Bordered Heegaard Floer Homology and the Surgery Exact Triangle

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Introduction

Heegaard Floer homology is a power topological invariant associated with a 3-manifold, but in practice it is difficult to compute because the definition involves counting pseudo-holomorphic curves. Bordered Heegaard Floer homology is a general conceptual framework that describes the Heegaard Floer homology of a 3-manifold $Y = Y_1 \bigcup Y_2$ in terms of invariants associated with bordered 3-manifolds Y_1 and Y_2. We give an application of bordered Heegaard Floer homology to deriving the surgery exact triangle.

Heegaard Diagrams

Definition. A (multi-pointed) **Heegaard diagram** $(\Sigma, \alpha, \beta, w, z)$ consists of the following data:

- A surface $\Sigma \in \mathbb{S}^3$ of genus $g \geq 0$, splitting $\mathbb{S}^3$ into two handlebodies U_0 and U_1, with Σ oriented as the boundary of U_0.
- A collection $\alpha = \{\alpha_1, \dots, \alpha_{g+k-1}\}$ consisting of $g+k-1$ pairwise disjoint, simple closed curves on Σ, such that each α_i bounds a properly embedded disk D_i^α in U_0, and the complement of these disks in U_0 is a union of k balls $B_1^\alpha, \dots, B_k^\alpha$.
- A curve collection $\beta = \{\beta_1, \dots, \beta_{g+k-1}\}$ with similar properties, bounding disks D_i^β in U_1 such that their complement is a union of k balls $B_1^\beta, \dots, B_k^\beta$.
- Two collections of points on Σ, denoted $w = \{w_1, \dots, w_k\}$ and $z = \{z_1, \dots, z_k\}$, all disjoint from each other and from the α and β curves.

One can construct a 3-manifold $Y(\mathcal{H})$ from a Heegaard diagram $\mathcal{H}$. Conversely, for each 3-manifold we can construct (possibly many) a Heegaard diagram from which we can recover the original manifold.

Examples of Heegaard Diagrams

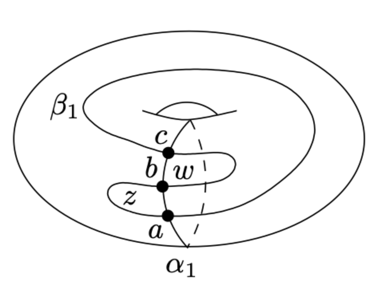

Figure 1. A Heegaard diagram associated to a torus. [3]

