# A Majority Voting Model on Branching Brownian Motion

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## Abstract

The use of voting models on Branching Brownian Motion to represent solutions to reaction - diffusion equations is a novel method that has become increasingly popular. Etheridge, Pennington, and Freeman first observed the beautiful connection between voting models on ternary Branching Brownian Motion and the Allen-Cahn equation. In this paper, we generalize one of their results (Theorem 2.5 in [Ali16]) for the majority voting scheme with $2m + 1$ voters. We present both a probabilistic and a PDE proof of this generalized result.

## Contents

# 1 Acknowledgements

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# 2 Introduction

Reaction-diffusion equations, or reaction-diffusion systems, are a class of partial differential equations (PDEs) that possess the potential to model a wide range of physical phenomena. The solutions $u(t,x)$ to these PDEs represent quantities such as concentrations, densities, populations, and we usually take $0 \geq u(t,x) \leq 1$. The typical form for such a PDE is

$$u_t = D\Delta u + f(u),$$

where $\Delta$ is the Laplacian operator, which describes "diffusion." The second term $f \in C^\infty(\mathbb{R})$ describes some process with a "change:" birth, death, chemical reaction, .... We state a few possibilities in relation to population dynamics. We can have functions that have some exponential growth, wherein

$$f(u) = au,$$

for $a$ a constant growth factor. We can have logistic growth functions of the form

$$f(u) = au\left(1 - \frac{u}{K}\right),$$

where $K$ denotes some limiting carrying capacity of our population. Lastly, we can have an equation that models the Allee effect, which gives us a logistic growth with the additional condition that if the population gets too low, it will quickly die out. These functions are of the form

$$f(u) = au\left(\frac{n}{K_0} - 1\right)\left(1 - \frac{n}{K}\right).$$

Some famous reaction diffusion equations are depicted in the table below.

| Equation Name | PDE | Example phenomena modelled |
|---|---|---|
| Fisher-KPP | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$ | Invasive species growth |
| ZFK | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\theta)$ | Flame propagation |
| Bicoid | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - au + j$ | Bicoid gradient formation |
| NWS | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^q$ | Taylor-Couette Flow |
| Allen-Cahn | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(2u-1)$ | Order-disorder transitions |

Recently, it has been proposed to model the solutions to these equations using a probabilistic model that relies upon Branching Brownian Motion (BBM) and voting trees. In this section, we will first delineate the model for the branching process, then introduce Branching Brownian Motion and finally we will consider voting models on Branching Brownian Motion to allow us to extend our probabilistic model to a larger class of reaction-diffusion equations of the type $u_t = u_{xx} + f(u)$, where $f(u)$ is a polynomial nonlinearity.

## 2.1 Branching Mechanism of BBM

We will first start with the branching mechanism for BBM. We begin with a particle, which has an associated exponential random variable $Y \sim \text{Exp}(1)$ that denotes the time $\tau$ at which this particle will disappear and give birth to some number of new, identical particles of the same type, and the process repeats. That is, the probability that the branching time $\tau$ is greater than a given time $t$ is $\mathbb{P}(\tau > t) = e^{-t}$.

**Definition 2.1.** *We define a **leaf** to be a particle that is alive at the time we are considering.*

A particle need not split into the same number of new particles every time, and we can denote by $p_k$, $k \geq 2$, the probability that given that a particle splits into $k$ new particles. However, after this section, we will always assume that particles always split into the same number of child particles.

The following proposition is an exercise which appears in [Ber14], [Bov13], and [Ryz21] and will be essential in our proof of Theorem 3.1.

**Proposition 2.1.** *Let $N(t)$ denote the number of leaves at time $t$. Then*

$$\mathbb{E}[N(t)] = e^{(\overline{N}-1)t},$$

*where $\overline{N} = \sum_{k\geq 2} kp_k$.*

*Proof of Proposition 2.1.* We will use the partition theorem to express this expectation as a renewal equation. Let $\tau$ denote the very first branching time, and let $f(t)$ denote this expectation. We split into the case in which our initial particle has not branched, or it has branched at some time $s \in [0, t]$.

$$\begin{aligned} f(t) = \mathbb{P}(\tau > t) + \sum_{k\geq 2} kp_k f(t-\tau)\mathbb{P}(\tau < t) &= e^{-t} + \overline{N}\int_0^t f(t-s)e^{-s}ds, \\ &= e^{-t} + \overline{N}e^{-t}\int_0^t f(w)e^{w}dw, \end{aligned}$$

where $w = t - s$. Then by the product rule,

$$\frac{df}{dt} = -e^{-t} + \overline{N}f(t) - \overline{N}e^{-t}\int_0^t f(w)e^{w}dw = -e^{-t} + \overline{N}f(t) + e^{-t} - f(t),$$

$$\implies \frac{df}{dt} = (\overline{N} - 1)f(t),$$

which, with the initial condition that $f(0) = 1$, gives that $\mathbb{E}[N(t)] = e^{(\overline{N}-1)t}$.

This branching process is an example of a Yule process, describing the birth of a population in some amount of time. $\square$

## 2.2 Branching Brownian Motion

This model becomes more interesting when we introduce the factor of Brownian Motion. We begin with our initial particle starting at some point $x \in \mathbb{R}$. It does Brownian Motion for time $\tau$, where $\tau$ is its first branching time, distributed exponentially, as before. At the branching time, it is replaced with some number of new identical child particles, each with their own independent exponentially distributed branching time clock. The Brownian Motion itself carries an interesting connection with the Heat Equation.

**Claim 2.1.** *$u(t, x) = \mathbb{E}[u_0(x + B_t)]$, denoted as $\mathbb{E}_x[u_0(B_t)]$, is a solution to the Heat Equation with initial condition $u(0, x) = u_0(x)$, where $B_t$ denotes Brownian Motion for time $t$.*

*Proof of Claim 2.1.* First, note that at time 0, the expectation is just $u_0$, so the initial condition is satisfied. Then,

$$\mathbb{E}_x[u_0(B_t)] = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(y) e^{-\frac{(x-y)^2}{4t}} dy = u_0 * K(x, t),$$

which is the convolution of $u_0$ and the heat kernel, $K(x, t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$ implying that this is indeed a solution to the heat equation. $\square$

## 2.3 McKean's Observation

As it would turn out, putting the structure of Brownian Motion and this tree branching mechanism together would lead to surprising connections in the mathematics of reaction-diffusion equations. Around 1975, McKean observed in [McK75] that if we took the case of dyadic BBM, then the quantity

$$u(t, x) = \mathbb{E}_x \left[ \prod_{i=1}^{N(t)} u_0(X_i) \right] \tag{1}$$

satisfies the Fisher-Kolomogorov-Petrovsky-Piskunov (Fisher-KPP) equation of the form, $u_t = u_{xx} + u(u - 1)$, with initial condition $u_0$, where here $x$ denotes the position of the starting particle, $N(t)$ denotes the number of leaves at time $t$, and $X_i$ denotes the location of the $i$th leaf at time $t$. To see why this is the case, we will introduce the very useful Duhamel's Principle:

**Theorem 2.1** (Duhamel's Principle)**.** *Let $\mathcal{L}$ be a linear differential operator. Then if $u(t,x) = (H(t,\cdot) * f)(x)$ takes $f(x)$ to the solution to $u_t = \mathcal{L}u$ with initial condition $u_0(x) = f(x)$, we have that*

$$u(t,x) = (H(t,\cdot) * f)(x) + \int_0^t (H(s,\cdot) * g(t-s,\cdot))(x)ds$$

*solves the equation $(u_t - \mathcal{L}u)(t,x) = g(t,x)$ with initial condition $u_0(x) = f(x)$. $H(t,x)$ is often called the Green's function for the differential operator $\mathcal{L}$.*

*Proof of Theorem 2.1.* We have

$$\begin{aligned} u_t - \mathcal{L}u = ((H(t,\cdot) * f)(x))_t - \mathcal{L}((H(t,\cdot) * f)(x)) \\ + \left(\frac{d}{dt} - \mathcal{L}\right) \int_0^t (H(s,\cdot) * g(t-s,\cdot))(x)ds. \end{aligned}$$

The first two terms vanish, since $(H(t,\cdot) * f)(x)$ solves $u_t = \mathcal{L}u$. With the Leibniz rule, we have that this becomes

$$(H(t,\cdot) * g(0,\cdot))(x) + \int_0^t \left(\frac{d}{dt} - \mathcal{L}\right) (H(s,\cdot) * g(t-s,\cdot))(x)ds = (H(t,\cdot) * g(0,\cdot))(x) = g(t,x),$$

where again we have used the fact that $(H(t,\cdot) * f)(x)$ solves $u_t = \mathcal{L}u$, and so we are done. $\square$

It is well known that $K(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/2t}$ is the Green's function for $\mathcal{L} = \partial^2_{xx}$. We claim that the Green's function for $\mathcal{L}u = \partial_{xx}u - u$ is $H(t,x) = e^{-t}K(t,x)$. Indeed, if $u(t,x)$ is the convolution of $H$ with some initial condition function $f(x)$, we see that

$$\begin{aligned} u_t(t,x) &= (e^{-t}(K(t,\cdot) * f)(x))_t, \\ &= e^{-t}((K(t,\cdot) * f)(x))_t - e^{-t}(K(t,\cdot) * f)(x), \\ &= (e^{-t}(K(t,\cdot) * f)(x))_{xx} - u(t,x), \\ &= (u_{xx} - u)(t,x). \end{aligned}$$

Now we show that we can apply Duhamel's Principle 2.1 to show that quantity 1 satisfies the Fisher-KPP equation. We condition on the first branching time $\tau$.

$$u(t,x) = \mathbb{E}_x \left[ \prod_{i=1}^{N(t)} u_0(X_i) \right] = \mathbb{E}_x[u_0(B_t)]e^{-t} + \int_0^t \mathbb{E}_x[u^2(t-s, B_s)]e^{-s}ds.$$

We notice now, that $\mathbb{E}_x[u_0(B_t)]e^{-t}$ is the convolution of $u_0$ with with $e^{-t}K(t,x) = H(t,x)$ and $\mathbb{E}_x[u^2(t-s, B_s)]e^{-s}ds$ is the convolution of $u^2(t-s,x)$ with $H(s,x)$, and so by Duhamel's Principle 2.1, we see that quantity 1 satisfies the PDE $u_t = u_{xx} - u + u^2$, as desired.

### 2.4 The Voting Model

Past using different expressions of the McKean type, the above work does not seem to extend very far. Thus we introduce the a tree voting structure to our model. At time $t$, we look at every particle that is alive and have it vote 0 or 1 with probability given by the initial condition evaluated on its position. That is, leaf $i$, which is at position $X_i(t)$ at time $t$ will vote 1 with probability $u_0(X_i(t))$. Then we propagate the vote backwards in genealogy based on some rule, e.g. majority voting, random group voting. We claim that the probability that the initial particle votes 1, henceforth denoted by $\mathbb{P}_x(V_0(t) = 1)$, gives us the solutions to some interesting PDEs .

