MINIMAL SURFACES AND BERNSTEIN'S THEOREM

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ABSTRACT. Given an entire graphical minimal hypersurface in $\mathbb{R}^{n+1}$, is it a hyperplane? This is Bernstein's problem, named after Sergey Bernstein, who proved it in the affirmative for the $n = 2$ case. This problem has since invited numerous developments in differential geometry and geometric measure theory, and after work of Fleming; De Giorgi; Almgren; Simons; and Bombieri, De Giorgi, and Giusti, the problem has since been resolved in the affirmative for $n \leq 7$ and negative for $n \geq 8$. However, the only known examples of non-flat entire minimal graphs for $n \geq 8$ are not given by polynomials, and it is still an open question whether or not there exist polynomial solutions to the minimal surface equation.

This is a report of progress made on finding polynomial solutions to the minimal surface equation during SURIM 2024.

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Date: Summer 2024.

Acknowledgements

I would like to thank Otis Chodosh, my advisor, for suggesting this topic and for patiently guiding me through the process. I would also like to thank Ben Foster, my DRP mentor, for teaching me the basics of minimal surface theory. Finally, I would like to thank Lernik Asserian for organizing SURIM 2024 and making this work possible.

1. Bernstein's problem

We begin by explaining Bernstein's problem. Let $\Omega \subseteq \mathbb{R}^n$ be an open set with C^2 smooth boundary, and let $u \in C^2(\Omega)$ be a C^2 smooth function. Then, denote by

$$\Gamma_u = \{(x, u(x)) \in \Omega \times \mathbb{R} \colon x \in \Omega\}$$

the *graph* of u. Note that Γ_u is a smooth hypersurface in $\mathbb{R}^{n+1}$ with boundary $\partial\Gamma_u$. We are interested in the case when Γ_u is area-minimizing. That is, when is it true that for all hypersurfaces $\Sigma \subseteq \mathbb{R}^{n+1}$ with boundary $\partial\Sigma = \partial\Gamma_u$, we have that $\text{Area}(\Gamma_u) \leq \text{Area}(\Sigma)$? Put another way, we wish to consider global minima of the functional

$$\text{Area} : \left\{\text{surfaces in } \mathbb{R}^{n+1} \text{ with boundary } \partial\Gamma_u\right\} \to \mathbb{R}.$$

From single-variable calculus, our first instinct is to consider critical points of this functional. But, unlike single-variable calculus, the set of surfaces with prescribed boundary has no obvious smooth or additive structure, so we must be a little more strategic.

1.1. Area of a graph and the minimal surface equation.

Recall that for any Riemannian manifold (M, g) of dimension n, there is a unique volume form ω_g given in oriented coordinates $(x^1, \dots, x^n)$ by

$$\omega_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n.$$

Now, let $u \in C^2(\Omega)$ as above, and denote the inclusion map $\iota \colon \Omega \to \mathbb{R}^{n+1}$ with $\iota(x) = (x, u(x))$. Let $(x^1, \dots, x^n)$ be the standard coordinates on $\Omega \subseteq \mathbb{R}^n$, and note that the area of Γ_u is given by

$$\text{Area}(\Gamma_u) = \int_\Omega \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n,$$

where g is the pullback metric $g = \iota^*\overline{g}$. Note that when dealing with subsets of Euclidean space, we will often drop the volume form in notation. We claim that we can write this formula in terms of u and its derivatives:

Proposition 1.1 (Area of a graph)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and $u \in C^2(\Omega)$. Then,*

$$\text{Area}(\Gamma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Proof. Let us denote $u_i = \partial_i u$. Note that $d\iota_x \colon T_x\Omega \to T_{\iota(x)}\mathbb{R}^{n+1}$ is given by the matrix

$$d\iota_x = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ u_1 & \cdots & u_n \end{bmatrix},$$

and hence the metric tensor is given by

$$g_{ij} = \delta_{ij} + u_i u_j.$$

Note that the eigenvalues of this matrix are $1 + |\nabla u|^2$ and 1, with eigenspaces span $\{\nabla u\}$ and span $\{\nabla u\}^\perp$, respectively. Hence, the determinant is $\det g_{ij} = 1 + |\nabla u|^2$, so the area is given by

$$\mathrm{Area}(\Gamma_u) = \int_\Omega \sqrt{\det g_{ij}} = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

□

Suppose $\eta \in C_c^\infty(\Omega)$ is a smooth function on Ω with compact support, so that $\eta|_{\partial\Omega} \equiv 0$. Then, $\Gamma_{u+t\eta}$ for $t \in \mathbb{R}$ has boundary $\partial\Gamma_{u+t\eta} = \partial\Gamma_u$, so $\Gamma_{u+t\eta}$ is a path in the space of hypersurfaces with boundary $\partial\Gamma_u$. So, instead of putting a smooth structure on the space of all hypersurfaces with fixed boundary, we can ask for Γ_u to be a critical point for the area functional with respect to all such compactly supported smooth variations:

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and $u \in C^2(\Omega)$. Then, Γ_u is a *critical point for the area functional*[1] if for all $\eta \in C_c^\infty(\Omega)$ with $\eta|_{\partial\Omega} = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(\Gamma_{u+t\eta}) = 0.$$

This condition yields the following equation for graphical hypersurfaces, called the *minimal surface equation* (or the MSE for short). This is the Euler-Lagrange equation for the area functional.

Proposition 1.3 (Minimal surface equation)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and $u \in C^2(\Omega)$. Then,*

$$\Gamma_u \textit{ is a critical point for the area functional} \iff \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$. Computing,

$$\begin{aligned} \left.\frac{d}{dt}\mathrm{Area}(\Gamma_{u+t\eta})\right|_{t=0} &= \int_\Omega \left.\frac{d}{dt}\sqrt{1 + |\nabla u + t\nabla\eta|^2}\right|_{t=0} \\ &= \int_\Omega \left.\frac{1}{2}\left(1 + |\nabla u + t\nabla\eta|^2\right)^{-1/2} 2\langle \nabla u + t\nabla\eta, \nabla\eta\rangle\right|_{t=0} \\ &= \int_\Omega \frac{\langle \nabla u, \nabla\eta\rangle}{\sqrt{1 + |\nabla u|^2}} \end{aligned}$$

The first equality comes from η being compactly supported, so that we can exchange the derivative with the integral; the second equality comes from taking the chain rule and the

[1]We will give a more general definition later, and show that that definition reduces to this one for graphs.

identity $(|x(t)|^2)' = 2\langle x(t), x'(t)\rangle$. Let $\nu\colon \Omega \to N\Gamma_u$ be the upwards unit normal map, given by

$$\nu(x) = \frac{(-\partial_1 u, \ldots, -\partial_n u, 1)}{\sqrt{1+|\nabla u|^2}}.$$

Then, integrating by parts,

$$\begin{aligned} \frac{d}{dt}\operatorname{Area}(\Gamma_{u+t\eta})\bigg|_{t=0} &= \int_\Omega \frac{\langle \nabla u, \nabla \eta\rangle}{\sqrt{1+|\nabla u|^2}} \\ &= \int_{\partial\Omega} \eta \frac{\langle \nu, \nabla u\rangle}{\sqrt{1+|\nabla u|^2}} - \int_\Omega \eta \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \\ &= -\int_\Omega \eta \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \end{aligned}$$

because $\eta|_{\partial\Omega} = 0$. Because our choice of η was arbitrary, this formula entails our result. □

The above proposition justifies the following definition:

Definition 1.4. A graphical hypersurface Γ_u is *minimal* if one of the following equivalent conditions holds:

(1) Γ_u is a critical point for the area functional,

(2) u satisfies the minimal surface equation.

Moreover, u is an *entire* solution to the MSE if $\Omega = \mathbb{R}^n$, in which case we say that Γ_u is an *entire minimal graph* or *entire graphical minimal hypersurface.*

Finally, we say that u is *affine* if it is a degree 1 polynomial, in which case Γ_u is a hyperplane. Then, we can state Bernstein's problem, as follows:

Question 1.5 (Bernstein's Problem)**.** *Let $u \in C^2(\mathbb{R}^n)$ be an entire solution to the minimal surface equation. Then, is u affine?*

Equivalently, is every entire graphical minimal hypersurface a hyperplane?

This paper will contain a series of results and technical tools on the road to Bernstein's theorem, along with some notes on polynomial solutions to the minimal surface equation. This report is a work-in-progress, though we will provide an outline below for the current version.

We begin by reviewing some preliminaries from Riemannian geometry, then we will move on to the (smooth) theory of minimal submanifolds and minimal hypersurfaces in particular. We collect some formulas and estimates useful for controlling curvature, then we will move on to discuss polynomial solutions to the minimal surface equation.

2. Preliminaries: Riemannian geometry

In this section, we state our notation and conventions for Riemannian geometry and measure theory, along with important theorems we will rely on later.