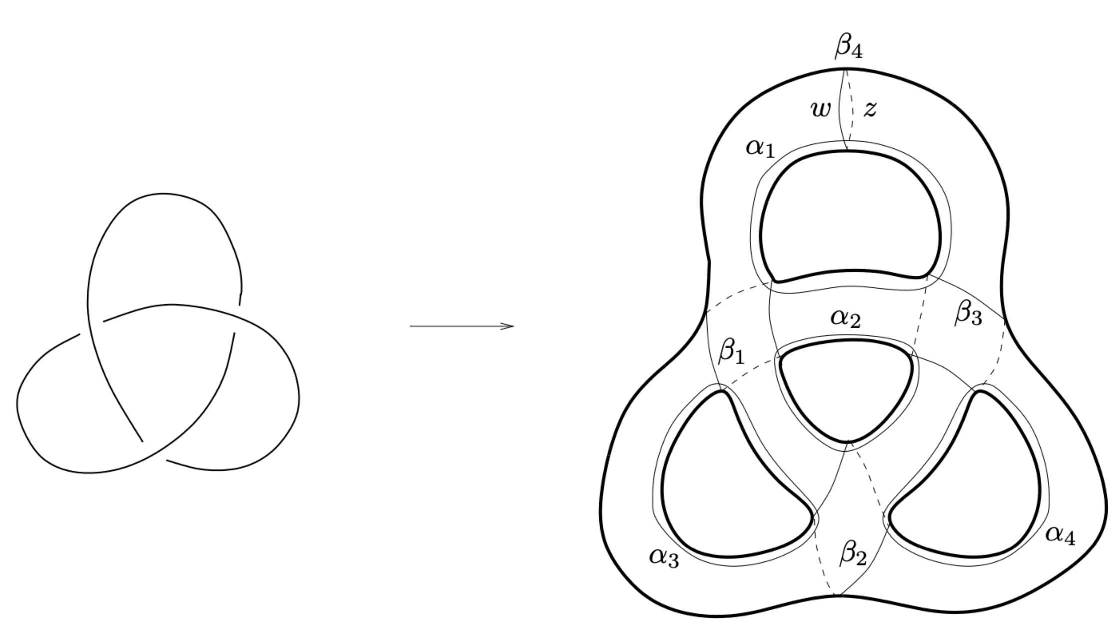

Figure 2. A Heegaard diagram for the boundary of a solid tubular neighbourhood of the planer projection of a knot. [3]

Heegaard Floer Homology

1. **Heegaard Floer homology** (in the simplest version) associates a bigraded chain complex $\widehat{CF}$ to a Heegaard diagram.
2. The chain complex $\widehat{CF}$ admits the **Maslov (homological)** grading and a grading by $Spin^c$ **structures**.
3. The generators of the chain complexes are intersection points of the two tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ generated by the α and β circles in the symmetric product $Sym(\Sigma)$ of Σ.
4. The chain map is given by counting pseucoholomorhic disks satisfying certain conditions.
5. The homology computed depends only on the manifold $Y(\mathcal{H})$, not the specific Heegaard diagram, and is hence a topological invariant.

Bordered Heegaard Diagram

Definition. A **bordered Heegaard diagram** is a quadruple $\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \beta, z)$ consisting of

1. a compact, oriented surface $\overline{\Sigma}$ with one boundary component, of some genus g;
2. a g-tuple of pairwise-disjoint circles $\beta = \{\beta_1, \dots, \beta_g\}$ in the interior of Σ;
3. a $(g+k)$-tuple of pairwise-disjoint curves $\overline{\alpha}$ in $\overline{\Sigma}$, split into $g-k$ circles $\alpha^{\mathbf{c}} = (\alpha_1^c, \dots, \alpha_{g-k}^c)$ in the interior of $\overline{\Sigma}$ and $2k$ arcs $\overline{\alpha^{\mathbf{a}}} = (\overline{\alpha_1^a}, \dots, \overline{\alpha_{2k}^a})$ in $\overline{\Sigma}$ with boundary on $\partial\overline{\Sigma}$ (and transverse to $\partial\overline{\Sigma}$); and
4. a point z in $(\partial\overline{\Sigma}) \setminus (\overline{\alpha} \cap \partial\overline{\Sigma})$,

such that the intersections are transverse and $\overline{\Sigma} \setminus \overline{\alpha}$ and $\overline{\Sigma} \setminus \beta$ are connected.

1. The boundary of each bordered Heegaard diagram is a **matched circle** $\mathcal{Z}$ that specifies a way to form a closed 2-manifold. There is a strand algebra $A(\mathcal{Z})$ which is an $\mathcal{A}_\infty$-algebra associated with the matched circle.
2. For each matched circle $\mathcal{Z}$, we can form a surface $F(\mathcal{Z})$.
3. For each bordered Heegaard diagram $\mathcal{H}$, we can form $Y(\mathcal{H})$ which is a 3-manifold with boundary $F(\mathcal{Z})$.
4. For two bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same boundary $\mathcal{Z}$, intuitively, $A(\mathcal{Z})$ describes how $Y(\mathcal{H}_1)$ and $Y(\mathcal{H}_2)$ are glued together along $F(\mathcal{Z})$.