**Example 2.1.** *Consider the dyadic branching case in which we have a parent node vote* $1$ *if and only if both child nodes vote* $1$*. Then the probability that the first node voted* $1$ *is the probability that all leaves voted* $1$*, which is given by* $u(t,x) = \mathbb{E}_x\left[\prod_{i=1}^{N(t)} u_0(X_i)\right]$*, which is the solution to the Fisher-KPP Equation.*

**Example 2.2.** *Consider the case of ternary branching in which we have a parent node vote* $1$ *if and only if the majority of its child nodes voted* $1$*. Then we can condition on the first branching time to get that our probability of the origin voting* $1$ *satisfies the renewal equation*

$$u(t,x) = \mathbb{E}[u_0(x+B_t)]\mathbb{P}(\tau > t) + \int_0^t \mathbb{E}_x\left[u(t-s,B_s)^3 + 3u(t-s,B_s)^2(1-u(t-s,B_s))\right]e^{-s}ds,$$

*which, using Duhamel's Principle 2.1, implies that*

$$u_t = u_{xx} - u + u^3 + 3u^2(1-u) = u_{xx} + u(1-u)(2u-1),$$

*which is the Allen-Cahn Equation.*

**Example 2.3.** *We can also ask what PDE the quantity* $\mathbb{E}_x\left[\prod_{i=1}^{N(t)} u_0(X_i)\right]$ *satisfies for general* $n$*-branching. Writing out a renewal equation and using Duhamel's Principle, we get that*

$$u(t,x) = \mathbb{E}[u_0(x+B_t)]e^{-t} + \int_0^t \mathbb{E}_x[u(t-s,B_s)^n]e^{-s}ds,$$

*giving us the PDE* $u_t = u_{xx} - u + u^n$*. This will be used in Lemma 2.3.*

We can expand even further on our voting model by introducing weights. That is, if $k$ out of $n$ children vote 1, then the parent node has probability $a_{kn}$ of voting 1. For example, in the Allen-Cahn case, we have that $a_{33} = a_{23} = 1$ and all other $a_{kn}$ are 0. Using these weights, we may write out our renewal equation and apply Duhamel's Principle again to find that $\mathbb{P}_x(V_0(t) = 1) = u(t,x)$ satisfies the PDE

$$u_t = u_{xx} - u + \sum_{k=0}^{n} \binom{n}{k} a_{kn} x^k (1-x)^{n-k}.$$

We will show that we can achieve a large class of degree $n$ polynomial nonlinearities. To see this, we need the following proposition:

**Proposition 2.2.** *For all* $0 \le m \le n$*, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{k}{m}}{\binom{n}{m}} x^k (1-x)^{n-k} = x^m.$$

*Proof of Proposition 2.2.* Note that

$$\binom{n}{k} \frac{\binom{k}{m}}{\binom{n}{m}} = \frac{n!}{k!(n-k)!} \cdot \frac{k!m!(n-m)!}{n!m!(k-m)!} = \binom{n-m}{k-m}.$$

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{k}{m}}{\binom{n}{m}} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n-m}{k-m} x^k (1-x)^{n-k},$$

$$= \sum_{k=m}^{n} \binom{n-m}{k-m} x^k (1-x)^{n-k} = \sum_{j=0}^{n-m} \binom{n-m}{j} x^{j+m} (1-x)^{n-j-m},$$

$$= x^m \sum_{j=0}^{n-m} \binom{n-m}{j} x^j (1-x)^{n-m-j} = x^m,$$

by the Binomial Theorem. □

Probabilistically, this is biasing by the number of groups of size $m$ that exist among the children that voted 1. Using this proposition, we get the following nice corollary.

**Corollary 2.1.** *If $g(u) = c_n u^n + c_{n-1} u^{n-1} + \cdots + c_1 u + c_0$, then for*

$$a_{kn} = \sum_{j=0}^{n} c_j \frac{\binom{k}{j}}{\binom{n}{j}},$$

*we have that*

$$\sum_{k=0}^{n} \binom{n}{k} a_{kn} u^k (1-u)^{n-k} = g(u).$$

*Provided that $0 \leq a_{kn} \leq 1$ for each $k = 0, 1, \ldots, n$, then a voting model with these weights will satisfy the PDE $u_t = u_{xx} - u + g(u)$.*

*Proof of Corollary 2.1.* This is a straightforward calculation that follows from simply taking linear combinations of expressions of the type in 2.2 □

**Example 2.4.** *For $n = 3$, we can let $a_{13} = \frac{1}{3}(1-\theta)$, $a_{23} = 1 - \frac{1}{3}\theta$, and $a_{33} = 1$ to get the ZFK equation, $u_t = u_{xx} + u(1-u)(u-\theta)$ for all $\theta \in (0, \frac{1}{2})$. Notice that for $\theta = \frac{1}{2}$, this is just the Allen Cahn equation.*

**Example 2.5.** *For $a_{kn} = \frac{k}{n}$, we obtain $g(u) = u$ and thus our respective PDE is the heat equation, $u_t = u_{xx}$. This represents an unbiased voting model, in the sense that the probability of voting 1 given that $k$ out of the $n$ children voted 1 is $\frac{k}{n}$.*

**Example 2.6.** *Let $a_{in} = \frac{i}{n}(1-a)$, for $i < n$ and $a_{nn} = 1$. These weights give us the Bicoid equation $u_t = u_{xx} - au$ for $a \in (0, 1)$. Recall that the solution to the Bicoid equation $u$ is given by $u = e^{-at} w$ where $w$ solves the heat equation. We can interpret this voting model by noticing that there is a slight bias towards zero. This makes logical sense since, in comparison to the unbiased case for the heat equation, the Bicoid equation has an extra coefficient that is at most 1 and is thus smaller.*

**Example 2.7.** *Take $n = q$, and let $a_{iq} = \frac{i}{q}(1 + \frac{a}{b})$, for $i < q$ with $a_{qq} = 1$. Then we get the Newell-Whitehead-Segel equation, $u_t = u_{xx} + au - bu^q$. To ensure our that our weights are small enough to represent probabilities, we require $\frac{a}{b} \leq \frac{1}{q-1}$ to make $0 \leq a_{iq} \leq 1$.*

## 2.5 Probabilistic Arguments for PDE Properties

We would like to be able to use these voting models to prove results about the corresponding PDEs. For instance, we might start with some interesting observations on the Fisher-KPP equation.

**Lemma 2.1.** *Given some initial condition $u_0(t,x) > 0$, the solution $u(t,x)$ to the Fisher-KPP equation is strictly positive for any time $t$.*

*Proof.* Probabilistically speaking, $u(t,x) = \mathbb{P}_x(V_0(t) = 1)$ is given by $\mathbb{E}\left[\prod_{i=1}^{N(t)} u_0(x+X_i)\right]$, which is the expected value of a positive product that should certainly be positive.

From a PDE perspective, applying the strong maximum principle gives us the result. □

**Lemma 2.2.** *Given some initial condition $u_0(x)$ such that $u_0$ is monotonically increasing in terms of $x$, we have that $u(t,x)$ is monotonically increasing with respect to $x$ for all $t$.*

*Proof.* Probabilistically speaking, it is immediate that the solution to Fisher-KPP is monotonically increasing given that $u_0(x)$ is monotonically increasing, since we can write is as $u(t,x) = \mathbb{E}\left[\prod_{i=1}^{N(t)} u_0(x+X_i)\right]$.
To see this from a PDE perspective, we first denote $w(t,x) = u_x(t,x)$. Then taking $\frac{d}{dx}$ on both sides of the Fisher-KPP equation, we get

$$w_t = w_{xx} + (1-2u)w.$$

Similarly, the strong maximum principle suggests that $u_x(t,x) \geq 0$ for all $t$. □

Moreover, using our probabilistic model, we can compare solutions to reaction-diffusion equations.

We do some preliminary work first. Recall that if our particles always branch into $n$ new particles, then

$$v = \mathbb{E}_x\left[\prod_{i=0}^{N_t} g(X_i)\right]$$

satisfies the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - v + v^n.$$

Hence, we get the following lemma:

**Lemma 2.3.** *Given some*

$$f(u) = \sum_{k=0}^{n} \binom{n}{k} a_{kn} u^k (1-u)^{n-k},$$

*with $a_{nn} = 1$ and $0 \leq a_{kn} \leq 1$ for $k = 0, 1, \ldots, n-1$ (representing weighted votes), then $v(t,x)$ satisfying the equation*

$$v_t = v_{xx} - v + v^n,$$

*is a subsolution to*

$$u_t = u_{xx} - u + f(u),$$

*where both PDEs have initial condition $\mathbb{1}_{(x<0)}$ and as a result $v(t,x) \leq u(t,x)$ for all $t, x$.*

*Proof of Lemma 2.3.* We present two proofs of this inequality. First, there is a simple proof that uses the maximal principle for parabolic PDEs. Let $\mathcal{L}$ denote the operator that sends $u$ to $u_t - u_{xx} + u - f(u)$. We claim that $v$ is a subsolution of $u$, or in other words that $\mathcal{L}v \leq 0$ for all $x, t \in \mathbb{R}$ with $t \geq 0$. Indeed, we have that

$$\mathcal{L}v = v_t - v_{xx} + v - f(v) = v^n - f(v) \leq 0,$$

by assumption.

Thus $v(t, x)$ is a subsolution, and so $v(t, x) \leq u(t, x)$.

Alternatively, we can obtain this inequality in a much neater way, using our probabilistic model. Note that $u(t, x) = \mathbb{P}_x(V_0(t) = 1)$. Clearly this is at least $\mathbb{P}$(all leaves vote 1) since $a_{nn} = 1$. Note that since we have the initial condition $\mathbb{1}_{(x \leq 0)}$, $P$(all leaves vote 1) $= \mathbb{P}_x(\max_{j \leq N_t} X_j < 0) = \mathbb{E}_x\left[\prod_{i=0}^{N_t} \mathbb{1}_{(X_i < 0)}\right]$ which is exactly $v(t, x)$, and so we have proved that $v(t, x) \leq u(t, x)$ once again but in a probabilistic way. □

# 3 Probabilistic Proof

In the next two sections, our goal is to prove a generalization of Theorem 2.5 in [Ali16]. The set up for this theorem is that our initial condition is, $\mathbb{1}_{(x \geq 0)}$, our particles split with an exponential clock of parameter $1/\varepsilon^2$ into $2m + 1$ particles, $m \geq 1$, and we propagate the vote back to the origin by majority voting.

**Theorem 3.1** (Main Theorem)**.** *Let $T^* \in (0, \infty)$. For all $k \in \mathbb{N}$, there exists $c_1(k)$ and $\varepsilon_1(k) > 0$ such that, for all times $t \in [0, T^*]$ and all $\varepsilon \in (0, \varepsilon_1)$,*

1. *for $z \geq c_1(k)\, \varepsilon |\log \varepsilon|$, we have $\mathbb{P}_z^\varepsilon[\mathbb{V}(\boldsymbol{B}(t)) = 1] \geq 1 - \varepsilon^k$,*
2. *for $z \leq -c_1(k)\, \varepsilon |\log \varepsilon|$, we have $\mathbb{P}_z^\varepsilon[\mathbb{V}(\boldsymbol{B}(t)) = 1] \leq \varepsilon^k$.*

We will need a number of lemmas to state our theorem, but we must first consider some properties of our majority voting model.

## 3.1 Properties of the Nonlinearity

Note that the probability for majority voting is given by

$$g(p_1, p_2, \ldots, p_{2m+1}) := \sum_{\substack{X \subseteq \{1, \ldots, 2m+1\} \\ |X| \geq m+1}} \prod_{i \in X} p_i \prod_{j \in X^C} (1 - p_j),$$

where $p_1, p_2, \ldots, p_{2m+1}$ are the respective probabilities that a child node votes 1.

