections of vector bundles. We omit the details.

□

From this we immediately get the following useful corollaries.

Corollary 5.7. *For a compact Riemannian manifold M,*

$$ker\ d_k = \mathcal{H}^k \otimes im\ d_{k-1} \tag{5.5}$$

and hence

$$H^k_{dR}(M) \cong \mathcal{H}^k \tag{5.6}$$

Proof. Let $\omega \in \ker d_k$. By the Hodge Decomposition Theorem, there exists $\alpha \in \mathcal{H}^k, \beta \in C^\infty(M, \Omega^{k+1})$ and $\gamma \in C^\infty(M, \Omega^{k-1})$ such that

$$\omega = \alpha + \delta\beta + d\gamma \tag{5.7}$$

Then, applying d to both sides we find that

$$0 = d\omega = d\alpha + d\delta\beta + d^2\gamma$$

By 5.4, $d\alpha = 0$ and $d^2\gamma = 0$ so $d\delta\beta = 0$ and hence

$$\langle d\delta\beta, \beta\rangle_g = \langle \delta\beta, \delta\beta\rangle_g = \|\delta\beta\|_g^2 = 0 \tag{5.8}$$

so $\omega = \alpha + d\gamma$, completing the first direction of the proof. Now suppose $\omega = \alpha + d\gamma$ for some $\alpha \in \mathcal{H}^k$ and $\gamma \in C^\infty(M, \Omega^{k-1})$. Then $d\omega = 0$ by 5.4, completing the proof of 5.5, from which 5.6 is immediate. □

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