An Example of a Bordered Heegaard Diagram

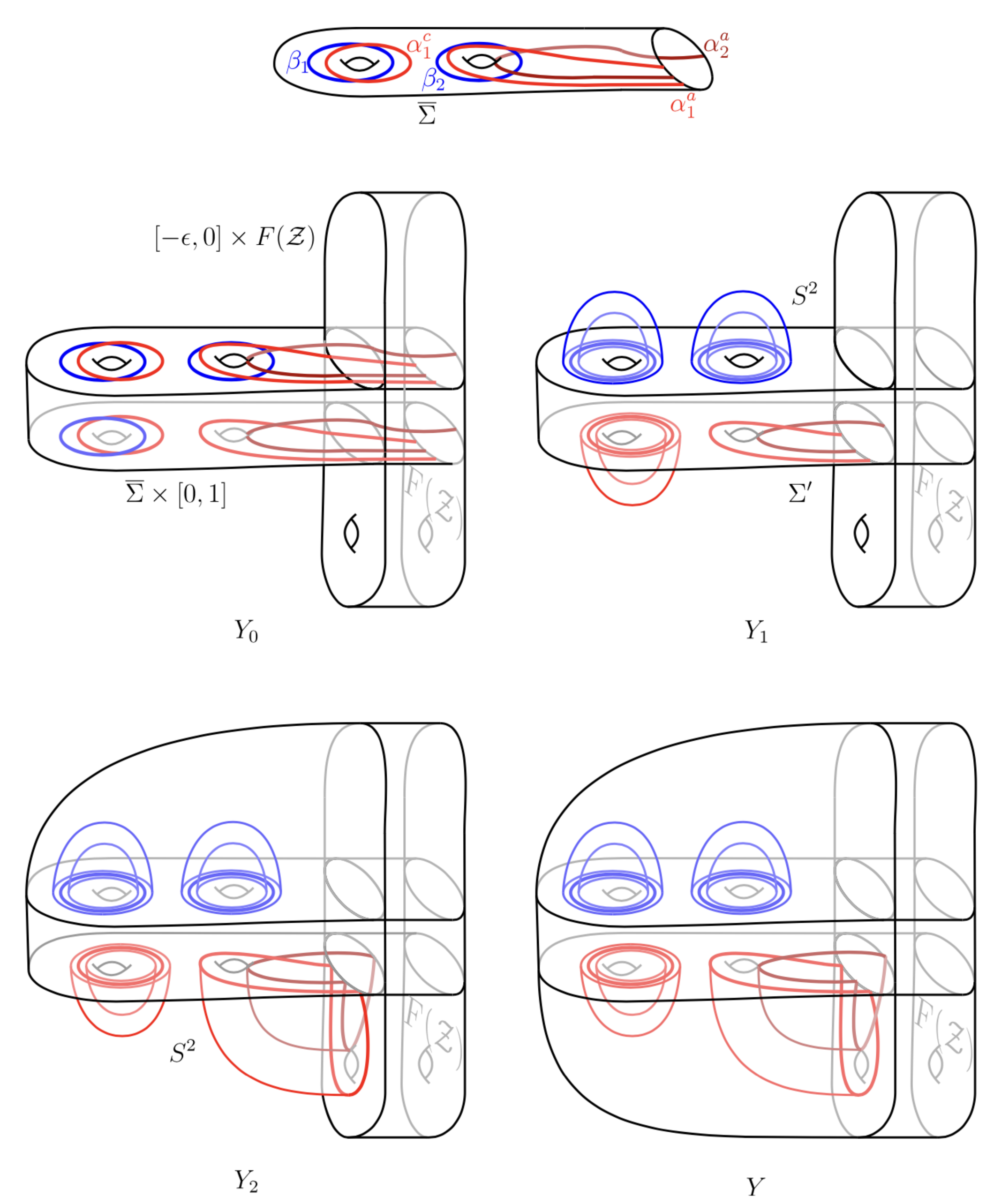

Figure 3. An example showing a bordered Heegaard diagram and a schematic illustration of how to construct $Y(\mathcal{H})$. [2]