**Claim 3.1.** $g(1 - p_1, 1 - p_2, \ldots, 1 - p_{2m+1}) = 1 - g(p_1, p_2, \ldots, p_{2m+1})$.

*Proof of Claim 3.1.* We can write $g$ as

$$g(p_1, p_2, \ldots, p_{2m+1}) = \sum_{\substack{Y \subseteq \{1,\ldots,2m+1\} \\ |Y| \le m}} \prod_{j \in Y} (1 - p_j) \prod_{i \in Y^C} p_i,$$

$$\implies g(1 - p_1, 1 - p_2, \ldots, 1 - p_{2m+1}) = \sum_{\substack{Y \subseteq \{1,\ldots,2m+1\} \\ |Y| \le m}} \prod_{j \in Y} p_j \prod_{i \in Y^C} (1 - p_i),$$

and so

$$g(p_1, \ldots, p_{2m+1}) + g(1 - p_1, \ldots, 1 - p_{2m+1}) = \sum_{X \subseteq \{1,\ldots,2m+1\}} \prod_{i \in X} p_i \prod_{j \in X^C} (1 - p_j),$$

$$= \prod_{i=1}^{2m+1} (p_i + (1 - p_i)) = 1.$$

Probabilistically, note that $1 - p_i$ denotes the probability of a child voting 0. So, if a majority of the children vote 0, then the parent node votes 0, which happens with probability $1 - g(p_1, \ldots, p_{2m+1})$. $\square$

We abuse notation and write

$$g(p) = \sum_{j=0}^{m} \binom{2m+1}{j} p^{2m+1-j} (1 - p)^j.$$

**Proposition 3.1.** *If*

$$g(x) = \sum_{j=0}^{m} \binom{2m+1}{j} x^{2m+1-j} (1 - x)^j,$$

*then*

$$g'(x) = (2m + 1) \binom{2m}{m} \left(x - x^2\right)^m,$$

*and*

$$g''(x) = (2m + 1) m \binom{2m}{m} \left(x - x^2\right)^{m-1} (1 - 2x).$$

*Proof of Proposition 3.1.* The second derivative very easily follows from the first, by the chain rule. We compute the first derivative as follows:

$$g'(x) = \sum_{j=0}^{m} \binom{2m+1}{j} \left( (2m + 1 - j) x^{2m-j} (1 - x)^j - j x^{2m+1-j} (1 - x)^{j-1} \right),$$

$$= \sum_{j=0}^{m} (2m + 1 - j) \binom{2m+1}{j} x^{2m-j} (1 - x)^j - \sum_{j=1}^{m} \binom{2m+1}{j} j x^{2m+1-j} (1 - x)^{j-1}.$$

But since $(2m + 1 - j) \binom{2m+1}{j} = (2m + 1) \binom{2m}{j}$ and $j \binom{2m+1}{j} = (2m + 1) \binom{2m}{j-1}$, we can re-index the second summation for this to become

$$= \sum_{j=0}^{m} (2m + 1) \binom{2m}{j} x^{2m-j} (1 - x)^j - \sum_{j=0}^{m-1} (2m + 1) \binom{2m}{j} x^{2m-j} (1 - x)^j,$$

$$= (2m+1)\binom{2m}{m}\left(x-x^2\right)^m,$$

by telescoping series. $\square$

**Corollary 3.1.** *On the interval* $\left[\frac{1}{2},1\right]$, $g'(x) \geq 0$ *and* $g''(x) \leq 0$.

*Proof of Corollary 3.1.* Clearly both $x(1-x) \geq 0$ and $1-2x \leq 0$ on the interval $\left[\frac{1}{2},1\right]$. $\square$

**Theorem 3.2** (Condorcet's Jury Theorem)**.** *For* $n \geq 1$, *let* $g_{2n-1}(p)$ *denote the probability of a majority of* $2n-1$ *particles voting* $1$, *given that each particle votes* $1$ *with probability* $p$ *and* $0$ *with probability* $1-p$. *If* $\frac{1}{2} \leq p \leq 1$, *then*

$$g_{2n+1}(p) \geq g_{2n-1}(p).$$

*Proof of Theorem 3.2.* We will create a bijection as follows: Let $X_1, X_2, \ldots, X_{2n+1}$ be $2n+1$ Bernoulli random variables with parameter $p \geq \frac{1}{2}$. Consider the random variables $S_i = 2X_i - 1$. Let $\mathcal{S}_{2n+1}$ denote the sum of the $S_i$. Our probability $g^{(n+1)}(p)$ is exactly $\mathbb{P}(\mathcal{S}_{2n+1} > 0)$. We will first consider the case of $2n-1$ voters and then add 2 new voters in. We get three cases:

- $\mathcal{S}_{2n-1} > 2$ and the two new voters do not influence the vote.
- $\mathcal{S}_{2n-1} = 1$ and at least one of the new voters needs to vote 1.
- $\mathcal{S}_{2n-1} = -1$ and both of the new voters need to vote 1.

This gives the following recursive structure to our probabilities:

$$\mathbb{P}(\mathcal{S}_{2n+1} > 0) = \mathbb{P}(\mathcal{S}_{2n-1} > 2) + (1-(1-p)^2)\mathbb{P}(\mathcal{S}_{2n-1} = 1) + p^2\mathbb{P}(\mathcal{S}_{2n-1} = -1).$$

Now, notice that $\mathbb{P}(\mathcal{S}_{2n-1} > 2) + \mathbb{P}(\mathcal{S}_{2n-1} = 1) = \mathbb{P}(\mathcal{S}_{2n-1} > 0)$, and so we can rewrite our equation as

$$\mathbb{P}(\mathcal{S}_{2n+1} > 0) - \mathbb{P}(\mathcal{S}_{2n-1} > 0) = p^2\mathbb{P}(\mathcal{S}_{2n-1} = -1) - (1-p)^2\mathbb{P}(\mathcal{S}_{2n-1} = 1).$$

To show that this last quantity is at least 0, we can rewrite it as

$$p^2\left(\binom{2n-1}{n}p^n(1-p)^{n-1}\right) - (1-p)^2\left(\binom{2n-1}{n}p^{n-1}(1-p)^n\right),$$

which is clearly nonnegative for $p \in \left[\frac{1}{2},1\right]$, and so we are done. $\square$

**Corollary 3.2** (Increasing in Composition)**.** *For all* $n \geq 0$ *and* $p \in \left[\frac{1}{2},1\right]$ *we have that* $g^{(n+1)}(p) \geq g^{(n)}(p)$.

*Proof of Corollary 3.2.* Since $g$ is increasing on the interval $\left[\frac{1}{2},1\right]$, and $\frac{1}{2} \leq g(p) \leq 1$ for $p \in \left[\frac{1}{2},1\right]$, we need only show that $g(p) \geq p$ for all $p \in \left[\frac{1}{2},1\right]$. But this is exactly an application of Theorem 3.2. $\square$

The last fact we will need about $g$ is an important theorem which states that when the input probabilities of $g$ are at least $\frac{1}{2}$, $g$ is at least the average of the input probabilities. This theorem is nontrivial and merits its own subsection.

### 3.2 Generalized Condorcet's Theorem

We wish to prove the following theorem:

**Theorem 3.3.** *For all $n \in \mathbb{N}$, we have that for all $\frac{1}{2} \leq p_1, p_2, \ldots, p_n \leq 1$*

$$g\left(g_1, g_2, \ldots, g_n\right) \geq \frac{1}{n}\left(p_1 + p_2 + \cdots + p_n\right).$$

**Corollary 3.3.** *Given $n$ Bernoulli random variables $X_1, X_2, \ldots, X_n$ with parameters $p_i \in \left[\frac{1}{2}, 1\right]$ for $1 \leq i \leq n$, then $\mathbb{P}\left[X_1 + \cdots + X_n > \frac{n}{2}\right] \geq \frac{1}{n}(p_1 + \cdots + p_n)$.*

*Proof of Corollary 3.3.* Recall that a Bernoulli random variable of parameter $p$ takes the value 1 with probability $p$ and the value 0 with probability $1-p$. There is a bijection between having a majority vote and having a majority of the Bernoulli variables be 1. If a majority of the Bernoulli variables are 1, then clearly their sum is strictly greater than $\frac{n}{2}$, and so we are done. □

Now to prove Theorem 3.3, we will prove the more general fact: Consider a voting model of $n$ people, each with possibly different probabilities greater than $\frac{1}{2}$ of voting 1 in which the group votes by randomly selecting a subcommittee of $k$ people, $1 \leq k \leq n$, and then allowing the majority vote of those $k$ people to decide how the group of $n$ votes. We will show that as $k$ increases, the probability of the group voting 1 increases. Our proof of this fact comes from [Dan03], and we provide it here for completeness.

*Proof of Theorem 3.3.* Define by $[n]$ the set $\{1, 2, \ldots, n\}$, and let $p_i$ denote the probability that the $i$th person votes 1, and $q_i = 1 - p_i$ denote the probability that the $i$th person votes 0. For a subset $E \subseteq [n]$, Let $M_{m,j}(E)$ denote the probability that exactly $m - j$ members of $E$ vote 1. Then

$$M_{m,j}(E) = \begin{cases} 1, & m = 0 \\ \displaystyle\sum_{\substack{E_1 \subseteq E \\ |E_1| = m-j}} \prod_{i \in E_1} p_i \cdot \prod_{j \in E \setminus E_1} q_j, & m \geq 1 \end{cases}$$

Denote by $S_{n,m,j}$ the sum of all $M_{m,j}(E)$ as $E$ ranges over all $\binom{n}{m}$ subsets of $[n]$ of size $m$. Let $M(E)$ denote the probability that a majority votes 1 in the subcommittee $E$. If $|E|$ is odd, this is defined as

$$M(E) = \sum_{j=0}^{\frac{|E|-1}{2}} M_{|E|,j}(E).$$

If $|E|$ is even, then we will define

$$M(E) = \sum_{j=0}^{\frac{|E|}{2}-1} M_{|E|,j}(E) + \frac{1}{2} M_{|E|,|E|/2}(E).$$

Then, we will define by $P(m)$ the probability of the group voting correctly given that they used a subcommittee of $m$ people. That is

$$P(m) = \frac{1}{\binom{n}{m}} \sum_{\substack{E\subseteq[n]\\ |E|=m}} M(E).$$

Note that $P(1) = \frac{1}{n}\sum_{i=1}^{n} p_i$ and $P(n) = g(p_1, \ldots, p_n)$ as defined previously. Therefore, the following theorem will suffice.

**Theorem 3.4.** *Suppose we have a committee of $n$ people with probabilities $p_1, p_2, \ldots, p_n$, each between $\frac{1}{2}$ and 1 of voting 1. Then*

$$P(0) \leq P(1) = P(2) \leq P(3) = P(4) \leq \cdots \leq P(n-1) \leq P(n),$$

*if $n$ is odd and*

$$P(0) \leq P(1) = P(2) \leq P(3) = P(4) \leq \cdots \leq P(n-1) = P(n),$$

*if $n$ is even.*

**Lemma 3.1.** *For a group of size $n$ and nonnegative integers $m, j, k$ and $r$, with $m+r \leq n$ and $0 \leq j \leq m$ and $0 \leq k \leq r$, we have*

$$\sum_{\substack{E,E'\subseteq[n],E\cap E'=\emptyset\\ |E|=m,|E'|=r}} M_{m,j}(E)\, M_{r,k}(E') = \binom{j+k}{k}\binom{m+r-j-k}{r-k} S_{n,m+r,j+k}.$$

*Proof of Lemma 3.1.* Recall that $S_{n,m+r,j+k}$ is the sum over all $M_{m+r,j+k}(F)$ as $F$ ranges over all subsets of size $m+r$. It is not hard to see that the product $M_{m,j}(E)\, M_{r,k}(E')$ contains terms of the form $\prod_{i\in F_1} p_i \cdot \prod_{i\in(E\cup E')\setminus F_1} q_i$ where $F_1 \subseteq E\cup E'$ and $|F_1| = m+r-j-k$. Now, let's think about it instead as first choosing a subset $F$ of size $m+r$ and then another subset $F_1 \subseteq F$ of size $m+r-j-k$. Then, we allow $E$ and $E'$ to range over some disjoint subsets of size $m$ and $r$, respectively, of $F$. Notice that once we select the set $E$, $E'$ is uniquely determined as $F\setminus E$.