2.1. **Riemannian manifolds.**

All of the material in this section is entirely standard, see e.g. [LeeRM] for details.

Definition 2.1. A C^k *smooth manifold* is a Hausdorff second countable topological space that is locally Euclidean of dimension n, with C^k transition functions.

We will most often refer to C^∞ smooth manifolds, but the generality will be useful later.

Recall that $T^{(k,l)}M$ is the bundle of (k,l)-tensors on M, and $\Gamma(T^{(k,l)}M)$ is the space of smooth sections of this bundle; i.e., smooth (k,l)-tensor fields. We denote by $\mathfrak{X}(M) = \Gamma(T^{(1,0)}M)$ the space of smooth vector fields. More generally, for a smooth vector bundle $\pi\colon E \to M$, we denote by $\Gamma(E)$ the space of smooth sections of E.

Definition 2.2. A *Riemannian metric* is a symmetric $(0,2)$-tensor that is positive definite: at each point $p \in M$ and $v, w \in T_pM$,

(1) $g_p(v,w) = g_p(w,v)$,

(2) $g_p(v,v) \geq 0$, with equality if and only if $v = 0$.

Definition 2.3. A *Riemannian manifold* is a pair (M,g) where M is a smooth manifold and g is a metric.

On such a manifold there are usually many notions of differentiation of sections, captured by the notion of a connection.

Definition 2.4. Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M. A *connection*, or a *covariant derivative*, is a map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying the following properties:

(1) $\nabla_X Y$ is $C^\infty(M)$-linear in X,

(2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y, and

(3) $\nabla_X Y$ satisfies the following product rule: for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $Y \in \Gamma(E)$,
$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

In the special case where we are considering the (k,l)-tensor bundle $T^{(k,l)}M \to M$, we can use a connection to define the total covariant derivative:

Definition 2.5. Let M be a smooth manifold and let ∇ be a connection on TM. The *total covariant derivative* $\nabla\colon \Gamma(T^{(k,l)}M) \to \Gamma(T^{(k,l+1)}M)$ is given by
$$\nabla F(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X) = (\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l).$$

In the even more special case of a Riemannian manifold (M, g), there is a unique choice of connection that is compatible with both the metric structure and the smooth structure.

Regarding compatibility with the metric structure, note that since $g \in \Gamma(T^{(0,2)}M)$ is a $(0,2)$-tensor, we can take its total covariant derivative ∇g. Recall that a tensor field $F \in \Gamma(T^{(k,l)}M)$ is called *parallel* if $\nabla F = 0$. A connection ∇ on a Riemannian manifold is then called a *metric connection*, or *compatible with the metric*, if g is parallel with respect to ∇: $\nabla g = 0$. Equivalently, the connection satisfies the following product rule over the metric:

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle .$$

The second condition is a bit more mysterious, though it can be seen as a compatibility with the smooth structure. Recall that the *Lie bracket* of two vector fields $X, Y \in \mathfrak{X}(M)$ is given by

$$[X, Y]f = X(Yf) - Y(Xf).$$

A connection is called *torsion-free* or *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

It is a wonderful fact that these two conditions are enough to specify a unique connection.

Theorem 2.6. *Let (M, g) be a Riemannian manifold. Then, there exists a unique connection ∇, called the* Levi-Civita connnection*, that is both compatible with the metric and torsion-free.*

Going forward, if (M, g) is a Riemannian manifold, we denote by ∇ the Levi-Civita connection. The connection allows us to define various notions of curvature, as a failure for second derivatives to commute.

Definition 2.7. Let (M, g) be a Riemannian manifold. The $(1,3)$-*curvature tensor* $R \in \Gamma(T^{(1,3)}M)$ is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

The $(0,4)$-*curvature tensor* (or *Riemann curvature tensor*) Rm, is given by lowering an index: $\mathrm{Rm} = R^\flat$, or

$$\mathrm{Rm}(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle .$$

The *Ricci curvature tensor* $\mathrm{Ric} \in \Gamma(T^{(0,2)}M)$ is given by taking the trace of the Riemann curvature tensor on the first and last index:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\left(Z \mapsto R(Z, X)Y\right).$$

2.2. **Riemannian submanifolds.**

Often, we would like to understand the induced Riemannian structure of a submanifold of a Riemannian manifold.

Definition 2.8. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds. We say that M is a *Riemannian submanifold* if $\iota \colon M \hookrightarrow \widetilde{M}$ is an embedding (such that M is an embedded submanifold of $\widetilde{M}$) and $\iota^* \widetilde{g} = g$: if $v, w \in T_pM$,

$$g_p(v, w) = \widetilde{g}_{\iota(p)}(d\iota_p(v), d\iota_p(w)).$$

If $\iota\colon M \looparrowright \widetilde{M}$ is merely an immersion, then we say that ι is a *Riemannian (or isometric) immersion*, and call M an *immersed Riemannian submanifold.*

Let $M \hookrightarrow \widetilde{M}$ be a Riemannian submanifold. Then, at each point we have $T_pM \subseteq T_p\widetilde{M}$, and given the metric structure we can define the *normal space at* p: $N_pM = T_pM^{\top}$. We can then define the *normal bundle* $NM \to M$ on M by taking $NM = \bigsqcup_{p\in M} N_pM$; it is straightforward to show that NM is a smooth manifold of dimension $\dim M + \operatorname{codim} M = \dim \widetilde{M}$. Given any smooth vector field $X \in \mathfrak{X}(M)$ (or more generally $\Gamma(T\widetilde{M}|_M)$), we can thus decompose it into its *normal* and *tangential* parts:

$$X = X^{\perp} + X^{\top},$$

where $X^{\perp} \in \Gamma(NM)$ and $X^{\top} \in \Gamma(TM)$. This gives us a splitting $T\widetilde{M}|_M = NM \oplus TM$ with

$$\pi^{\perp}(X) = X^{\perp} \in \Gamma(NM) \qquad \text{and} \qquad \pi^{\top}(X) = X^{\top} \in \Gamma(TM) = \mathfrak{X}(M).$$

We can thus decompose the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$ into its normal and tangential parts, for vector fields in $\mathfrak{X}(M)$.

Definition 2.9. The *(vector) second fundamental form* $\vec{\mathrm{II}} \in \Gamma(T^{(1,2)}M)$ is the map $\vec{\mathrm{II}}\colon \mathfrak{X}(M)\times \mathfrak{X}(M) \to \Gamma(NM)$ given by

$$\vec{\mathrm{II}}(X,Y) = (\nabla_X Y)^{\perp}.$$

Magically, the tangential part turns out to simply be equal to the Levi-Civita connection ∇ on the submanifold (M,g) with the induced metric $g = \iota^*\widetilde{g}$, by the *Gauss formula*:

Proposition 2.10 (Gauss formula)**.** *Let $(M,g) \hookrightarrow (\widetilde{M},\widetilde{g})$ be a Riemannian submanifold, with Levi-Civita connections ∇ and $\widetilde{\nabla}$, respectively. Then, for any $X,Y \in \mathfrak{X}(M)$,*

$$\widetilde{\nabla_X Y} = \nabla_X Y + \vec{\mathrm{II}}(X,Y).$$

We now recall several results and definitions regarding Riemannian submanifolds; for more detail see Lee's *Introduction to Riemannian Manifolds*, Chapter 8.

Proposition 2.11 (Gauss Equation)**.** *Let $\iota\colon (M,g) \looparrowright (\widetilde{M},\widetilde{g})$ be an isometric immersion, and let $X,Y,Z,W \in \mathfrak{X}(M)$. Then,*

$$\widetilde{\mathrm{Rm}}(W,X,Y,Z) = \mathrm{Rm}(W,X,Y,Z) - \left\langle \vec{\mathrm{II}}(W,Z)\vec{\mathrm{II}}(X,Y)\right\rangle + \left\langle \vec{\mathrm{II}}(W,Y),\vec{\mathrm{II}}(X,Z)\right\rangle.$$

Definition 2.12. Let $\iota\colon (M^k,g) \looparrowright (\widetilde{M}^n,\widetilde{g})$ be an isometric immersion. Then, the (vector) *mean curvature*, denoted $\vec{H} \in \Gamma(NM)$, is defined as

$$\vec{H} = \frac{1}{k}\operatorname{tr}_g \vec{\mathrm{II}}.$$

If $(E_i)_{i=1}^k$ is an orthonormal frame for M, then

$$H = \frac{1}{k}\sum_{i=1}^{k} \vec{\mathrm{II}}(E_i,E_i).$$

Definition 2.13. Let $\iota\colon (M^k, g) \looparrowright (\widetilde{M}^n, g)$ be an isometric immersion. Then, the *norm squared of the second fundamental form*, denoted $|\vec{\mathrm{I\!I}}|^2\colon M \to \mathbb{R}$, is defined as

$$|\vec{\mathrm{I\!I}}|^2 = \sum_{i=1}^{k} \left|\vec{\mathrm{I\!I}}(E_i, E_i)\right|^2$$

for an orthonormal frame $(E_i)_{i=1}^k$.

