Polynomial solutions to the minimal surface equation

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Introduction

Given a C^1 closed curve $\Gamma \subseteq \mathbb{R}^3$, does there exist a surface Σ with boundary $\partial\Sigma = \Gamma$ with minimal area with respect to all such surfaces? More generally, if $\Gamma^{n-1} \subseteq \mathbb{R}^{n+1}$ is a closed submanifold and

$$\mathcal{S} = \left\{\Sigma^n \subseteq \mathbb{R}^{n+1} \text{ smooth submanifold with boundary } \partial\Sigma = \Gamma^{n-1}\right\},$$

then is the infimum $\inf_{\Sigma\in\mathcal{S}} \operatorname{Area}(\Sigma)$ achieved? This problem, Plateau's problem, is intimately related to the geometry of soap bubbles and films, which locally minimize area with respect to various boundary or volume constraints.

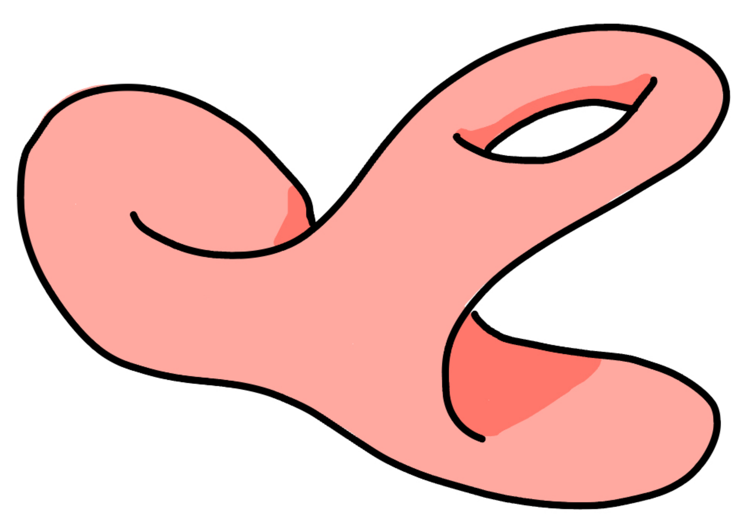

Figure 1. A surface $\Sigma \subseteq \mathbb{R}^3$ with boundary $\partial\Sigma = \Gamma$.

Often, a functional has an associated differential equation, called its *Euler-Lagrange equation*, where critical points of the functional correspond to solutions of the Euler-Lagrange equation. The minimal surface equation (MSE) arises as the Euler-Lagrange equation of the area functional

$$\operatorname{Area}(\Gamma_u) = \int_\Omega \sqrt{1 + |\nabla u|^2},$$

where $\Omega \subseteq \mathbb{R}^n$ is open and $\Gamma_u = \left\{(x, u(x)) \in \mathbb{R}^{n+1} \colon x \in \Omega\right\}$ is the graph of $u \in C^2(\Omega \to \mathbb{R})$. That is, $u \in C^2(\Omega \to \mathbb{R})$ is a critical point for the area functional if and only if it solves the *minimal surface equation* (MSE):

$$\operatorname{div}\left(\left(1 + |\nabla u|^2\right)^{-\frac{1}{2}} \nabla u\right) = 0.$$

In either case, we say that Γ_u is a *minimal surface*.

Bernstein's theorem

The MSE is a quasilinear second-order elliptic PDE, and a moment's thought shows that *affine functions* $u(x) = a \cdot x + b$ are *entire* solutions to the MSE (i.e., $\Omega = \mathbb{R}^n$). A famous problem of Sergey Bernstein asks if these are the only entire solutions. Through joint work of Bernstein, Fleming, Almgren, De Giorgi, Simons, and Bombieri–De Giorgi–Giusti, the Bernstein problem has since been resolved:

***Theorem* 1** (Bernstein's theorem)**.** *Let $u \colon C^2(\mathbb{R}^n \to \mathbb{R})$ be an entire solution to the MSE.*

1. *If $n \leq 7$, then u must be affine.*
2. *If $n \geq 8$, there exist non-affine solutions to the MSE.*

The non-affine solutions for $n \geq 8$ constructed by Bombieri, De Giorgi, and Giusti are known to be analytic but not polynomial, prompting the question:

***Question* 2.** *Do there exist non-affine polynomial solutions to the minimal surface equation?*

By Bernstein's theorem, such candidate polynomial solutions must have at least 8 variables. Yet, little more is known about existence (or non-existence) of polynomial solutions to the MSE.