Notice that since $E$ and $F_1$ must have $m-j$ elements in common, we first select $m-j$ elements for $E$ in $\binom{m+r-j-k}{m-j}$ ways. Then, we select the remaining $j$ elements from $F\setminus F_1$, which we can do in $\binom{j+k}{j}$ ways. We reiterate that since we have chosen $E$, $E'$ is uniquely determined. Therefore, the term

$$\prod_{i\in F_1} p_i \cdot \prod_{i\in F\setminus F_1} q_i$$

appears exactly $\binom{m+r-j-k}{m-j}\binom{j+k}{j}$ times. Summing over all possible sets $F$ and $F_1$ will give us $S_{n,m+r,j+k}$. Therefore, our result is $\binom{m+r-j-k}{m-j}\binom{j+k}{j}S_{n,m+r,j+k}$, as desired. □

**Lemma 3.2.** *For a group of size $n$, we have that*

$$S_{n,2k+1,k} \geq S_{n,2k+1,k+1},$$

*where $1 \leq k \leq \frac{n-1}{2}$ with equality if and only if $p_1 = p_2 = \cdots = p_n = \frac{1}{2}$.*

*Proof of Lemma 3.2.* It suffices to show that $M_{2k+1,k}(E) \geq M_{2k+1,k+1}(E)$ for $E$ a subset of size $2k+1$ in $[n]$. Note that

$$M_{2k+1,k}(E) = \sum_{\substack{E_1 \subseteq E \\ |E_1|=k+1}} \prod_{i \in E_1} p_i \cdot \prod_{i \in E \setminus E_1} q_i.$$

Since each $p_i$ is at least $\frac{1}{2}$, we have that

$$\prod_{i \in E_1} p_i \cdot \prod_{i \in E \setminus E_1} q_i \geq \prod_{i \in E_1 \setminus \{i_0\}} p_i \cdot \prod_{i \in (E \setminus E_1) \cup \{i_0\}} q_i,$$

where $E_1 \subseteq E, i_0 \in E_1$. Then, summing over each $i_0 \in E_1$, we have that

$$(k+1) \cdot \prod_{i \in E_1} p_i \cdot \prod_{i \in E \setminus E_1} q_i \geq \prod_{i \in E \setminus E_1} q_i \cdot \sum_{i=1}^{|E_1|} \frac{q_i}{p_i} \prod_{i \in E_1} p_i = \prod_{i \in E \setminus E_1} q_i \cdot M_{k+1,1}(E_1).$$

Summing over all sets $E_1$ with $|E_1| = k+1$, we have that

$$(k+1) M_{2k+1,k}(E) \geq (k+1) M_{2k+1,k+1}(E).$$

□

*Proof of Theorem 3.4.* Now we will finish this proof by two parts. First we show that

$$P(2k+1) = P(2k+2),$$

for all $0 \leq k \leq \frac{n}{2} - 1$. Recall that

$$P(2k+1) = \frac{1}{\binom{n}{2k+1}} \sum_{\substack{E \subseteq [n] \\ |E|=2k+1}} M(E).$$

Notice that

$$(n-2k-1) \sum_{\substack{E \subseteq [n] \\ |E|=2k+1}} M(E) = \sum_{\substack{E \subseteq [n] \\ |E|=2k+1}} M(E) \sum_{i \in [n]/E} 1 = \sum_{\substack{E \subseteq [n] \\ |E|=2k+1}} M(E) \sum_{i \in [n]/E} (p_i + q_i),$$

$$= \sum_{\substack{E,E' \subseteq [n], E \cap E' = \emptyset \\ |E|=2k+1, |E'|=1}} M(E) \left(M_{1,0}(E') + M_{1,1}(E')\right),$$

$$= \sum_{j=0}^{k} \sum_{\substack{E,E' \subseteq [n], E \cap E' = \emptyset \\ |E|=2k+1, |E'|=1}} M_{2k+1,j}(E) M_{1,0}(E') + \sum_{j=0}^{k} \sum_{\substack{E,E' \subseteq [n], E \cap E' = \emptyset \\ |E|=2k+1, |E'|=1}} M_{2k+1,j}(E) M_{1,1}(E').$$

Now using Lemma 5.3 with $m = 2k+1, r = 1, k = 0$, and then once again with $m = 2k+1, r = 1, k = 1$, we obtain

$$= \sum_{j=0}^{k} (2k+2-j) S_{n,2k+2,j} + \sum_{j=1}^{k+1} j S_{n,2k+2,j}$$

which telescopes to

$$= (2k+2)\left(\sum_{j=0}^{k} S_{n,2k+2,j} + \frac{1}{2}S_{n,2k+2,k+1}\right).$$

Therefore, we find that

$$\begin{aligned}
P\left(2k+1\right) &= \frac{2k+2}{\binom{n}{2k+1}(n-2k-1)}\left(\sum_{j=0}^{k} S_{n,2k+2,j} + \frac{1}{2}S_{n,2k+2,k+1}\right), \\
&= \frac{1}{\binom{n}{2k+2}}\left(\sum_{j=0}^{k} S_{n,2k+2,j} + \frac{1}{2}S_{n,2k+2,k+1}\right) = P\left(2k+2\right).
\end{aligned}$$

Lastly, and most importantly, we show that $P(2k) \leq P(2k+1)$. Similarly to the previous proof, we get that

$$(n-2k)\sum_{\substack{E\subseteq[n] \\ |E|=2k}} M(E) = \sum_{j=0}^{k-1}\sum_{\substack{E\subseteq[n] \\ |E|=2k}} M_{2k,j}(E)\sum_{i\in[n]\setminus E}(p_i+q_i) + \frac{1}{2}\sum_{\substack{E\subseteq[n] \\ |E|=2k}} M_{2k,k}(E)\sum_{i\in[n]\setminus E}(p_i+q_i),$$

$$\begin{aligned}
&= \sum_{j=0}^{k-1}\sum_{\substack{E,E'\subseteq[n],E\cap E'=\emptyset \\ |E|=2k,|E'|=1}} (M_{2k,j}(E)M_{1,0}(E') + M_{2k,j}(E)M_{1,1}(E')) \\
&\qquad + \frac{1}{2}\sum_{\substack{E,E'\subseteq[n],E\cap E'=\emptyset \\ |E|=2k,|E'|=1}} (M_{2k,k}(E)M_{1,0}(E') + M_{2k,k}(E)M_{1,1}(E')).
\end{aligned}$$

Applying Lemma 3.1, this gives

$$\begin{aligned}
(n-2k)\sum_{E\subseteq[n],|E|=2k} M(E) &= \sum_{j=0}^{k-1}(2k+1-j)S_{n,2k+1,j} + \sum_{j=0}^{k-1}(j+1)S_{n,2k+1,j+1} \\
&\qquad + \frac{1}{2}(k+1)S_{n,2k+1,k} + \frac{1}{2}(k+1)S_{n,2k+1,k+1}, \\
&= (2k+1)\sum_{j=0}^{k-1} S_{n,2k+1,j} + \frac{3k+1}{2}S_{n,2k+1,k} + \frac{k+1}{2}S_{n,2k+1,k+1}, \\
&= (2k+1)\sum_{j=0}^{k} S_{n,2k+1,j} + \frac{k+1}{2}(S_{n,2k+1,k+1} - S_{n,2k+1,k}).
\end{aligned}$$

This gives that

$$P(2k) = \frac{1}{\binom{n}{2k}(n-2k)}\left((2k+1)\sum_{j=0}^{k} S_{n,2k+1,j} + \frac{k+1}{2}(S_{n,2k+1,k+1} - S_{n,2k+1,k})\right),$$

$$= \frac{1}{\binom{n}{2k+1}} \left( \sum_{j=0}^{k} S_{n,2k+1,j} + \frac{k+1}{2(2k+1)} (S_{n,2k+1,k+1} - S_{n,2k+1,k}) \right),$$

but since $P(2k+1) = \frac{1}{\binom{n}{2k+1}} \sum_{j=0}^{k} S_{n,2k+1,j}$, we have that

$$P(2k+1) - P(2k) = \frac{1}{\binom{n}{2k+1}} \cdot \frac{k+1}{2(2k+1)} (S_{n,2k+1,k} - S_{n,2k+1,k+1}),$$

and so we are done by Lemma 3.2. □

### 3.3 Necessary Lemmas

Now we are ready to begin stating and proving the lemmas we will need to prove Theorem 3.1.

**Definition 3.1.** *Define a* ***time-labelled*** $n$***-tree*** *to be the the genealogical tree formed through our branching mechanism wherein branching particles split into* $n \geq 2$ *child particles.*

Denote by $\mathcal{T}_n(\mathbf{B}(t))$ the time-labelled $n$-tree traced out by Branching Brownian Motion up to time $t$. Since the context usually is clear, we will drop the $n$ subscript. Then we denote by $\mathbb{P}_z^t(\mathcal{T})$ the probability of voting 1 at the origin, given that $\mathcal{T}_n(\mathbf{B}(t)) = \mathcal{T}$ and our branching motion started at $z$.

Now if we have branching at time $\tau \leq t$ into subtrees $\mathcal{T}_1, \ldots, \mathcal{T}_{2m+1}$, then we have that

$$\mathbb{P}_z^t(\mathcal{T}) = \mathbb{E}_z\left[g\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_1), \ldots, \mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_{2m+1})\right)\right],$$

where $B_\tau$ denotes Brownian Motion for time $\tau$. We shall denote this expression by $\mathbb{E}_z[g\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)]$.

**Lemma 3.3.** *For any time-labelled $n$-tree, any time $t \geq 0$ and any $z \geq 0$,*

$$\mathbb{P}_z^t(\mathcal{T}) \geq P_z[B_t \geq 0].$$

*Proof of Lemma 3.3.* We proceed by induction on the number of branching events. Clearly, if there are no branching events, then $\mathbb{P}_z^t = P_z[B_t \geq 0]$. Now assume that this holds for all time-labelled $n$-trees up to $\ell \geq 1$ branching events. Define a function $h : [0,1]^{2m+1} \to [0,1]$ by

$$h(p_1, p_2, \ldots, p_{2m+1}) = g(p_1, p_2, \ldots, p_{2m+1}) - \frac{1}{2m+1}(p_1 + p_2 + \cdots + p_{2m+1}).$$

From 3.3 we have that for $\frac{1}{2} \leq p_1, p_2, \ldots, p_{2m+1} \leq 1$ we have $h \geq 0$. Also note that, using 3.1, we have

$$h(1-p_1, \ldots, 1-p_{2m+1}) = g(1-p_1, \ldots, 1-p_{2m+1}) - \frac{1}{2m+1}(2m+1-p_1-\cdots-p_{2m+1}),$$

$$= g(1-p_1, \ldots, 1-p_{2m+1}) - 1 + \frac{1}{2m+1}(p_1 + \cdots + p_{2m+1}),$$

$$= -g(p_1, \ldots, p_{2m+1}) + \frac{1}{2m+1}(p_1 + \cdots + p_{2m+1}) = -h(p_1, p_2, \ldots, p_{2m+1}).$$

Now, for our inductive step, we have that

$$\begin{aligned}\mathbb{P}_z^t(\mathcal{T}) &= \mathbb{E}_z\left[g\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_1),\ldots,\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_{2m+1})\right)\right],\\ &= \mathbb{E}_z\left[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_1),\ldots,\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_{2m+1})\right)\right] + \frac{1}{2m+1}\sum_{i=1}^{2m+1}\mathbb{E}_z\left[\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_i)\right].\end{aligned}$$

We will show that the first term of this expression is nonnegative, from which induction will complete our proof.