Suppose $N \in \Gamma(NM)$ is a normal vector field. Then, by contracting with the metric we get a 1-form $\langle -, N\rangle : \Gamma(NM) \to C^\infty(M)$. (Note that this map is defined on all of $\Gamma(T\widetilde{M}|_M)$, but the tangential components do not matter.) Hence, we can post-compose this with $\vec{\mathrm{I\!I}}$ to get a scalar-valued second fundamental form in the direction of N:

Definition 2.14. Let $\iota\colon (M^k, g) \looparrowright (\widetilde{M}^n, \widetilde{g})$ be an isometric immersion, and let $N \in \Gamma(NM)$ be a normal vector field. The *scalar second fundamental form in the direction of* N, denoted $\mathrm{I\!I}_N\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is defined by

$$\mathrm{I\!I}_N(X, Y) = \left\langle \vec{\mathrm{I\!I}}(X, Y), N \right\rangle.$$

The *Weingarten map in the direction of* N, denoted $W_N\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the self-adjoint map corresponding to this bilinear form, defined by

$$\langle W_N(X), Y\rangle = \mathrm{I\!I}_N(X, Y).$$

Remark 2.15. The above definitions amount to the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{X}(M) \times \mathfrak{X}(M) & \xrightarrow{\ \vec{\mathrm{I\!I}}\ } & \Gamma(NM) \\ {\scriptstyle W_N \times \mathrm{id}}\downarrow & \searrow^{\mathrm{I\!I}_N} & \downarrow{\scriptstyle \langle -, N\rangle} \\ \mathfrak{X}(M) \times \mathfrak{X}(M) & \xrightarrow[\langle -,-\rangle]{} & C^\infty(M) \end{array}$$

Now let us suppose that $\iota\colon (\Sigma^n, g) \looparrowright (M^{n+1}, \widetilde{g})$ is an isometric immersion. Moreover, let us take Σ to be *two-sided*:

Definition 2.16. Let $\iota\colon \Sigma^n \looparrowright (M^{n+1}, \widetilde{g})$ be an isometric immersion. Then, Σ is *two-sided* if $N\Sigma$ is trivial. Equivalently, Σ admits a global smooth unit normal $\nu \in \Gamma(N\Sigma)$.

In the codimension 1 two-sided case, $N\Sigma \to \Sigma$ is a rank 1 trivial bundle, so $N\Sigma \cong \Sigma \times \mathbb{R}$. Then, for a choice of unit normal $\nu \in \Gamma(N\Sigma)$ (i.e., a choice of basis for $N_x\Sigma$ at each point $x \in \Sigma$), we get an isomorphism between their spaces of sections $\langle -, \nu\rangle : \Gamma(N\Sigma) \to C^\infty(\Sigma)$. Thus, we can define canonical (up to sign) *scalar* versions of $\vec{\mathrm{I\!I}}$ and $\vec{H}$:

Definition 2.17 (Scalar second fundamental form and mean curvature)**.** Let $\iota\colon \Sigma^n \looparrowright (M^{n+1}, \widetilde{g})$ be an isometric two-sided immersion. Then, the *scalar second fundamental form*, denoted $\mathrm{I\!I}\colon \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$, is defined by

$$\mathrm{I\!I}(X, Y) = \left\langle \vec{\mathrm{I\!I}}(X, Y), \nu \right\rangle.$$

The *scalar mean curvature* is given by

$$H = \left\langle \vec{H}, \nu \right\rangle = \frac{1}{n} \left\langle \operatorname{tr}_g \vec{\mathrm{II}}, \nu \right\rangle .$$

Definition 2.18 (Shape operator of a hypersurface)**.** Let $\iota\colon \Sigma^n \looparrowright (M^{n+1}, g)$ be an isometric two-sided immersion. The *shape operator*, denoted $s\colon \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, is the Weingarten map in the direction of ν:

$$\langle sX, Y\rangle = \left\langle \vec{\mathrm{II}}(X,Y), \nu \right\rangle .$$

Definition 2.19 (Principal curvatures)**.** Let $\iota\colon \Sigma^n \looparrowright (M^{n+1}, g)$ be an isometric two-sided immersion. At a point $x \in \Sigma$, the shape operator $s_x\colon T_x\Sigma \to T_x\Sigma$ is a linear endomorphism. The *principal curvatures at x*, denoted $\kappa_1, \dots, \kappa_n$, are the eigenvalues of s_x.

Proposition 2.20. *The scalar mean curvature is given by*

$$H = \frac{1}{n} \sum_{i=1}^{n} \kappa_i .$$

Proof. This is immediate from the definition; pick an orthonormal frame (E_i) that is an eigenbasis at x. □

3. Minimal submanifolds

3.1. **First variation formula.**

Among the Riemannian submanifolds of a given ambient manifold $(\widetilde{M}, \widetilde{g})$, there are special classes of submanifolds, called *minimal submanifolds*, which are critical points for the area functional. In Section 1 we proved that in the graphical case (i.e., $\Sigma = \Gamma_u$), minimality can be expressed either by being a critical point of the area functional or by u solving the minimal surface equation. In this section, we will derive a similar result to define minimal submanifolds in general.

In this section, $\Sigma = (\Sigma^k, g)$ and $M = (M^n, \widetilde{g})$ are Riemannian manifolds, with Levi-Civita connections ∇ and $\widetilde{\nabla}$, respectively.

Definition 3.1. Let $F_t\colon \Sigma \looparrowright M$ be a one-parameter family of isometric immersions, for $t \in (-\varepsilon, \varepsilon)$. We say that F_t is a *variation.* Furthermore, we say that F_t is:

(1) *compactly supported* if $F_t \equiv \mathrm{id}$ outside of some compact subset of Σ,

(2) *normal* if its variation field $\partial_t F_t \in \Gamma(N\Sigma_t)$, where $\Sigma_t \coloneqq F_t(\Sigma) \subseteq M$.

Theorem 3.2 (First variation formula)**.** *Let $F_t\colon \Sigma^k \to (M^n, g)$ be a smooth compactly supported normal variation such that $N = \partial_t F_t|_{t=0}$. Then,*

$$\left.\frac{d}{dt} \operatorname{Area}(F_t(\Sigma))\right|_{t=0} = -\int_\Sigma \left\langle \vec{H}, N \right\rangle .$$

As an immediate corollary, we get two equivalent definitions for a minimal submanifold:

Definition 3.3. Let $\iota\colon (\Sigma^k, g) \looparrowright (M^n, \widetilde{g})$ be an isometric immersion. Then, Σ is a *minimal submanifold* if either of the following conditions holds:

(1) (critical point of area functional) For every compactly supported normal variation F_t, $\frac{d}{dt}\operatorname{Area}(F_t(\Sigma))\big|_{t=0} = 0$,

(2) (mean curvature) The mean curvature vector $\vec{H} \equiv 0$ vanishes identically.

Note that in the codimension 1 two-sided case, any variation's variation field is a multiple of the chosen unit normal at $t = 0$, and hence considering compactly supported normal variations is equivalent to considering compactly supported smooth functions, so our strategy in Section 1 is equivalent to the case here. Indeed, we can come to the minimal surface equation by computing the mean curvature of a graph explicitly; see Section 3.2 for details.

3.2. **Mean curvature of graphical hypersurfaces.**

In this section, we collect a number of facts about graphical minimal hypersurfaces.

In this section, (Σ^n, g) and $(M^{n+1}, \widetilde{g})$ are Riemannian manifolds, with Levi-Civita connections ∇ and $\widetilde{\nabla}$, respectively. We let $\iota\colon \Sigma \to M$ be a two-sided immersion with unit normal ν.

Lemma 3.4. *Let $U \subseteq M$ be an open set, and let $F\colon U \to \mathbb{R}$ be a locally defining function for Σ: 0 is a regular value for F and $F^{-1}(0) = U \cap \Sigma$. Let $N = |\nabla F|^{-1}\nabla F$ be the normalized gradient. Then,*

(1) *The scalar second fundamental form is*
$$\mathrm{II}(X,Y) = -\frac{\widetilde{\nabla}^2 F(X,Y)}{|\nabla F|},$$
where $\widetilde{\nabla}^2 F(X,Y) = \widetilde{\nabla}(\widetilde{\nabla} F)(X,Y) = \widetilde{\nabla}_X \widetilde{\nabla}_Y F - \widetilde{\nabla}_{\widetilde{\nabla}_X Y} F$ is the covariant Hessian.

(2) *The scalar mean curvature of Σ is given by*
$$H = -\frac{1}{n}\operatorname{div}_{\widetilde{g}}(N) = -\frac{1}{n}\operatorname{div}_{\widetilde{g}}\left(\frac{\nabla F}{|\nabla F|}\right),$$
where $\operatorname{div}_{\widetilde{g}}(X) = \operatorname{tr}(\widetilde{\nabla} X)$, considering $\widetilde{\nabla} X \in \Gamma(T^{(1,1)}\widetilde{M}) = \operatorname{End}(T\widetilde{M})$ by $\widetilde{\nabla} X\colon Y \mapsto \widetilde{\nabla}_Y X$.