Bordered Heegaard Floer Homology

1. **Bordered Heegaard Floer homology** associates to each bordered Heegaard diagrams the $\mathcal{A}_\infty$-algebra $A(\mathcal{Z})$ and two $\mathcal{A}_\infty$-modules $\widehat{CFD}$ and $\widehat{CFA}$.
2. Intuitively, $\mathcal{A}_\infty$-algebras generalize the notion of graded differential algebras, but contain more information of higher degree homotopies. In particular, there is a boundary map associated to an $\mathcal{A}_\infty$ algebra inducing the homology.
3. For two bordered Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ of the same bordered 3-manifold Y, it can be shown that the $\mathcal{A}_\infty$-modules $\widehat{CFD}(\mathcal{H})$ and $\widehat{CFA}(\mathcal{H})$ are pseudo-isomorphic, respecitvely, to $\widehat{CFD}(\mathcal{H}')$ and $\widehat{CFA}(\mathcal{H}')$, i.e. their induces homologies are isomorphic.
4. Therefore, bordered Heegaard Floer homology associates topological invariants for bordered manifolds up to pseudo-isomorphisms.

The Pairing Theorem

The following **pairing theorem** specifies how the Heegaard Floer homology of a 3-manifold can be described in terms of the bordered Floer homology of its components.

Theorem. Let Y_1 and Y_2 be two 3-manifolds with parameterized boundary $\partial Y_1 = F = -\partial Y_2$, where F is specified by the pointed matched circle $\mathcal{Z}$. Fix corresponding bordered Heegaard diagrams for Y_1 and Y_2. Let Y be the closed 3-manifold obtained by gluing Y_1 and Y_2 along F. Then $\widehat{CF}(Y)$ is homotopy equivalent to the A_∞ tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$. In particular,

$$\widehat{HF}(Y) \cong H_*(\widehat{CFA}(Y_1) \hat{\oplus}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y_2)).$$

The Surgery Exact Triangle

The pairing theorem immediately implies the following **surgery exact triangle** as a corollary.

Corollary. There is a long exact sequence relating the manifolds M_∞, M_{-1}, and M_0, which corrspond to the results of applying ∞, -1, and 0 surgeries to the 3-manifold M, respectively.

$$\cdots \widehat{HF}_n(M_\infty) \to \widehat{HF}_n(M_{-1}) \to \widehat{HF}_n(M_0) \to \widehat{HF}_{n-1}(M_\infty) \cdots$$

The proof relies on the pairing theorem and the following proposition [2].

Proposition. If $0 \to N_1 \xrightarrow{\phi^1} N_2 \xrightarrow{\psi^1 P} N_3 \to 0$ is a short exact sequence of type D over a dg-algebra and $\mathcal{M}$ is a bounded $\mathcal{A}_\infty$-module, then there is an exact sequence in homology

$$H_*(\mathcal{M} \boxtimes N_1) \to H_*(\mathcal{M} \boxtimes N_2) \to H_*(\mathcal{M} \boxtimes N_3) \to H_*(\mathcal{M} \boxtimes N_1)[-1] \to 0$$

.

We give a quick sketch of proof:

1. Construct a short exact sequence of $\mathcal{A}_\infty$-algebras associated to bordered Heegaard diagrams that correspond to performing each surgery.
2. We Recover the homology long exact sequence using the pairing theorem and the above proposition.

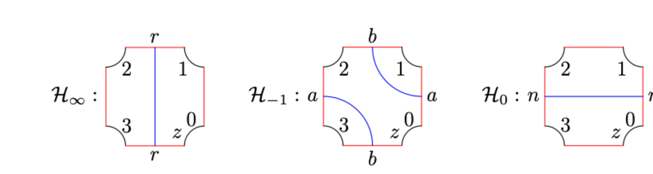

Figure 4. The bordered Heegaard Diagrams in step 2, with generators for chain complexes. [2]

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