Recall that $\mathbb{P}_z^t(\mathcal{T}) = 1 - \mathbb{P}_{-z}^t(\mathcal{T})$, and that $h(1-p_1,\ldots,1-p_{2m+1}) = -h(p_1,\ldots,p_{2m+1})$. Therefore,

$$\begin{aligned}\mathbb{E}_z\left[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_1),\ldots,\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_{2m+1})\right)\right] &= -\mathbb{E}_z\left[h\left(\mathbb{P}_{-B_\tau}^{t-\tau}(\mathcal{T}_1),\ldots,\mathbb{P}_{-B_\tau}^{t-\tau}(\mathcal{T}_{2m+1})\right)\right],\\ \iff \mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)] &= -\mathbb{E}_z[h\left(\mathbb{P}_{-B_\tau}^{t-\tau}(\mathcal{T}\star)\right)].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)] &= \mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}\{B_\tau \geq 0\}] + \mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}\{B_\tau < 0\}],\\ &= \mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}\{B_\tau \geq 0\}] - \mathbb{E}_z[h\left(\mathbb{P}_{-B_\tau}^{t-\tau}(\mathcal{T}\star)\right)\mathbb{1}\{B_\tau < 0\}],\\ &= \int_0^\infty h\left(\mathbb{P}_x^{t-\tau}(\mathcal{T}\star)\right)\left(\phi_{z,2\tau}(x) - \phi_{z,2\tau}(-x)\right)dx,\end{aligned}$$

where we denote by $\phi_{z,2\tau}$ the probability distribution of a Gaussian with mean $z$ and variance $2\tau$, in accordance with our Brownian Motion.

Now, recall that we have an increasing initial condition, and that $\mathbb{P}_0^t(\mathcal{T}) = \frac{1}{2}$. That is, the probability of voting 1 at the origin given that the origin started exactly in the middle is $\frac{1}{2}$ by a symmetry argument. Therefore, for $x, t \geq 0$ we have that $\mathbb{P}_x^t(\mathcal{T}) \geq \frac{1}{2}$, and so since $h$ is nonnegative on the interval $\left[\frac{1}{2}, 1\right]$, so is $\mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)]$.

Finally, since $z \geq 0$ and $x \geq 0$, we have that $\phi_{z,2\tau}(x) - \phi_{z,2\tau}(-x) \geq 0$ (that is, starting at a positive location $z \geq 0$, it is more likely for us to end in a positive location than a negative location).

Hence $\mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right) \geq 0$ and so

$$\mathbb{P}_z^t(\mathcal{T}) = \mathbb{E}_z[h\left(\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}\star)\right)] + \frac{1}{2m+1}\sum_{j=1}^{2m+1}\mathbb{E}_z[\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_i)] \geq \frac{1}{2m+1}\sum_{j=1}^{2m+1}\mathbb{E}_z[\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_i)].$$

But using the inductive hypothesis, we have that

$$\mathbb{E}_z[\mathbb{P}_{B_\tau}^{t-\tau}(\mathcal{T}_i)] \geq \mathbb{E}_z[\mathbb{P}_{B_\tau}[B_{t-\tau} \geq 0]] = \mathbb{P}_z[B_t \geq 0],$$

which completes the proof of our lemma. □

Now we will show that having a small voting bias $\varepsilon$ at the leaves of our tree magnifies to be greater than $1-\varepsilon^k$ as we propagate the vote back through an order of $\mathcal{O}(|\log\varepsilon|)$ rounds of voting.

**Lemma 3.4.** *For all $k \in \mathbb{N}$, there exist $A(k) < \infty$ such that, for all $\varepsilon \in (\frac{1}{2}, 1]$, and $n \geq A(k)\,|\log \varepsilon|$, we have*

$$g^{(n)}\left(\frac{1}{2}+\varepsilon\right) \geq 1-\varepsilon^k. \tag{2}$$

*Proof of Lemma 3.4.* First, we prove that it takes an order of $\mathcal{O}(|\log \varepsilon|)$ iterations to get

$$g^{(n)}\left(\frac{1}{2}+\varepsilon\right) \geq \frac{1}{2}+\delta, \tag{3}$$

for some well-chosen $\delta > 0$.

Next, we show that an additional $n$ iterations, where $n$ is of the order of $\mathcal{O}(\log(k|\log \varepsilon|))$ are needed to achieve

$$g^{(n)}\left(\frac{1}{2}+\delta\right) \geq 1-\varepsilon^k. \tag{4}$$

From 3.1, we know that $g$ is monotonically increasing, so combining the two parts above will lead to the result we need. First, we will write $1-g(1-x)$ as an explicit polynomial in $x$. Let $a_0, a_1, \ldots, a_n$ be real numbers such that

$$1-g(1-x) = \sum_{i=0}^{n} a_i x^i.$$

We note that $a_0 = a_1 = 0$, and so for all $x \in (0,1)$, we have

$$1-g(1-x) \leq Cx^2,$$

where $C = |a_2| + |a_3| + \cdots + |a_n|$. Let $\delta = \frac{1}{2} - \frac{1}{2C}$, and let $0 < \xi < 1$ be the unique real number for which $g\left(\frac{1}{2}+\xi\right) = \frac{1}{2}+\delta$. We know that $\xi$ is unique, since $g$ is nondecreasing on the interval $\left[\frac{1}{2}, 1\right]$. Let $\frac{\delta}{\xi} = b > 1$ be the unique real number such that

$$g\left(\frac{1}{2}+\xi\right) = \frac{1}{2}+b\xi.$$

We also know from 3.1 that $g$ is concave down, and so for all $b < g'\left(\frac{1}{2}\right)$ we have

$$g\left(\frac{1}{2}+\varepsilon\right) \geq \frac{1}{2}+b\varepsilon,$$

for all $\varepsilon \in [0, \delta)$. Figure 1 depicts the choices of our variables $b$ and $\delta$ graphically. We must first check that our choice of $b$ is well - defined. That is, we need $g'\left(\frac{1}{2}\right) > 1$. Indeed, since we know that for $m \geq 1$

$$g'\left(\frac{1}{2}\right) = (2m+1)\binom{2m}{m}\left(\frac{1}{4}\right)^m > \left(\sum_{i=0}^{2m}\binom{2m}{i}\right)\left(\frac{1}{2}\right)^{2m} = 1.$$

Now, while $g^{(n)}\left(\frac{1}{2}+\varepsilon\right) - \frac{1}{2} < \delta$, we will have

$$g^{(n+1)}\left(\frac{1}{2}+\varepsilon\right) - \frac{1}{2} \geq b\left(g^{(n)}\left(\frac{1}{2}+\varepsilon\right) - \frac{1}{2}\right) \geq b^n\varepsilon.$$

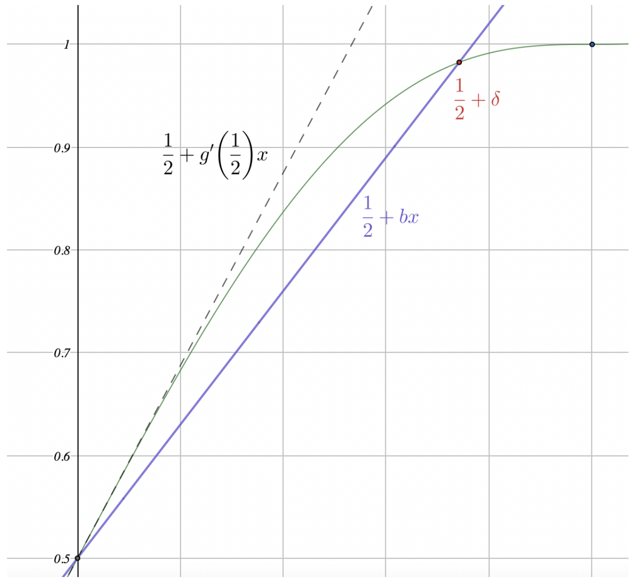


Figure 1: Depiction of $g\left(\frac{1}{2}+x\right)$ and $\frac{1}{2}+bx$ with intersection point $\left(\xi, \frac{1}{2}+\delta\right)$.

It immediately follows that $\mathcal{O}\left(|\log \varepsilon|\right)$ iterations are sufficient for the first step. Finally, invoking the fact that $1-g(1-x) \leq Cx^2$, we have that

$$
\begin{aligned}
1-g^{(n+1)}\left(\frac{1}{2}+\delta\right) &\leq C\left(1-g^{(n)}\left(\frac{1}{2}+\delta\right)\right)^2, \\
&\leq C^{1+2+\cdots+2^{n-1}}\left(\frac{1}{2}-\delta\right)^{2^n} = \frac{1}{C}\left(C\left(\frac{1}{2}-\delta\right)\right)^{2^n}.
\end{aligned}
$$

Notice that $C\left(\frac{1}{2}-\delta\right)<1$ for our choice of $\delta=\frac{1}{2}-\frac{1}{2C}$, therefore, we know that the right hand side goes to $\varepsilon^k$ with $\mathcal{O}\left(\log\left(k|\log \varepsilon|\right)\right)$ iterations, and so we are done. ☐

Note that the work we did for Lemma 3.4 is valid only for large regular trees, where we define a $k$-level regular tree to be $\mathcal{T}_k^{reg}=\bigcup_{j\leq k}\{1,2,\ldots,n\}^j$. In more colloquial terms, a regular tree is a tree which is "complete." A necessary and sufficient condition for our tree to be complete is that every leaf has the same number of nodes in its genealogy. Our next lemma will show the unintuitive fact that, with high probability, there exists a large regular $n$-tree within $\mathcal{T}(\mathbf{B}(t))$. We will also define for $\ell \in \mathbb{R}$, $\mathcal{T}_\ell^{reg}=\mathcal{T}_{\lceil \ell \rceil}^{reg}$.

**Lemma 3.5.** *Let $k\in\mathbb{N}$ and let $A=A(k)$ be as in Lemma 3.4. Then there exists $a_1=a_1(k)$ and $\varepsilon_1=\varepsilon_1(k)$ such that, for all $\varepsilon\in(0,\varepsilon_1)$ and $t\geq a_1\varepsilon^2|\log\varepsilon|$,*

$$
\mathbb{P}^{\varepsilon}\left[\mathcal{T}(\boldsymbol{B}(t)) \supseteq \mathcal{T}^{reg}_{A(k)|\log\varepsilon|}\right] \geq 1-\varepsilon^k.
$$

*Proof of Lemma 3.5.* It suffices to show that the probability of the complement is smaller than $\varepsilon^k$. Our method of proof will be to notice that for an $\ell$ level regular tree if $X_1+X_2+\cdots+X_\ell>t$, where $X_1,\ldots,X_\ell$ denote the exponential branching times from some chain of branching events that led to a given leaf of our regular tree, and $t$ is the current time, then the $\ell$ level tree does not exist yet at time $t$.