Now, suppose $\Sigma = \Gamma_u$ is a graphical hypersurface, for $u \in C^2(\Omega \to \mathbb{R})$ for $\Omega \subseteq \mathbb{R}^n$ open. Let us write $(x,y) \in \Omega \times \mathbb{R}$. Then, $F(x,y) = y - u(x)$ is a (globally) defining function for Γ_u, since $\Gamma_u = F^{-1}(0)$. Since
$$dF = \frac{\partial F}{\partial y}\,dy - \frac{\partial F}{\partial x^i}\,dx^i = dy - \frac{\partial F}{\partial x^i}\,dx^i,$$
we can see that dF vanishes nowhere and hence 0 is a regular value. Then, note by $\nabla F = (-\nabla u, 1)$, so the mean curvature of Γ_u is given by
$$H = -\frac{1}{n}\operatorname{div}\left(\frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}\right)$$

$$= -\frac{1}{n}\left(\frac{\partial}{\partial y}\frac{1}{\sqrt{1+|\nabla u|^2}} + \sum_{i=1}^{n}\frac{\partial}{\partial x^i}\left(\frac{-u_i}{\sqrt{1+|\nabla u|^2}}\right)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)$$
$$= \frac{1}{n}\operatorname{div}_{\mathbb{R}^n}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Therefore, we recover the minimal surface equation:

$$H = \frac{1}{n}\operatorname{div}_{\mathbb{R}^n}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

4. Curvature estimates

4.1. **Bochner formula.** We recall the Euclidean Bochner formula:

Proposition 4.1. *Let $u \in C^3(\mathbb{R}^n)$. Then,*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle\nabla u, \nabla\Delta u\rangle.$$

Proof. The proof is a straightforward computation:

$$\frac{1}{2}\Delta|\nabla u|^2 = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\left(\frac{\partial u}{\partial x^j}\right)^2$$
$$= \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x^i}\left(2\frac{\partial u}{\partial x^j}\frac{\partial^2 u}{\partial x^i\partial x^j}\right)$$
$$= \sum_{i,j}\left(\frac{\partial^2 u}{\partial x^i\partial x^j}\right)^2 + \left(\frac{\partial u}{\partial x^j}\frac{\partial^3 u}{\partial x^i\partial x^i\partial x^j}\right)$$
$$= \sum_{i,j}\left(\frac{\partial^2 u}{\partial x^i\partial x^j}\right)^2 + \left(\frac{\partial u}{\partial x^j}\frac{\partial^3 u}{\partial x^j\partial x^i\partial x^i}\right)$$
$$= |\mathrm{Hess}_u|^2 + \langle\nabla u, \nabla\Delta u\rangle.$$

□

Note that the final step relies on the fact that mixed partials commute, but otherwise all of these computations are purely local. Hence, we can expect for there to be a Bochner formula for an arbitrary Riemannian manifold.

Proposition 4.2 (Bochner formula)**.** *Let (M, g) be a Riemannian manifold, and let $u \in C^3(M)$. Then,*

$$\frac{1}{2}\Delta |\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

Before we prove this, we recall a couple facts.

Definition 4.3. Let $\omega \in \Gamma(T^{(k,l)}M)$ be a (k, l)-tensor field, and $X, Y \in \mathfrak{X}(M)$. Then, the *second covariant derivative* with respect to X and Y is

$$\nabla^2_{X,Y}\omega = (\nabla^2\omega)(\ldots, Y, X).$$

If $k = l = 0$ and $u \in C^2(M)$, the second covariant derivative is called the *(covariant) Hessian*:

$$\mathrm{Hess}_u(X, Y) = \nabla^2_{X,Y} u = (\nabla^2 u)(Y, X).$$

Definition 4.4. Let $\omega \in \Gamma(T^{(k,l)}M)$ be a (k, l)-tensor field. Then, the *Laplacian operator* $\Delta\colon \Gamma(T^{(k,l)}M) \to \Gamma(T^{(k,l)}M)$ is defined by

$$\Delta\omega = \mathrm{tr}_g\, \nabla^2\omega,$$

where the trace is taken with respect to the last two entries. Note that the coefficients of ω need only be C^2 regular.

Lemma 4.5. *Let $u \in C^2(M)$.*

(1) *The second covariant derivative of any tensor field $F \in \Gamma(T^{(k,l)}M)$ can be computed as*

$$\nabla^2_{X,Y} F = \nabla_X \nabla_Y F - \nabla_{\nabla_Y X} F.$$

(2) *The Hessian can be computed as*

$$\mathrm{Hess}_u(X, Y) = Y(Xu) - (\nabla_Y X)u = \langle \nabla_Y \nabla u, X\rangle\,.$$

(3) *The Hessian is symmetric:* $\mathrm{Hess}_u(X, Y) = \mathrm{Hess}_u(Y, X)$.

(4) *If $u \in C^3(M)$, the Laplacian and gradient satisfy the following commutation relation:*

$$\langle [\Delta, \nabla]u, -\rangle = \langle \Delta\nabla u - \nabla\Delta u, -\rangle = \mathrm{Ric}(\nabla u, -).$$

(5) *If $u \in C^3(M)$, the Laplacian of the norm squared of the gradient satisfies the following product rule:*

$$\frac{1}{2}\Delta |\nabla u|^2 = \langle \Delta\nabla u, \nabla u\rangle + |\mathrm{Hess}_u|^2\,.$$

Proof. (1) See LeeRM, Prop 4.21.

(2) Let $X, Y \in \mathfrak{X}(M)$.

$$\begin{aligned}\mathrm{Hess}_u(X, Y) &= \nabla^2_{X,Y}\\ &= (\nabla_Y \nabla u)(X)\\ &= Y(\nabla u)(X) - \nabla u(\nabla_Y X)\\ &= Y(Xu) - \nabla_{\nabla_Y X} u\end{aligned}$$

$$= Y(Xu) - (\nabla_Y X)u.$$

Similarly,

$$\begin{aligned}\operatorname{Hess}_u(X,Y) &= (\nabla_Y \nabla u)(X)\\ &= \langle \nabla_Y \nabla u, X\rangle .\end{aligned}$$

(3) From the first equation for the Hessian,

$$\begin{aligned}\operatorname{Hess}_u(X,Y) - \operatorname{Hess}_u(Y,X) &= (\nabla_X Y)u - (\nabla_Y X)u + Y(Xu) - X(Yu)\\ &= [X,Y]u - [X,Y]u = 0\end{aligned}$$

since the connection is torsion-free.

(4) Since this formula is purely local, we can compute it on a local orthonormal frame (E_i) about $x \in M$, all parallel at x. Note that at x, since $g_{ij} = \delta_{ij}$,

$$\langle \Delta \nabla u, E_k\rangle = \delta^{ji}\left\langle \nabla^2_{E_j,E_i}\nabla u, E_k\right\rangle \quad \text{and} \quad \langle \nabla \Delta u, E_k\rangle = \nabla_{E_k}(\Delta u) = \nabla_{E_k}\left(\delta^{ij}\nabla^2_{E_i,E_j}u\right).$$

(Since we will eventually trace with respect to i and j, their order does not matter here—they are simply reversed in the first term for ease in the computation.) Therefore, we get

$$\begin{aligned}\left\langle \nabla^2_{E_j,E_i}\nabla u, E_k\right\rangle - \nabla_{E_k}(\nabla^2_{E_i,E_j}u) &= \left\langle \nabla_{E_j}(\nabla_{E_i}\nabla u), E_k\right\rangle - \nabla_{E_k}(\nabla^2_{E_j,E_i}u)\\ &= \nabla_{E_j}\langle \nabla_{E_i}\nabla u, E_k\rangle - \nabla_{E_k}(\nabla_{E_j}\nabla_{E_i}u)\\ &= \nabla_{E_j}\langle \nabla_{E_k}\nabla u, E_i\rangle - \nabla_{E_k}\nabla_{E_j}\langle \nabla u, E_i\rangle\\ &= \left\langle \nabla_{E_j}\nabla_{E_k}\nabla u, E_i\right\rangle - \nabla_{E_k}\left\langle \nabla_{E_j}\nabla u, E_i\right\rangle\\ &= \left\langle \nabla_{E_j}\nabla_{E_k}\nabla u, E_i\right\rangle - \left\langle \nabla_{E_k}\nabla_{E_j}\nabla u, E_i\right\rangle\\ &= \operatorname{Rm}(E_j, E_k, \nabla u, E_i).\end{aligned}$$

Note that we use the parallel assumption to say that $\nabla_{E_i}E_j = 0$ for all i,j, so that by compatibility $\nabla_{E_i}\langle X, E_j\rangle = \langle \nabla_{E_i}X, E_j\rangle$. Also, note that we use the symmetry of the Hessian to swap $\langle \nabla_Y \nabla u, X\rangle = \langle \nabla_X \nabla u, Y\rangle$.