The p-Laplacian

Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in C^2(\Omega \to \mathbb{R})$. Denote by $u_i = \frac{\partial u}{\partial x^i}$. The usual Laplacian $\Delta u = \sum_{i=1}^n u_{ii}$ is the Euler-Lagrange equation for the *energy functional* $\frac{1}{2}\int_\Omega |\nabla u|^2$. In general, for $p \in [1, \infty)$, we can define the *p-Laplacian operator* Δ_p to be the Euler-Lagrange equation corresponding to the energy functionals

$$J_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p,$$

where $|\nabla u|^p = \left(u_1^2 + \cdots + u_n^2\right)^{p/2}$. Similarly, we define Δ_∞ via the Euler-Lagrange equation for the functional $\operatorname{ess\,sup}_\Omega |\nabla u|$. Explicitly, these differential operators are given by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{p-2} \Delta u + (p-2)\, |\nabla u|^{p-4} \Delta_\infty u$$

$$\Delta_\infty u = \operatorname{Hess}_u(\nabla u, \nabla u) = \sum_{i,j=1}^n u_i u_{ij} u_j.$$

The MSE can actually be written in terms of p-Laplacians: after expanding the divergence and multiplying by $\left(1 + |\nabla u|^2\right)^{3/2}$, we have that u satisfies the MSE if and only if

$$0 = (1 + |\nabla u|^2)\Delta u - \Delta_\infty u = \Delta u + |\nabla u|^3 \Delta_1 u.$$

Let $Lu \coloneqq |\nabla u|^3 \Delta_1 u$, so that the MSE can be written as $\Delta u + Lu = 0$. This operator is a *normalized mean curvature* operator: If $Lu = 0$, and 0 is a regular value for u, then $u^{-1}(0) \subseteq \mathbb{R}^n$ is a minimal hypersurface. In this form, the operator $\Delta + L$ usefully sends polynomials to polynomials, so solving the minimal surface equation for an arbitrary polynomial can be translated to an algebraic problem of solving a large number of (cubic) equations in the coefficients.

Tangent cones at ∞

A subset $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda C = C$ for all $\lambda > 0$. Historically, questions about the regularity of solutions to Plateau's problem became reduced to questions about the existence of non-flat area minimizing cones.

To solve Plateau's problem one takes an area-minimizing sequence (i.e., a sequence $\Sigma_i \in \mathcal{S}$ such that $\operatorname{Area}(\Sigma_i) \to \inf_{\Sigma' \in \mathcal{S}} \operatorname{Area}(\Sigma')$), and extracts a limit as Caccioppoli sets in codimension 1 (or as integral rectifiable currents in arbitrary codimension). The limit may be singular, but since blow-ups at singular points give non-flat area-minimizing cones, proving nonexistence or classification results about such cones can prove regularity of these candidate solutions to Plateau's problem.