First, we will find some bounds on the distribution of the sum of the $n$ independent, exponentially distributed, branching times. To this end, we will apply Cramér's Theorem. Notice that for $X \sim \text{Exp}(1)$, we have

$$M_X(\lambda) = \mathbb{E}[e^{\lambda X}] = \begin{cases} \frac{1}{1-\lambda} & \lambda < 1 \\ \infty & \lambda \geq 1 \end{cases}$$

Then for $a \geq 1$ the Legendre transform is

$$\Psi^*(a) = \sup_{\lambda \geq 0}(\lambda a - \log M_X(\lambda)) = \sup_{0 \leq \lambda < 1}(\lambda a + \log(1-\lambda).$$

We can easily check that since the derivative is $a - \frac{1}{1-\lambda}$ and the second derivative is non positive everywhere, that the maximum is achieved at $\lambda = \frac{a-1}{a}$, and this maximum is $a - 1 - \log a$. Now denote by $S_n$ the sum $X_1 + X_2 + \cdots + X_n$, where the $X_i$ are i.i.d exponential branching times with parameter 1. Cramér's Theorem tells us that

$$\lim_{n\to\infty}\left(-\frac{1}{n}\log \mathbb{P}[S_n \geq na]\right) = \Psi^*(a) = a - 1 - \log a, \qquad a \geq 1.$$

In particular, we then know that

$$\lim_{n\to\infty}\left(\frac{1}{n}\log \mathbb{P}[S_n \geq na]\right) = \Psi^*(a) = 1 + \log a - a,$$

and so for sufficiently large $n$, we have

$$\frac{1}{n}\log \mathbb{P}[S_n \geq na] \leq \frac{3}{2} + \log a - a.$$

Now we will use this inequality to bound the probability that a given leaf of an $A|\log\varepsilon|$ level regular tree is not contained in our time-labelled $2m+1$-tree. Using a union bound, since our regular tree has $n^{\lceil A|\log\varepsilon|\rceil}$ children, $\mathbb{P}^{\varepsilon}\left[\mathcal{T}(\mathbf{B}(t)) \not\supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}\right]$ is bounded by $(2m+1)^{\lceil A|\log\varepsilon|\rceil}\mathbb{P}\left(\text{one leaf of } \mathcal{T}^{reg}_{A|\log\varepsilon|} \text{ is not in } \mathcal{T}(\mathbf{B}(t))\right)$. In terms of everything we have above, for times $t \geq a\varepsilon^2\lceil A|\log\varepsilon|\rceil$, this becomes

$$\begin{aligned}\mathbb{P}^{\varepsilon}\left[\mathcal{T}(\mathbf{B}(t)) \not\supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}\right] &\leq (2m+1)^{\lceil A|\log\varepsilon|\rceil}\mathbb{P}\left[\varepsilon^2 S_{\lceil A|\log\varepsilon|\rceil} \geq t\right], \\ &\leq (2m+1)^{\lceil A|\log\varepsilon|\rceil}\mathbb{P}\left[\varepsilon^2 S_{\lceil A|\log\varepsilon|\rceil} \geq a\varepsilon^2\lceil A|\log\varepsilon|\rceil\right], \\ &= \exp\left(\lceil A|\log\varepsilon|\rceil\left(\log(2m+1) + \frac{1}{\lceil A|\log\varepsilon|\rceil}\log\mathbb{P}\left[S_{\lceil A|\log\varepsilon|\rceil} \geq a\lceil A|\log\varepsilon|\rceil\right]\right)\right),\end{aligned}$$

where we must scale $S_n$ by $\varepsilon^2$, since in practice our branching times are exponential random variables with parameter $\frac{1}{\varepsilon^2}$. Now, by Cramér's Theorem, we can choose $\varepsilon_1(k)$ sufficiently small (corresponding to our sufficiently large $n$ above) so that for all $\varepsilon \in (0, \varepsilon_1)$ we have

$$\frac{1}{\lceil A|\log\varepsilon|\rceil}\log\mathbb{P}\left[S_{\lceil A|\log\varepsilon|\rceil} \geq a\lceil A|\log\varepsilon|\rceil\right] \leq -a + \frac{3}{2} + \log a.$$

Finally, we choose $a \geq 1$ sufficiently large so that $-a + \frac{3}{2} + \log a \leq -\log(2m+1) - \frac{k}{A}$, which, since the right hand side is constant and the left hand side is unbounded below and decreasing, is clearly possible. Putting this into our exponential above, we get

$$\mathbb{P}^{\varepsilon}\left[\mathcal{T}(\mathbf{B}(t)) \not\supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}\right] \leq \exp(-k|\log\varepsilon|) = \varepsilon^k,$$

for small $\varepsilon$ and $t \geq a\varepsilon^2\lceil A|\log\varepsilon|\rceil$. Letting $a_1 = a(A+1)$ then completes our proof. $\square$

The last thing we will do is bound the maximal displacement of particles in our branching process for small times. Let $N(s)$ denote the set of particles alive in the historic process $\mathbf{B}(s)$.

**Lemma 3.6.** *Let $k \in \mathbb{N}$, and let $a_1(k)$ be as in the lemma above. Then there exists $d_1(k)$ and $\varepsilon_1(k)$ such that for all $\varepsilon \in (0, \varepsilon_1(k))$ and all $s \leq a_1\varepsilon^2|\log\varepsilon|$, we have*

$$\mathbb{P}^{\varepsilon}_x[\exists i \in N(s) : |B_i(s) - x| \geq d_1(k)\varepsilon|\log\varepsilon|] \leq \varepsilon^k.$$

*That is, the probability that there is a particle that has moved further than $d_1(k)\varepsilon|\log\varepsilon|$ is very small.*

*Proof of Lemma 3.6.* We will denote the quantity $a_1\varepsilon^2|\log\varepsilon|$ by $\delta_1$. Let $Z$ be a $\mathcal{N}(0,1)$ distributed random variable. Note that $N(s)$ is itself a random variable. Therefore, we can condition on $N(s)$ and get

$$\begin{aligned}\mathbb{P}^{\varepsilon}_x\left[\exists i \in N(s) : |B_i(s) - x|\right] \geq d_1\varepsilon|\log\varepsilon|] &= \mathbb{E}^{\varepsilon}_x\left(\mathbb{1}_{\exists i \in N(s): |B_i(s)-x|] \geq d_1\varepsilon|\log\varepsilon|}\right), \\ &= \mathbb{E}^{\varepsilon}\left[\mathbb{E}_x\left(\mathbb{1}_{(\exists i \in N(s): |B_i(s)-x|] \geq d_1\varepsilon|\log\varepsilon|)}|N(s)\right)\right].\end{aligned}$$

Now we bound this by taking a sum over the indicators for each $i$ in $N(s)$ and using the linearity of expectation we have

$$\mathbb{P}^{\varepsilon}_x\left[\exists i \in N(s) : |B_i(s) - x|\right] \geq d_1\varepsilon|\log\varepsilon|] \leq \mathbb{E}^{\varepsilon}\left[\mathbb{E}_x\left(\sum_{i=0}^{|N(s)|}\mathbb{1}_{(|B_i(s)-x| \geq d_1\varepsilon|\log\varepsilon|)} \mid N(s)\right)\right].$$

But since our Brownian Motions are i.i.d, this becomes

$$\mathbb{E}^{\varepsilon}\left[|N(s)|\mathbb{E}_x\left[\mathbb{1}_{(|B(s)-x| \geq d_1\varepsilon|\log\varepsilon|)}\right]\right] = \mathbb{E}^{\varepsilon}\left[|N(s)|\right]\mathbb{E}_x\left[\mathbb{1}_{(|B(s)-x| \geq d_1\varepsilon|\log\varepsilon|)}\right].$$

Since our expectation on the right does not depend on $N(s)$, it is in fact just the probability that Brownian Motion in time $s$ travels further than $d_1\varepsilon|\log\varepsilon|$. Therefore, our desired probability is bounded above by

$$= \mathbb{E}^{\varepsilon}\left[|N(s)|\right]\mathbb{P}\left(\sqrt{2s}|Z| \geq d_1\varepsilon|\log\varepsilon|\right).$$

Now we finish our bounding with some computations. Recall from Proposition 2.1 that $\mathbb{E}^{\varepsilon}\left[|N(s)|\right] = e^{2ms/\varepsilon^2}$, and so for $s \leq \delta_1$ we have

$$\begin{aligned}\mathbb{P}^{\varepsilon}_x[\exists i \in N(s) : |B_i(s) - x|] \geq d_1(k)\varepsilon|\log\varepsilon|] &\leq \mathbb{E}^{\varepsilon}[|N(s)|]\mathbb{P}(\sqrt{2s}|Z| \geq d_1\varepsilon|\log\varepsilon|), \\ &= e^{\frac{2ms}{\varepsilon^2}}\mathbb{P}\left[\sqrt{2s}|Z| \geq d_1\varepsilon|\log\varepsilon|\right], \\ &\leq e^{\frac{2m\delta_1}{\varepsilon^2}}\mathbb{P}\left[\sqrt{2\delta_1}|Z| \geq d_1\varepsilon|\log\varepsilon|\right], \\ &= \frac{1}{\varepsilon^{2ma_1}}\mathbb{P}\left[\sqrt{2a_1}|Z| \geq d_1|\log\varepsilon|^{1/2}\right].\end{aligned}$$

But note that for $d_1$ large enough, we have that $(0, \infty)$,

$$\mathbb{P}\left[\sqrt{2a_1}|Z| \geq d_1|\log\varepsilon|^{1/2}\right] = \int_{\frac{d_1|\log\varepsilon|^{1/2}}{\sqrt{2a_1}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

$$\leq \int_{\frac{d_1|\log\varepsilon|^{1/2}}{\sqrt{2a_1}}}^{\infty} xe^{-x^2/2} = -e^{-x^2/2}\Big|_{\frac{d_1|\log\varepsilon|^{1/2}}{\sqrt{2a_1}}}^{\infty} = \exp\left(-\frac{d_1^2}{4a_1}|\log\varepsilon|\right),$$

which implies that our above probability is bounded above by

$$\frac{1}{\varepsilon^{2ma_1}} \exp\left(-\frac{d_1^2}{4a_1}|\log\varepsilon|\right) = \varepsilon^{\frac{d_1^2}{4a_1} - 2ma_1}.$$

Finally, we are done by choosing $d_1(k)$ large enough so that $\frac{d_1^2}{4a_1} - 2ma_1 \geq k$, since then, since $\varepsilon < 1$, this will be bounded above by $\varepsilon^k$, and completes our proof. $\square$

Now we are finally read to put our four lemmas together and prove Theorem 3.1.

### 3.4 Finishing the Proof of Theorem 3.1

*Proof of Theorem 3.1.* First, note that if the first statement is true, then by symmetry of our Brownian Motion the second one is also true, since $u(t, x) \leq \varepsilon^k$ is the same as $1 - u(t, x) \geq 1 - \varepsilon^k$. So, it suffices to prove just the first statement. That is, we will show that for all $t \in [0, T^*]$ and $\varepsilon \in (0, \varepsilon_1)$ and $z \geq c_1(k)\varepsilon|\log\varepsilon|$, for $\varepsilon_1$, $T^*$, and $c_1(k)$ chosen appropriately, we have $\mathbb{P}_z^\varepsilon[\mathbb{V}(\mathbf{B}(t)) = 1] \geq 1 - \varepsilon^k$.