Tracing with respect to i and j and using the symmetry of the Ricci tensor, we get that

$$\langle \Delta \nabla u, E_k\rangle - \langle \nabla \Delta u, E_k\rangle = \operatorname{Ric}(\nabla u, E_k),$$

as desired.

(5): Note that the LHS is

$$\frac{1}{2}\Delta |\nabla u|^2 = \frac{1}{2}\operatorname{tr}_g \nabla^2 \langle \nabla u, \nabla u\rangle .$$

Once again, let (E_i) be a local orthonormal frame, parallel at a point $x \in M$. Computing the untraced version,

$$\begin{aligned}\frac{1}{2}\nabla^2_{E_i,E_j}\langle \nabla u, \nabla u\rangle &= \frac{1}{2}\nabla_{E_i}\nabla_{E_j}\langle \nabla u, \nabla u\rangle\\ &= \nabla_{E_i}\left\langle \nabla_{E_j}\nabla u, \nabla u\right\rangle\\ &= \left\langle \nabla_{E_i}\nabla_{E_j}\nabla u, \nabla u\right\rangle + \left\langle \nabla_{E_j}\nabla u, \nabla_{E_i}\nabla u\right\rangle\\ &= \left\langle \nabla^2_{E_i,E_j}\nabla u, \nabla u\right\rangle + \left\langle \nabla_{E_i}\nabla u, \nabla_{E_j}\nabla u\right\rangle .\end{aligned}$$

Tracing with respect to i and j, we get the desired formula

$$\frac{1}{2}\Delta |\nabla u|^2 = \langle \Delta \nabla u, \nabla u\rangle + |\mathrm{Hess}_u|^2 .$$

□

Proof of Bochner formula. The Bochner formula is now a quick corollary of the above lemma:

$$\frac{1}{2}\Delta |\nabla u|^2 = \langle \Delta \nabla u, \nabla u\rangle + |\mathrm{Hess}_u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

□

4.2. **Simons identity.** Recall that the Bochner formula came out of analyzing the curvature term of the commutator $[\Delta, \nabla]$. Now recall that for a two-sided immersion $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$, the Gauss map $\nu\colon \Sigma \to S^n$ is related to the second fundamental form as follows. For an orthonormal frame (E_i) at a point $x \in \Sigma$,

$$\langle d\nu_x(E_i), E_j\rangle = \langle \nabla_{E_i}\nu, E_j\rangle = -\langle \nu, \nabla_{E_i}E_j\rangle = -h_{ij},$$

and hence we have that (up to raising an index and sign), $\nabla\nu = \mathrm{II}$. Hence, we can hope for there to be a Bochner-type formula for $\Delta |\mathrm{II}|^2$.

Proposition 4.6 (Simons identity)**.** *Let $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a two-sided immersed hypersurface. Then,*

$$\frac{1}{2}\Delta |h|^2 = |\nabla h|^2 - |h|^4 + |\mathrm{Hess}_h|^2 + H_\nu \,\mathrm{tr}(h^3).$$

Here, h^3 is the symmetric (0,2)-tensor with components $(h^3)_{ij} = h_{ik}h^{kl}h_{lj}$.

Lemma 4.7. *Let $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a two-sided immersed hypersurface with unit normal ν, and let (E_i) be a local orthonormal frame with dual coframe (ε^i).*

(1) *(Codazzi equation) The tensor $\nabla h = h_{ij;k}\,\varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k$ is a symmetric (0,3)-tensor.*

(2) *The tensor $\nabla^2 h$ is a (0,4)-tensor symmetric in its first three indices.*

(3) *(Gauss equation) For any four smooth vector fields $W, X, Y, Z \in \mathfrak{X}(\Sigma)$,*

$$\begin{aligned}\mathrm{Rm}(W,X,Y,Z) &= \langle \mathrm{II}(W,Z), \mathrm{II}(X,Y)\rangle - \langle \mathrm{II}(W,Y), \mathrm{II}(X,Z)\rangle \\ &= h(W,Z)h(X,Y) - h(W,Y)h(X,Z).\end{aligned}$$

(4) *(Commutator of second derivative of (0,2)-tensor field) For any (0,2)-tensor field $a \in \Gamma(T^{(0,2)}\Sigma)$ and vector fields $W, X, Y, Z \in \mathfrak{X}(\Sigma)$,*

$$(\nabla^2_{X,Y}a)(Z,W) - (\nabla^2_{Y,X}a)(Z,W) = a(R(X,Y)Z,W) + a(X, R(Z,W)Y).$$

(5) *(Laplacian of scalar second fundamental form) Let $H_\nu = \langle H, \nu\rangle$ be the scalar mean curvature, and let h^2 be the tensor given by $h^2(X,Y) = \sum_{k=1}^n h(E_k,X)h(E_k,Y)$, which has components $(h^2)_{ij} = h_i{}^k h^k{}_j$. Then,*

$$\Delta h = \mathrm{Hess}_{H_\nu} + H_\nu h^2 - |h|^2 h.$$

Proof. (1) Recall that $h = \langle \mathrm{I\!I}(-,-), \nu \rangle$ is a symmetric $(0,2)$-tensor, so $h_{ij;k} = h_{ji;k}$. It thus suffices to check that $h_{ij;k} - h_{kj;i} = 0$. Without loss of generality, we can compute at a point $x \in \Sigma$ where (E_i, ν) is parallel at x with respect to the ambient connection $\overline{\nabla}$ of $\mathbb{R}^{n+1}$. Computing,

$$
\begin{aligned}
h_{ij;k} &= (\nabla_{E_k} h)(E_i, E_j) \\
&= E_k h(E_i, E_j) - h(\nabla_{E_k} E_i, E_j) - h(E_i, \nabla_{E_k} E_j) \\
&= -E_k \left\langle \overline{\nabla}_{E_i} \nu, E_j \right\rangle + \left\langle \overline{\nabla}_{\nabla_{E_k} E_i} \nu, E_j \right\rangle + \left\langle \overline{\nabla}_{E_i} \nu, \nabla_{E_k} E_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{E_k} \overline{\nabla}_{E_i} \nu, E_j \right\rangle - \left\langle \overline{\nabla}_{E_i} \nu, \overline{\nabla}_{E_k} E_j \right\rangle + \left\langle \overline{\nabla}_{\nabla_{E_k} E_i} \nu, E_j \right\rangle + \left\langle \overline{\nabla}_{E_i} \nu, \nabla_{E_k} E_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{E_k} \overline{\nabla}_{E_i} \nu, E_j \right\rangle .
\end{aligned}
$$

Hence

$$
h_{ij;k} - h_{kj;i} = \left\langle \overline{\nabla}_{E_i} \overline{\nabla}_{E_k} \nu - \overline{\nabla}_{E_k} \overline{\nabla}_{E_i} \nu, E_j \right\rangle = \overline{\mathrm{Rm}}(E_i, E_k, \nu, E_j) = 0,
$$

since $\overline{\mathrm{Rm}} \equiv 0$.

(2) This is a straightforward consequence of (1): for X, Y, Z, W parallel at a point,

$$
(\nabla^2 h)(X, Y, Z, W) = (\nabla^2_{W,Z} h)(X, Y) = \nabla_W((\nabla_Z h)(X, Y)) = \nabla_W((\nabla h)(X, Y, Z)).
$$

(3) After taking an arbitrary smooth extension to a neighborhood in $\mathbb{R}^{n+1}$, this is an immediate consequence of the Gauss equation [Prop somewhere else in these notes] and the fact that the Riemann curvature tensor $\overline{\mathrm{Rm}}$ on $\mathbb{R}^{n+1}$ vanishes identically.