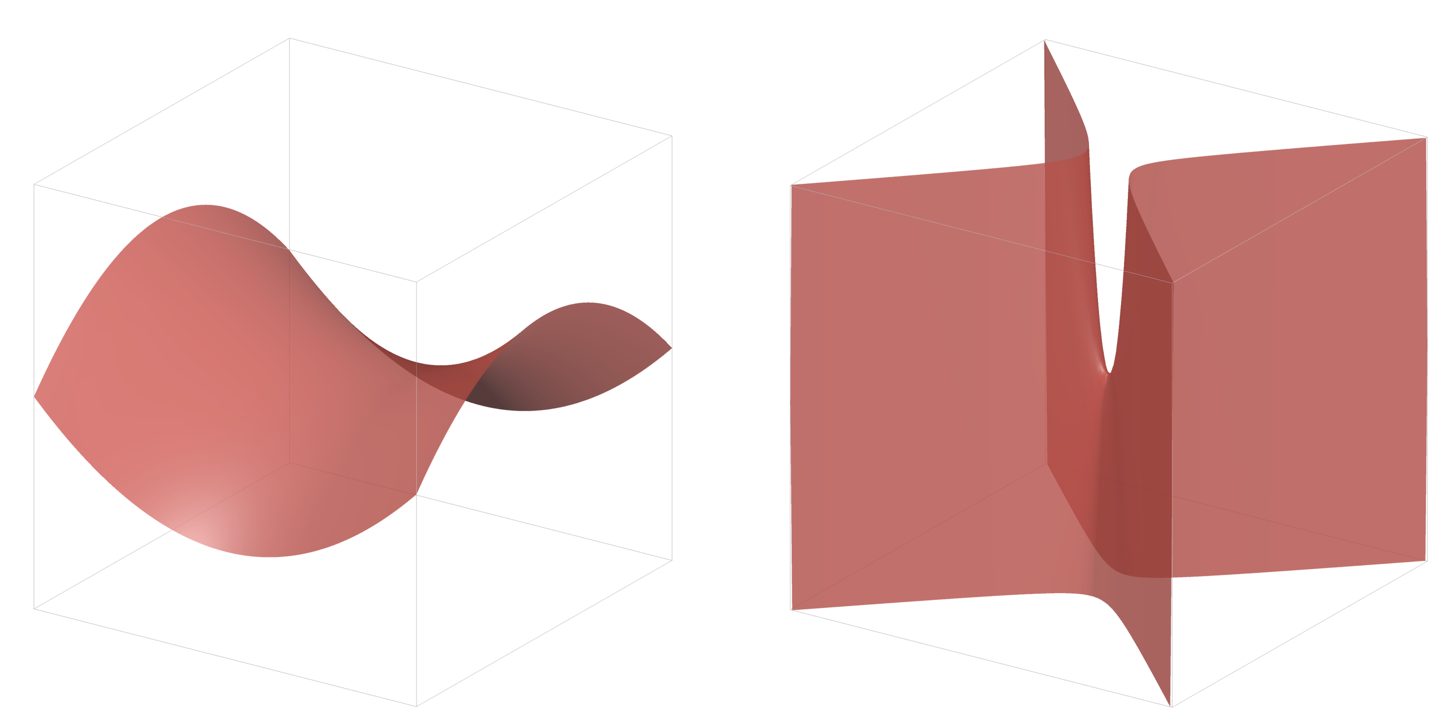

Figure 2. Blow-down of the graph of $x^2 - y^2$ at 0. The tangent cone at infinity is $C \times \mathbb{R}$, where $C = \left\{(x, y) \in \mathbb{R}^2 \colon x^2 - y^2 = 0\right\}$ is the cross.

Moreover, for entire minimal graphs $\Gamma_u \subseteq \mathbb{R}^{n+1}$, blowing down gives an area-minimizing cone called the *tangent cone at infinity*. This cone splits a line, giving an area-minimizing cone $C' \subseteq \mathbb{R}^n$. By a celebrated theorem of Simons, such a cone must be flat for $n \leq 7$, providing a resolution for Bernstein's problem in $n \leq 7$:

***Theorem* 3** (Simons 1968)**.** *Let $C^{n-1} \subseteq \mathbb{R}^n$ be a stable minimal cone. If $n \leq 7$, then C is a hyperplane.*

Polynomial solutions to the minimal surface equation

Simons's theorem gives a *de facto* uniqueness result for tangent cones in low dimension, but in general uniqueness of tangent cones remains a deep unsolved problem. However, in the case of polynomials we can explicitly compute the tangent cone at infinity:

***Theorem* 4** (Guo 2024 (2.9))**.** *Let $u \in \mathbb{R}[x_1, \ldots, x_n]$ solve the MSE, and write $u = u_m + \cdots + u_1$, for $u_i \in \mathbb{R}[x_1, \ldots, x_n]$ homogeneous of degree i. Then, there exists an irreducible homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ such that $p^k \mid u_m$ for some $k \geq 1$ odd, such that the tangent cone of Γ_u at infinity is $Z(p) \times \mathbb{R}$.*

Hence, in a similar vein, we can hope to rule out polynomial solutions to the MSE by classifying such *algebraic minimal cones* $Z(p) = p^{-1}(0) \subseteq \mathbb{R}^n$. Although some progress has been made on this front for $\deg p = 2, 3$, the picture in general is still open.

However, Guo was able to leverage a clever choice of Jacobi field on the minimal hypersurfaces $\{u_m = t\}_{t\in\mathbb{R}}$ to prove upper bounds on $\deg p$, which proves the following nonexistence theorem:

***Theorem* 5** (Guo 2024 (5.6))**.** *Suppose $u \in \mathbb{R}[x_1, \ldots, x_8]$ solves the MSE and $\deg u = m \geq 2$. Then, $Z(p) \subseteq \mathbb{R}^8$ is a cubic (i.e., $\deg p = 3$), strictly stable, area minimizing cone which is not strictly minimizing.*

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