For all $\varepsilon < \frac{1}{2}$ we define $z_\varepsilon$ to be a real number such that $\mathbb{P}[B_{T^*} \geq -z_\varepsilon] = \frac{1}{2} + \varepsilon$. Just to get a sense as to what $z_\varepsilon$ is, we can write that

$$\int_{-z_\varepsilon}^{\infty} \frac{1}{\sqrt{4\pi T^*}} e^{-x^2} dx = \frac{1}{2} + \varepsilon \implies \int_0^{z_\varepsilon} \frac{1}{\sqrt{4\pi T^*}} e^{-x^2/2} dx = \varepsilon.$$

But for small enough $\varepsilon$, $e^{-x^2/2}$ asymptotically approaches 1, and so our integral tells us that

$$\frac{z_\varepsilon}{\sqrt{4\pi T^*}} \sim \varepsilon \implies z_\varepsilon \sim \varepsilon\sqrt{4\pi T^*},$$

as $\varepsilon \to 0$. Let $\varepsilon_1(k)$ be sufficiently small so that Lemma 3.5 and Lemma 3.6 hold for $\varepsilon \in (0, \varepsilon_1(k))$. Let $d_1(k)$ be given by Lemma 3.6, and shrink $\varepsilon_1$ so that for all $\varepsilon \in (0, \varepsilon_1)$ we have

$$z_\varepsilon \leq d_1(k)\varepsilon|\log\varepsilon|.$$

Let $a_1(k)$ be such that Lemma 3.5 holds and let

$$\delta_1 = \delta_1(k, \varepsilon) = a_1(k)\varepsilon^2|\log\varepsilon|.$$

Now, we claim that $c_1(k) = 2d_1(k)$ is a sufficient constant so that our theorem holds for all $z \geq c_1\varepsilon|\log\varepsilon|$. If we have a small time $t \in (0, \delta_1)$ and $z \geq c_1\varepsilon|\log\varepsilon| = 2d_1\varepsilon|\log\varepsilon|$, then

$$\mathbb{P}_z^\varepsilon[\mathbb{V}(\mathbf{B}(t)) = 0] \leq \mathbb{P}_z^\varepsilon[\exists i \in N(t) \text{ such that } |B_i(t) - z| \geq c_1\varepsilon|\log\varepsilon|],$$

$$\leq \mathbb{P}_z^\varepsilon[\exists i \in N(t) \text{ such that } |B_i(t) - z| \geq d_1 \varepsilon |\log \varepsilon|] \leq \varepsilon^k,$$

where the first inequality holds since this means we must have at least one particle that is negative. The second inequality follows from the fact that $c_1 = 2d_1$, and the third follows from Lemma 3.6.

Now, suppose that $t \in [\delta_1, T^*]$ and $z \geq c_1 \varepsilon |\log \varepsilon|$. Let $\mathcal{T}_{\delta_1} = \mathcal{T}(\mathbf{B}(\delta_1))$ denote the time-labelled tree of Branching Brownian Motion up to time $\delta_1$. Define

$$p_{t-\delta_1}(z) = \mathbb{P}_z^\varepsilon[\mathbb{V}(\mathbf{B}(t - \delta_1)) = 1].$$

We write $\{\mathbf{B}(\delta_1) > z_\varepsilon\}$ for the event that $B_i(\delta_1) > z_\varepsilon$ for all $i \in N(\delta_1)$. Then, by the Markov Property of $\mathbf{B}$ applied at time $\delta_1$, and the monotonicity of our initial condition we have.

$$\mathbb{P}_z^\varepsilon[\mathbb{V}(\mathbf{B}(t)) = 1] = \mathbb{P}_z^\varepsilon\left[\mathbb{V}_{p_{t-\delta_1}(B_{\delta_1})}(\mathbf{B}(\delta_1)) = 1\right] \geq \mathbb{P}_z^\varepsilon\left[\left\{\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1)) = 1\right\} \cap \{\mathbf{B}(\delta_1) > z_\varepsilon\}\right],$$

$$\geq \mathbb{P}_z^\varepsilon\left[\left\{\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1)) = 1\right\} \cap \{\mathbf{B}(\delta_1) > d_1\varepsilon|\log\varepsilon|\}\right].$$

Now using the fact that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \implies \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$, from the fact that probabilities are bounded above by 1, we have that our expression is bounded below by

$$\geq \mathbb{P}_z^\varepsilon\left[\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1)) = 1\right] + \mathbb{P}(\mathbf{B}(\delta_1) > d_1\varepsilon|\log\varepsilon|) - 1.$$

Now, we bound the complement of $\mathbb{P}(\mathbf{B}(\delta_1) > d_1\varepsilon|\log\varepsilon|)$ by Lemma 3.6.

$$\mathbb{P}(\mathbf{B}(\delta_1) > d_1\varepsilon|\log\varepsilon|) = 1 - \mathbb{P}(\exists i \in N(t) \text{ such that } z - B_i(t) \geq d_1\varepsilon|\log\varepsilon|),$$

$$\geq 1 - \mathbb{P}(\exists i \in N(t) \text{ such that } |B_i(t) - z| \geq d_1\varepsilon|\log\varepsilon|) \geq 1 - \varepsilon^k,$$

where here we have used the fact that our initial starting point $z \geq c_1\varepsilon|\log\varepsilon| = 2d_1\varepsilon|\log\varepsilon|$. This implies that our initial expression is bounded below by

$$= \mathbb{P}_z^\varepsilon\left[\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1)) = 1\right] - \varepsilon^k.$$

From Lemma 3.3 we have that

$$p_{t-\delta_1}(z_\varepsilon) \geq P_{z_\varepsilon}[B_{t-\delta_1} \geq 0] \geq \frac{1}{2} + \varepsilon,$$

since $t - \delta_1 < T^*$.

Now, if each of $p_1, p_2, \ldots, p_{2m+1}$ is at least $\frac{1}{2}$, then we know that

$$g(p_1, \ldots, p_{2m+1}) \geq \frac{1}{2m+1}(p_1 + \cdots + p_{2m+1}) \geq \min\{p_1, \ldots, p_{2m+1}\}.$$

Therefore, if each leaf of $\mathcal{T}_{\delta_1}$ votes 1 independently with probability at least $\frac{1}{2} + \varepsilon$, and we have that $\mathcal{T}_{\delta_1} \supseteq \mathcal{T}^{reg}_{A|\log\varepsilon|}$, then each of the leaves of $\mathcal{T}^{reg}_{A|\log\varepsilon|}$ votes 1 independently with probability at least $\frac{1}{2} + \varepsilon$. Therefore

$$\mathbb{P}_z^\varepsilon[\mathbb{V}(\mathbf{B}(t)) = 1] \geq \mathbb{P}_z^\varepsilon\left[\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1)) = 1\right] - \varepsilon^k,$$

$$\geq \mathbb{P}_z^{\varepsilon}\left[\left(\mathbb{V}_{p_{t-\delta_1}(z_\varepsilon)}(\mathbf{B}(\delta_1))=1\right)\cap\left(\mathcal{T}_{\delta_1}\supseteq\mathcal{T}^{reg}_{A|\log\varepsilon|}\right)\right]-\varepsilon^k,$$
$$=\mathbb{P}_z^{\varepsilon}\left[\left(\mathbb{V}_{z_\varepsilon}(\mathcal{T}^{reg}_{A|\log\varepsilon|})=1\right)\cap\left(\mathcal{T}_{\delta_1}\supseteq\mathcal{T}^{reg}_{A|\log\varepsilon|}\right)\right]-\varepsilon^k,$$
$$\geq \mathbb{P}_z^{\varepsilon}\left[\mathbb{V}_{z_\varepsilon}(\mathcal{T}^{reg}_{A|\log\varepsilon|})=1\right]+\mathbb{P}_z^{\varepsilon}\left[\mathcal{T}_{\delta_1}\supseteq\mathcal{T}^{reg}_{A|\log\varepsilon|}\right]-1-\varepsilon^k,$$
$$\geq \mathbb{P}_z^{\varepsilon}\left[\mathbb{V}_{z_\varepsilon}(\mathcal{T}^{reg}_{A|\log\varepsilon|})=1\right]-2\varepsilon^k\geq g^{(\lceil A|\log\varepsilon|\rceil)}\left(\frac{1}{2}+\varepsilon\right)-2\varepsilon^k\geq 1-3\varepsilon^k,$$

and our proof is complete. □

# 4 PDE Proof

Consider the solution $u\,(t,x)$ to the equation

$$u_t=u_{xx}+f\,(u)\,. \tag{5}$$

In our case,

$$f\,(u)=g\,(u)-u=\sum_{j=0}^{m}\binom{2m+1}{j}u^{2m+1-j}\,(1-u)^j-u.$$

It is easy to observe that $f\,(0)=f\,(1)=0$. Moreover,

$$f'\,(u)=g'\,(u)-1=(2m+1)\binom{2m}{m}\left(u-u^2\right)^m-1.$$

This implies that $f$ is decreasing on $[0,a]$, increasing on $[a,b]$, and decreasing on $[b,1]$ where $a<b$ are the roots of $f'\,(u)=0$. So $f$ has only one root $\alpha$ in the interval $(0,1)$. Moreover $f'\,(0)\,,f'\,(1)<0$, $f\,(u)<0$ for $u\in(0,\alpha)$ and $f\,(u)>0$ for $u\in(\alpha,1)$. In particular, we have $g\,(u)=1-g\,(1-u)$ and thus $f\,(u)=-f\,(1-u)$. This implies that $f(\frac{1}{2})=-f(1-\frac{1}{2})=0$ and thus $f(\frac{1}{2})=0$.

We define traveling wave solutions to be solutions of form $u(t,x)=U(x-ct)$ where $c$ is the speed of the traveling wave. In particular, for our type of equations $U$ satisfies the ODE

$$U''+cU+f(U)=0.$$

With the above properties of $f$, we know that equation (5) is of the bistable type [Roq11], and so we know that there exists a traveling wave solution $U$ such that $U\,(-\infty)=0,U\,(\infty)=1$ and $U$ is monotonically increasing, which is illustrated in chapter 4.4 of [Fif79]. Moreover, there exists a unique speed $c$ and a unique travelling wave up to translation [Per15]. In the graph below, we can see how our choice of $f(u)$ behaves for several values of $n$.

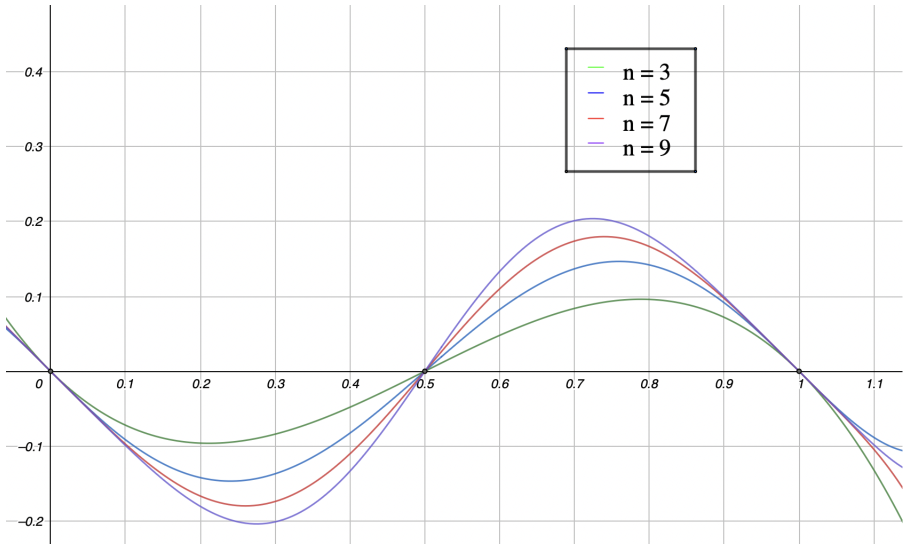


Figure 2: Nonlinearity $f(u)$ for several values of $n$.