(4) Without loss of generality let us assume $X, Y, Z, W \in \mathfrak{X}(\Sigma)$ are parallel at a point x. Then at that point,

$$
\begin{aligned}
(\nabla^2_{X,Y} a)(Z, W) &= (\nabla_X(\nabla_Y a))(Z, W) \\
&= \nabla_X \left((\nabla_Y a)(Z, W) - a(\nabla_Y Z, W) - a(Z, \nabla_Y W) \right) \\
&= \nabla_X \nabla_Y a(Z, W) - (\nabla_Y a)(\nabla_X Z, W) - (\nabla_Y a)(Z, \nabla_X W) \\
&\quad - (\nabla_X a)(\nabla_Y Z, W) + a(\nabla_X \nabla_Y Z, W) + a(\nabla_Y Z, \nabla_X W) \\
&\quad - (\nabla_X a)(Z, \nabla_Y W) + a(\nabla_X Z, \nabla_Y W) + a(Z, \nabla_X \nabla_Y W) \\
&= \nabla_X \nabla_Y a(Z, W) + a(\nabla_X \nabla_Y Z, W) + a(Z, \nabla_X \nabla_Y W).
\end{aligned}
$$

Then, since $a(Z, W)$ is a function and X, Y are parallel, $\nabla_X \nabla_Y a(Z, W) = \nabla^2_{X,Y}(a(Z, W))$ is the Hessian of $a(Z, W)$ and $\nabla_X \nabla_Y a(Z, W) = \nabla_Y \nabla_X a(Z, W)$ is symmetric. Therefore, the only terms remaining in the difference of second derivatives is

$$
\begin{aligned}
&(\nabla^2_{X,Y} a)(Z, W) - (\nabla^2_{Y,X} a)(Z, W) \\
&= a(\nabla_X \nabla_Y Z, W) + a(Z, \nabla_X \nabla_Y W) - a(\nabla_Y \nabla_X Z, W) - a(Z, \nabla_Y \nabla_X W) \\
&= a(R(X, Y)Z, W) + a(Z, R(X, Y)W).
\end{aligned}
$$

(5) Let (x^i) be a local normal coordinate patch, parallel at a point $x \in \Sigma$. Moreover, changing coordinates if necessary, assume that (∂_i) diagonalizes the second fundamental form. Recall that $\nabla^2 h$ is symmetric in the first three arguments; let us compute the commutator of the interchange permutation (13)(24).

$$
h_{ij;kk} - h_{kk;ij} = h_{ik;jk} - h_{ik;kj}
$$

$$
\begin{aligned}
&= h(R(\partial_i, \partial_k)\partial_j, \partial_k) + h(R(\partial_i, \partial_k)\partial_k, \partial_j) \\
&= \sum_{l=1}^{n} R_{ikjl} h_{lk} + R_{ikkl} h_{lj} \\
&= \sum_{l=1}^{n} h_{il} h_{kj} h_{lk} - h_{ij} h_{kl} h_{lk} + h_{il} h_{kk} h_{lj} - h_{ik} h_{kl} h_{lj}.
\end{aligned}
$$

Tracing with respect to k, since ∂_i diagonalizes the second fundamental form we can restrict to only the indices where $k = l$.

$$
\begin{aligned}
\sum_{k=1}^{n} h_{ij;kk} - h_{kk;ij} &= \sum_{k=1}^{n} h_{kk} h_{ik} h_{jk} - h_{kk}^2 h_{ij} + h_{kk} h_{ik} h_{jk} - h_{kk} h_{ik} h_{jk} \\
&= \sum_{k=1}^{n} h_{kk} h_{ik} h_{jk} - h_{kk}^2 h_{ij}.
\end{aligned}
$$

Hence, we get that

$$
\begin{aligned}
(\Delta h)(\partial_i, \partial_j) - \mathrm{Hess}_{H_\nu}(\partial_i, \partial_j) &= \sum_{i=1}^{n} h_{ij;kk} - h_{kk;ij} \\
&= \sum_{k=1}^{n} h_{kk} h_{ik} h_{jk} - h_{kk}^2 h_{ij}. \\
&= H_\nu h^2(\partial_i, \partial_j) - |h|^2 h(\partial_i, \partial_j).
\end{aligned}
$$

□

Proof of Simons identity. Note that $\Delta |h|^2 = \sum_{i,j,k} (h_{ij})^2_{;kk}$. Computing,

$$
\begin{aligned}
\frac{1}{2} \sum_{k=1}^{n} (h_{ij}^2)_{;kk} &= \sum_{k=1}^{n} (h_{ij;k} h_{ij})_{;k} \\
&= \sum_{k=1}^{n} h_{ij} h_{ij;kk} + h_{ij;k}^2 \\
&= \sum_{k=1}^{n} h_{ij} \left(h_{kk;ij} + h_{kk} h_{ik} h_{jk} - h_{kk}^2 h_{ij} \right) + h_{ij;k}^2 \\
&= \sum_{k=1}^{n} h_{ij} h_{kk;ij} + h_{kk} h_{ij} h_{ik} h_{jk} - h_{kk}^2 h_{ij}^2 + h_{ij;k}^2.
\end{aligned}
$$

Summing with respect to i and j, we get

$$
\begin{aligned}
\frac{1}{2} \Delta |h|^2 &= \frac{1}{2} \sum_{i,j,k=1}^{n} (h_{ij}^2)_{;kk} \\
&= \sum_{i,j,k=1}^{n} h_{ij} h_{kk;ij} + h_{kk} h_{ij} h_{jk} h_{ki} - h_{kk}^2 h_{ij}^2 + h_{ij;k}^2 \\
&= |\mathrm{Hess}_h|^2 + H_\nu \operatorname{tr}(h^3) - |h|^4 + |\nabla h|^2,
\end{aligned}
$$

as desired. ☐

Corollary 4.8 (Simons identity for minimal hypersurfaces)**.** *Let $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a minimal two-sided immersed hypersurface. Then,*

$$\frac{1}{2}\Delta\left|h\right|^2 = \left|\nabla h\right|^2 - \left|h\right|^4 .$$

4.3. **Simons and Kato inequality.**

Proposition 4.9. *Let (M,g) be a Riemannian manifold. For any (k,l)-tensor $\omega \in \Gamma(T^{(k,l)M})$,*

$$\frac{1}{2}\nabla\left|\omega\right|^2 = \left|\omega\right|\nabla\left|\omega\right| \qquad \text{and} \qquad \frac{1}{2}\Delta\left|\omega\right|^2 = \left|\omega\right|\Delta\left|\omega\right| + \left|\nabla\omega\right|^2 .$$

Proof. Let (x^i) be a regular coordinate patch at $x \in M$, such that (∂_i) is an orthonormal frame and (dx^i) is the orthonormal coframe. Then in these coordinates,

$$\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\right)^2_{;m} = 2\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^{i_1\ldots i_k}_{j_1\ldots j_l;m}$$

and

$$\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\right)^2_{;mm} = 2\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^{i_1\ldots i_k}_{j_1\ldots j_l;m}\right)_{;m} = 2\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l;mm}\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\right) + 2\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l;m}\right)^2$$

$$\begin{aligned}\nabla\left|a\right|^2 &= \nabla\left(\sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_l}}\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\right)^2\right)\\ &= \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_l}}\nabla_m\left(\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\right)^2 dx^m\\ &= \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_l}} 2\omega^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^{i_1\ldots i_k}_{j_1\ldots j_l;m}\, dx^m\end{aligned}$$

☐

Proposition 4.10 (Simons inequality)**.** *Let $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a minimal two-sided immersion. Then, on the set where $\left|h\right|^2 \neq 0$,*

$$\left|\nabla h\right|^2 \geq \left(1+\frac{2}{n}\right)\left|\nabla|h|\right|^2 ,$$

or equivalently

$$\left|h\right|\Delta\left|h\right| + \left|h\right|^4 \geq \frac{2}{n}\left|\nabla\left|h\right|\right|^2 .$$

Theorem 4.11 (Schoen–Simon–Yau)**.** *Let $\iota\colon \Sigma^n \looparrowright \mathbb{R}^{n+1}$ be a stable minimal two-sided immersion. For all $p \in \left[2, 2+\frac{\sqrt{2}}{n}\right)$, there exists a constant $C = C(n,p) > 0$ such that for all $\eta \in C^{0,1}_c(\Sigma \to [0,\infty))$,*

$$\int_\Sigma \left|A\right|^{2p}\eta^{2p} \leq C(n,p)\int_\Sigma \left|\nabla\eta\right|^{2p} .$$

5. Polynomial solutions to the minimal surface equation

Recall that the mean curvature of a graphical hypersurface $\Sigma = \Gamma_u$ is given by

$$H = \frac{1}{n}\,\mathrm{MSE}(u) = \frac{1}{n}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Expanding the divergence and multiplying by $(1+|\nabla u|^2)^{3/2}$, we get the following formula.

Proposition 5.1. *Let $u \in C^2(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ an open domain. Then*

$$n(1+|\nabla u|^2)^{3/2}H = (1+|\nabla u|^2)^{3/2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \Delta u(1+|\nabla u|^2) - \mathrm{Hess}_u(\nabla u, \nabla u).$$

Proof. The proof is a straightforward computation. Denote $u_i \coloneqq \partial_i u$ and $u_{ij} \coloneqq \partial_i \partial_j u$.

$$\begin{aligned}
\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= \sum_{i=1}^n \frac{\partial}{\partial x^i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right) \\
&= \sum_{i=1}^n \frac{u_{ii}\left(1+|\nabla u|^2\right)^{1/2} - u_i \frac{1}{2}\left(1+|\nabla u|^2\right)^{-1/2}\left(\sum_{j=1}^n 2u_j u_{ij}\right)}{1+|\nabla u|^2} \\
&= \sum_{i=1}^n \frac{u_{ii}}{(1+|\nabla u|^2)^{1/2}} - \sum_{i,j=1}^n \frac{u_i u_{ij} u_j}{(1+|\nabla u|^2)^{3/2}}.
\end{aligned}$$

The result follows from the observation that

$$\mathrm{Hess}_u(\nabla u, \nabla u) = \sum_{i,j=1}^n u_i u_{ij} u_j,$$

and hence

$$(1+|\nabla u|^2)^{3/2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \left(1+\sum_{k=1}^n u_k^2\right)\sum_{i=1}^n u_{ii} - \sum_{i,j=1}^n u_i u_{ij} u_j.$$

□

So, we can let MSE be the differential operator on the RHS of the above proposition, and test for minimality using this operator instead:

$$\Gamma_u \text{ is minimal} \iff \mathrm{MSE}(u) = \Delta u(1+|\nabla u|^2) - \mathrm{Hess}_u(\nabla u, \nabla u) = 0. \tag{5.1}$$

The advantage of this operator is that it maps polynomials to polynomials. Let $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ and let $\mathbb{R}[x] \coloneqq \mathbb{R}[x^1, \dots, x^n]$, and let us use multi-index notation $I = (i_1, \dots, i_n)$ with $|I| = i_1 + \cdots + i_n$ to write $a_I = a_{i_1,\dots,i_n} \in \mathbb{R}$ and $x^I = (x^1)^{i_1} \cdots (x^n)^{i_n}$. Then, for $u \in \mathbb{R}[x]$, u_i, u_{ij} are polynomials, so by the formula

$$\mathrm{MSE}(u) = \left(1+\sum_{k=1}^n u_k^2\right)\sum_{i=1}^n u_{ii} - \sum_{i,j=1}^n u_i u_{ij} u_j,$$

it follows that $\mathrm{MSE}(u) \in \mathbb{R}[x]$. But, we can say more by decomposing the minimal surface operator into the sum of two quasilinear operators, as follows.

5.1. **Homogeneous functions.** Given a polynomial $u \in \mathbb{R}[x]$ with $\deg u = m$, we can decompose it by degree:

$$u = \sum_{|I| \leq m} a_I x^I = \sum_{k=0}^{m} \sum_{|I|=k} a_I x^I = \sum_{k=0}^{m} u^{(k)},$$

where each $u^{(k)} = \sum_{|I|=k} a_I x^I$ is the degree k factor of u. These factors are all homogeneous, in the following sense.

Definition 5.2. Let $X, Y \in \mathsf{Vect}_F$ be vector spaces over a field F, and let $d \in \mathbb{Z}$. A function $f\colon X \to Y$ is *homogeneous of degree* d or d-homogeneous if for all $x \in X$ and $\lambda \in F^\times = F \setminus \{0\}$,

$$f(\lambda x) = \lambda^d f(x).$$

If in particular $F = \mathbb{R}$ and $d \in \mathbb{R}$, $f\colon X \to Y$ is *positively homogeneous of degree* d if for all $x \in X$ and $\lambda \in \mathbb{R}_{>0}$, $f(\lambda x) = \lambda^d f(x)$.

Note that positive homogeneity is required when d is not an integer, to rule out non-uniqueness of q-th roots.

Henceforth, we will work exclusively over $F = \mathbb{R}$.

Proposition 5.3 (Properties of homogeneous functions)**.** *Let $X, Y \in \mathsf{Vect}_\mathbb{R}$, and let $u, u'\colon X \to Y$ be homogeneous of degrees $d, d' \in \mathbb{Z}$, respectively.*

(1) *If $d = d'$, then $u + u'$ is homogeneous of degree d.*

(2) *If $Y = \mathbb{R}$, the function $uu'\colon X \to \mathbb{R}$ defined by pointwise multiplication is homogeneous of degree $d + d'$.*

(3) *If $u\colon X \to Y$ and $u'\colon Y \to Z$ are homogeneous of degrees d, d', then $u' \circ u$ is homogeneous of degree dd'.*

(4) *If $X = Y = \mathbb{R}$, and $u \in C^1(\mathbb{R} \to \mathbb{R})$, then its derivative $\frac{du}{dt}$ is homogeneous of degree $d - 1$.*

(5) *If $X = \mathbb{R}^n$, $Y = \mathbb{R}$, and $u \in C^1(\mathbb{R}^n \to \mathbb{R})$, then $\nabla u\colon \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $d - 1$.*

(6) *If $c \in \mathbb{R}^\times$, then $cu\colon X \to Y$ is d-homogeneous.*

(7) *If $Y = \mathbb{R}$ and $\alpha \in \mathbb{Z}_{\geq 0}$, then u^α is $d\alpha$-homogeneous.*

Proof. (1): For any $\lambda \in \mathbb{R}^\times$ and $x \in X$,

$$(u + u')(\lambda x) = u(\lambda x) + u'(\lambda x) = \lambda^d (u(x) + u'(x)).$$

(2): Similarly,

$$u(\lambda x) u'(\lambda x) = \lambda^d u(x) \lambda^{d'} u'(x) = \lambda^{d+d'} u(x) u'(x).$$

(3): Computing,

$$u'(u(\lambda x)) = u'(\lambda^d u(x)) = \lambda^{dd'} u'(u(x)).$$

(4): Computing, substituting $h = \lambda s$,

$$\frac{du}{dt}(\lambda x) = \lim_{h\to 0} \frac{u(\lambda x + h) - u(\lambda x)}{h} = \lim_{s\to 0} \frac{u(\lambda x + \lambda s) - u(\lambda x)}{\lambda s} = \lambda^{d-1}\frac{du}{dt}(x).$$

(5): Using (4),

$$\nabla u(\lambda x) = (u_1(\lambda x), \ldots, u_n(\lambda x)) = \lambda^{d-1}(u_1(x), \ldots, u_n(x)) = \lambda^{d-1}\nabla u(x).$$

(6): Note that the multiplication map $Y \to Y$ given by $y \mapsto cy$ is homogeneous of degree 1. The result then follows from (3).

(7): Apply (2) inductively α times. □

For the sake of completeness, we list a series of properties of positively homogeneous functions.

Proposition 5.4 (Properties of positively homogeneous functions)**.** *Let $X, Y \in \mathsf{Vect}_{\mathbb{R}}$, and let $u, u' : X \to Y$ be positively homogeneous of degrees $d, d' \in \mathbb{R}$, respectively.*

(1) *If $d = d'$, then $u + u'$ is homogeneous of degree d.*

(2) *If $Y = \mathbb{R}$, the function $uu' : X \to \mathbb{R}$ defined by pointwise multiplication is homogeneous of degree $d + d'$.*

(3) *If $u : X \to Y$ and $u' : Y \to Z$ are homogeneous of degrees d, d', then $u' \circ u$ is homogeneous of degree dd'.*

(4) *If $X = Y = \mathbb{R}$, and $u \in C^1(\mathbb{R} \to \mathbb{R})$, then its derivative $\frac{du}{dt}$ is homogeneous of degree $d - 1$.*

(5) *If $X = \mathbb{R}^n$, $Y = \mathbb{R}$, and $u \in C^1(\mathbb{R}^n \to \mathbb{R})$, then $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $d - 1$.*

(6) *If $c \in \mathbb{R}^\times$, then $cu : X \to Y$ is positively homogeneous of degree d.*

(7) *If $Y = \mathbb{R}$ and $\alpha \in \mathbb{R}$, then u^α is positively homogeneous of degree αd.*

(8) *If $(X, |-|)$ is a normed vector space and $Y = \mathbb{R}$, then the norm $|-| : X \to \mathbb{R}$ is positively homogeneous of degree 1.*

Proof. The proofs are exactly the same as above, but in certain cases the domain of definition may shrink to a cone, so that the resulting function is only a partial function.

(8): This follows immediately from the definition of a norm. □

5.2. **The p-Laplacian.** Recall that the minimal surface equation was the Euler-Lagrange equation for the area functional. Similarly, we note that the usual Laplacian is the Euler-Lagrange equation for the energy functional

$$J_2(u) = \frac{1}{2}\int_\Omega |\nabla u|^2,$$

where once again $\Omega \subseteq \mathbb{R}^n$. In general, for $p \in [1, \infty)$, we can define the energy functional

$$J_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p.$$

Computing the first variation with respect to a variation $\eta \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\frac{d}{dt} J_p(u + t\eta) &= \frac{1}{p} \int_\Omega \frac{d}{dt} |\nabla u + t\nabla \eta|^p \\
&= \frac{1}{p} \int_\Omega \frac{d}{dt} \left(\sum_{i=1}^n (u_i + t\eta_i)^2 \right)^{\frac{p}{2}} \\
&= \frac{1}{p} \int_\Omega \frac{p}{2} \left(\sum_{i=1}^n (u_i + t\eta_i)^2 \right)^{\frac{p-2}{2}} \left(\sum_{i=1}^n 2(u_i + t\eta_i)\eta_i \right).
\end{aligned}$$

Evaluating at $t = 0$,

$$\begin{aligned}
\frac{d}{dt} J_p(u + t\eta)\bigg|_{t=0} &= \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \eta \rangle \\
&= - \int_\Omega \eta \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right).
\end{aligned}$$

Hence, we can define the p-Laplacian as the Euler-Lagrange equation for J_p:

Definition 5.5. Let $\Omega \subseteq \mathbb{R}^n$, $u \in C^2(\Omega)$, and $p \in [1, \infty)$. The *p-Laplacian* operator Δ_p is defined to be the Euler-Lagrange equation for the functional J_p:

$$\Delta_p u = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right).$$

Note that for $p = 2$, we get $\Delta_2 = \Delta$, the usual Laplacian. We also define the ∞-Laplacian:

Definition 5.6. Let $\Omega \subseteq \mathbb{R}^n$, $u \in C^2(\Omega)$. The *∞-Laplacian* Δ_∞ is the operator

$$\Delta_\infty u = \mathrm{Hess}_u(\nabla u, \nabla u) = \sum_{i,j=1}^n u_i u_{ij} u_j.$$

We will later elaborate on the sense in which the ∞-Laplacian is the Euler-Lagrange equation for the functional $\operatorname{ess\,sup}_\Omega |\nabla u|$.

We list below some basic properties of the p-Laplacian.

Proposition 5.7. *The p-Laplacian Δ_p satisfies the following properties:*

(1) $\Delta_p u = |\nabla u|^{p-2} \Delta_2 u + (p-2) |\nabla u|^{p-4} \Delta_\infty u$,

(2) $\Delta_\infty u = \frac{1}{2} \left\langle \nabla u, \nabla |\nabla u|^2 \right\rangle$,

(3) *If u is homogeneous of degree d, then $\Delta_\infty u$ is homogeneous of degree $3d - 4$.*

(4) *If u is homogeneous of degree d, then $\Delta_2 u$ is homogeneous of degree $d - 2$.*

(5) *If u is homogeneous of degree d, then $\Delta_p u$ is homogeneous of degree $(d-1)(p-1) - 1$.*

Proof. (1): This follows by the product rule for the divergence:

$$\Delta_p u = |\nabla u|^{p-2} \Delta_2 u + \left\langle \nabla u, \nabla |\nabla u|^{p-2} \right\rangle$$

$$= |\nabla u|^{p-2} \Delta_2 u + \sum_{i=1}^n u_i \left(\frac{\partial}{\partial x^i} \left(\sum_{j=1}^n u_j^2 \right)^{\frac{p-2}{2}} \right)$$
$$= |\nabla u|^{p-2} \Delta_2 u + \sum_{i,j=1}^n u_i \left(\frac{p-2}{2} |\nabla u|^{\frac{p-4}{2}} 2 u_j u_{ji} \right)$$
$$= |\nabla u|^{p-2} \Delta_2 u + (p-2) |\nabla u|^{p-4} \Delta_\infty u.$$

(2): Computing, we get

$$\left\langle \nabla u, \nabla |\nabla u|^2 \right\rangle = \sum_{i=1}^n u_i \frac{\partial}{\partial x^i} \left(\sum_{j=1}^n u_j^2 \right) = 2 \sum_{i,j=1}^n u_j u_{ij} u_j = 2\Delta_\infty u.$$

(3): Examining the formula, each term $u_i u_{ij} u_j$ is homogeneous of degree $(d-1) + (d-2) + (d-1) = 3d - 4$. Thus, their sum is homogeneous of degree $3d - 4$.

(4): Again examining the formula, each term u_{ii} is homogeneous of degree $d - 2$, and the result follows.

(5): Examining the terms in the equation from (1), the first term is homogeneous of degree $(d-1)(p-2) + (d-2)$, and the second term is homogeneous of degree $(d-1)(p-4) + (3d-4)$. But, these terms are both equal to

$$dp - d - p = (d-1)(p-1) - 1.$$

□

Lemma 5.8 (Euler's identity). *Let $d \in \mathbb{Z}$ and let $u \in C^1(\mathbb{R}^n \to \mathbb{R})$ be a positively homogeneous function of degree d. Then, u satisfies the PDE*

$$d \cdot u(x) = \langle x, \nabla u(x) \rangle .$$

Proof. Since u is homogeneous, we have for all $t > 0$, $u(tx) = t^d u(x)$. Differentiating both sides with respect to t, we have

$$du_{tx}(x) = dt^{d-1} u(x).$$

Notice that the LHS is equal to $\langle x, \nabla u(tx) \rangle$, so taking $t \to 1$, we get

$$\langle x, \nabla u(x) \rangle = d \cdot u(x),$$

as desired. □

Lemma 5.9 (No nonzero homogeneous solutions to ∞-Laplace equation). *Let $u \colon \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function of degree $d \in \mathbb{Z} \setminus \{0, 1\}$, and let $\Delta_\infty u = 0$. Then, $u = 0$.*

Proof. Pick $x \in S^{n-1}$ such that $u(x) = \max_{y \in S^{n-1}} u(y)$, which exists since S^{n-1} is compact. By the Lagrange multiplier theorem, we must have that $\nabla u(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

By Lemma 6.7(2) and Euler's identity, we have that

$$0 = \frac{1}{2} \left\langle \nabla u(x), \nabla |\nabla u|^2 (x) \right\rangle = \frac{1}{2} \lambda \left\langle x, \nabla |\nabla u|^2 (x) \right\rangle = \lambda(d-1) |\nabla u|^2 (x) = \lambda^3 (d-1),$$

since $|\nabla u|^2$ is homogeneous of degree $2(d-1)$. Since $d \neq 1$ by assumption, we have that $\lambda = 0$, so $\nabla u(x) = 0$. Applying Euler's identity once more, we get that

$$0 = \langle x, \nabla u(x) \rangle = du(x),$$

and since $d \neq 0$ by assumption, $u(x) = 0$. Repeating this argument with $-u$, we get that $\max |u| = 0$, so $u \equiv 0$. □

Corollary 5.10 (2-out-of-3 lemma). *Let $p \in [1,2) \cup (2,\infty)$ and $u \in C^2(\Omega \to \mathbb{R})$ is homogeneous of degree $d \in \mathbb{Z} \setminus \{0,1\}$, and suppose that two of the quantities $\Delta_p u, \Delta u, \Delta_\infty u$ vanish. Then, the third vanishes, and in particular $u \equiv 0$.*

Proof. The result follows simply by observation of the formula in Proposition 6.7(1) and an application of Lemma 6.9. □

5.3. **Polynomial solutions to the minimal surface equation.** Recall that the minimal surface equation is given by

$$\text{MSE}(u) = \Delta u(1 + |\nabla u|^2) - \text{Hess}_u(\nabla u, \nabla u) = 0.$$

Now note that the 1-Laplacian is given by

$$\Delta_1 u = |\nabla u|^{-1} \Delta u - |\nabla u|^{-3} \Delta_\infty u,$$

and thus

$$|\nabla u|^3 \Delta_1 u = |\nabla u|^2 \Delta u - \Delta_\infty u.$$

Let us define $Lu = |\nabla u|^3 \Delta_1 u$, so that we can rewrite the MSE as

$$\text{MSE}(u) = \Delta u + |\nabla u|^3 \Delta_1 u = \Delta u + Lu.$$

The operator L is a normalized mean curvature vector:

$$Lu = |\nabla u|^3 \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Hence, we have the geometric interpretation that locally, on points where $\nabla u \neq 0$, $Lu = 0$ if and only if $u^{-1}(0)$ is a minimal hypersurface.

The advantage of this decomposition is that if u is a homogeneous polynomial of degree d, then Lu and Δu are homogeneous of degree $3d - 4$ and $d - 2$, respectively. Thus, if u is homogeneous of $d \geq 2$, then $Lu + \Delta u = 0$ implies that $Lu = \Delta u = 0$, since the two terms have degrees $3d - 4 \neq d - 2$, respectively. In view of Corollary 6.10, it follows that $\Delta_\infty u = 0$ and hence $u \equiv 0$, which shows immediately that there are no homogeneous solutions to the minimal surface equation. In fact, we can do better:

Proposition 5.11. *Suppose $u = u^m + \cdots + u^k$ is a solution to the MSE, with u^i homogeneous of degree i. Then, if $k \geq \lceil \frac{m}{3} \rceil + 1$, then $u \equiv 0$.*

Proof. By inspection, the degree $3m - 4$ term of MSE(u) is $Lu^m = 0$. Then, note that Lu has terms of degrees $3k-4, \ldots, 3m-4$. Moreover, Δu^m is degree $m-2$, but since $m-2 < 3k-4$ by assumption, $\Delta u^m = 0$ and hence $\Delta_\infty u^m = 0$. Thus, $u^m \equiv 0$. By induction, it follows that $u \equiv 0$. □