We first prove that solutions to bistable equations converge to a translation of the traveling wave exponentially fast in time.

Consider the change of coordinates $z = x - ct$. We get the equation

$$N(v) = v_t - cv_z - v_{zz} - f(v) = 0, \tag{6}$$

where $v(z,t) = u(x,t) = u(z+ct,t)$.

**Lemma 4.1.** *With initial condition $u(0,x) = \phi(x) = \mathbb{1}_{(x\geq 0)}$, there exist constants $z_1, z_1', q_0, \mu$, the last two positive, such that*

$$U(z - z_1) - q_0 e^{-\mu t} \leq v(z,t) = u(x,t) \leq U\left(z + z_1'\right) + q_0 e^{-\mu t}. \tag{7}$$

*Here $U(x)$ is the traveling wave solution to (5) where $f(0) = f\left(\frac{1}{2}\right) = f(1) = 0$, $f'(0), f'(1) < 0$, and $f(u) < 0$ for $u \in (0,\alpha)$ while $f(u) > 0$ for $u \in (\alpha, 1)$.*

*Proof of Lemma 4.1.* We first consider the left side of the inequality, and we want to find $\zeta(t), q(t)$ such that

$$v_0(z,t) = \max\{0, U(z - \zeta(t)) - q(t)\}$$

is a subsolution.

First, let $q_0 > 0$ be any real number such that $\alpha < 1 - q_0$. Then we choose $z^*$ sufficiently large such that $U(z - z^*) - q_0 \leq \phi(z)$ for all $z$, which can be achieved by choosing $z^*$ such that $U(-z^*) = q_0$. In this case, we have $U(z - z^*) - q_0 \leq 0$ for all $z \leq 0$ by monotonicity of $U$ and also $U(z - z^*) - q_0 < U(z - z^*) \leq 1$ for all positive $z$.

Let

$$\psi(u,q) = \begin{cases} \frac{f(u-q)-f(u)}{q} & , q > 0 \\ -f'(u) & , q = 0. \end{cases}$$

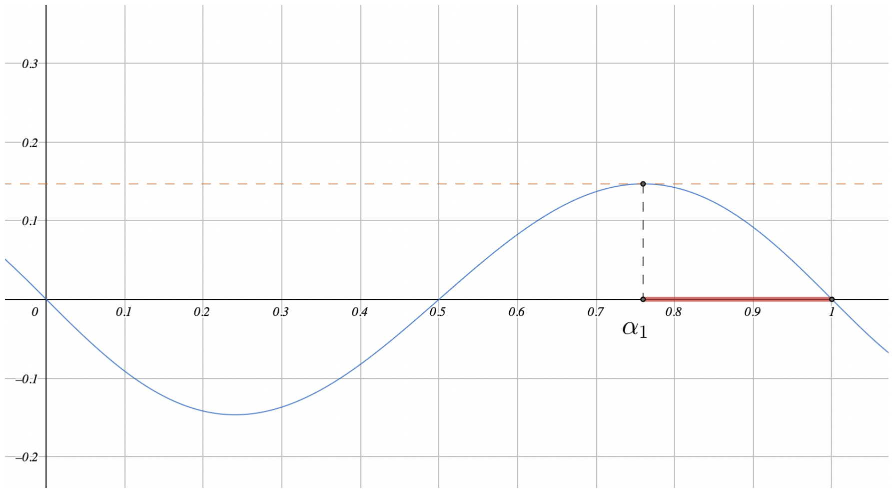


Figure 3: Critical point on $f(u)$ where the derivative begins to be negative. In particular, there is a small interval around 1 where the derivative starts to be negative.

For $0 < q \leq q_0$, there is $\alpha_1$ such that $\alpha \leq \alpha_1 < 1 - q_0 \leq 1 - q < 1$ implies $\psi(1, q) > 0$. This is because $f'(1) < 0$ suggests that there exist an interval $(\alpha_1, 1)$ where $f$ is decreasing. Also $\psi(1, 0) = -f'(1) > 0$. Notice that $\psi$ is continuous, there must exist some $\mu > 0$ such that $\phi(1, q) \geq 2\mu$ for $0 \leq q \leq q_0$. Also by continuity, there exist $\delta > 0$ such that $\psi(u, q) \geq \mu$ for $1 - \delta \leq u \leq 1$ and $0 \leq q \leq q_0$. So in this region we have

$$f(u - q) - f(u) \geq \mu q.$$

Let $\rho = z - \zeta(t)$, for $v_0 > 0$, equation (6) implies

$$N(v_0) = -\zeta'(t)\, U'(\rho) - cU'(\rho) - q'(t) - U''(\rho) - f(U - q).$$

Recall that $U'' + cU' + f(U) = 0$, we have

$$N(v_0) = -\zeta'(t)\, U'(\rho) - q'(t) + f(U) - f(U - q).$$

Therefore, for $U \in [1 - \delta, 1], q \in [0, q_0]$, we have

$$N(v_0) \leq -\zeta' U' - q' - \mu q \leq -\left(q' + \mu q\right),$$

provided that $\zeta' \geq 0$ since we already have $U' \geq 0$. Choosing $q(t) = q_0 e^{-\mu t}$, we have $N(v_0) \leq 0$ for $U \in (1 - \delta, 1)$, i.e. $v_0$ is a subsolution on this interval.

Similarly, we can run the same argument for $U \in [0, \delta], U \geq q$ using the fact that $f'(0) < 0$ by potentially choosing smaller $\delta$ and $\mu$ in the proof above.

The remaining case is when $U \in [\delta, 1-\delta]$. In this compact interval, we may assume $U'(z) \geq \beta$ for some $\beta > 0$.We also take $k > 0$ such that: $f(U) - f(U - q) \leq kq$ . Hence, we have

$$N(v_0) \leq -\beta\zeta' - q' + kq.$$

Setting $z_2 = \frac{-q_0(\mu+k)}{\mu\beta}$, $z_1 = z^* - z_2$, and $\zeta = z_1 + z_2 e^{-\mu t}$, we have

$$\zeta'(t) = \frac{-q' + kq}{\beta} = \frac{(\mu + k)\, q}{\beta} > 0.$$

Clearly, this implies $N(v_0) \le 0$ and we can now conclude that $v_0$ is a subsolution on the whole interval. Also notice that $\zeta(t)$ is an increasing function such that $\lim_{t\to\infty} \zeta(t) = z_1$. By monotonicity of $U$, we have

$$u(x,t) = v(z,t) \ge v_0(z,t) \ge U(z - z_1) - q(t) = U(z - z_1) - q_0 e^{-\mu t}.$$

Now consider the other direction where we want to find $\zeta(t), q(t)$ such that

$$v_1(z,t) = \min\{1, U(z + \zeta(t)) + q(t)\}$$

is a supersolution.

For any $q_0 > 0$ with $\alpha < 1 - q_0$, we choose $z^{**}$ with $U(z^{**}) = 1 - q_0$ to make sure that $U(z + z^{**}) + q_0 \ge \phi(z)$ for all $z$.

By the same argument, we can choose $\mu > 0$ such that there exist $\delta > 0$ with $f(u) - f(u+q) \ge \mu q$ for $u \in [1-\delta, 1]$, $u + q \le 1$ and $q \in [0, q_0]$. Hence, for such $u, q$ we have

$$\begin{aligned} N(v_1) &= \zeta' U' + q' - cU' - U'' - f(U+q), \\ &= \zeta' U' + q' + f(U) - f(U+q), \\ &\ge q' + \mu q = 0, \end{aligned}$$

where $q(t) = q_o e^{-\mu t}$ and we must have $\zeta' \ge 0$.

For $u \in [\delta, 1-\delta]$, there exist $\beta > 0$ such that $U' \ge \beta$ and $0 < k < \mu$ such that $f(u) - f(u+q) \ge kq$. Then we have

$$N(v_1) \ge \beta\zeta' + q' + kq.$$

Setting $z_2' = \frac{-q_0(\mu-k)}{\mu\beta}$, $z_1' = z^{**} - z_2'$, and $\zeta(t) = z_1' + z_2' e^{-\mu t}$, we have

$$\zeta'(t) = \frac{(\mu - k)\, q}{\beta} > 0.$$

This implies $N(v_1) \ge 0$ on the whole interval. Since $\zeta$ is increasing, monotonicity of $U$ implies that

$$u(x,t) = v(z,t) \le v_1(z,t) \le U\left(z + z_1'\right) + q_0 e^{-\mu t}.$$

$\square$

Now we focus on the traveling wave solutions to our specific equations, and we can show that the speed $c$ of our traveling waves is zero, which means that our solutions $u(t,x)$ will converge to the steady-state solutions.

We begin with a small lemma

**Lemma 4.2.**

$$\int_0^1 (g(x) - x)\, dx = 0,$$

*where is as before,* $g = \sum_{j=0}^{m} \binom{2m+1}{j} x^{2m+1-j}(1-x)^j$

*Proof of Lemma 4.2.* It suffices to show that $\int_0^1 g(x)\, dx = \frac{1}{2}$.

$$\int_0^1 g(x)\, dx = \int_0^1 \sum_{j=0}^{m} \binom{2m+1}{j} x^{2m+1-j} (1-x)^j\, dx = \sum_{j=0}^{m} \binom{2m+1}{j} \int_0^1 x^{2m+1-j} (1-x)^j\, dx,$$

$$= \sum_{j=0}^{m} \binom{2m+1}{j} \beta(2m+2-j, j+1) = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{\Gamma(2m+2-j)\Gamma(j+1)}{\Gamma(2m+3)},$$

$$= \sum_{j=0}^{m} \binom{2m+1}{j} \frac{(2m+1-j)!j!}{(2m+2)!} = \frac{1}{2m+2} \sum_{j=0}^{m} \frac{\binom{2m+1}{j}}{\binom{2m+1}{j}} = \frac{1}{2}.$$

Alternatively, note that $g(1-x) = 1 - g(x)$. hence $\int_0^1 = \int_0^{1/2} g(x) + \int_0^{1/2} (1-g(x))dx = \int_0^{1/2} 1dx = \frac{1}{2}$, as desired. □

This lemma allows us to state the following theorem:

**Theorem 4.1.** *Traveling wave solutions to*

$$u_t = u_{xx} + f(u),$$

*where* $f(u) = g(u) - u$ *have speed* $c = 0$.

*Proof of Theorem 4.1.* Recall that a traveling wave solution is of the form $u(x - ct)$. Therefore, rewritten in terms of our traveling wave, our equation reads

$$-cu' = u'' + g(u) - u.$$

Now, we can multiply by $u'$ and integrate over the real line to get

$$0 = \int_0^1 (g(u) - u)du + \frac{1}{2}(u')^2\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} c(u')^2.$$

Recall our boundary conditions, which state that $u'(-\infty) = u'(\infty) = 0$. Therefore

$$0 = \int_0^1 (g(u) - u)\, du + \int_{-\infty}^{\infty} c\,(u')^2 \implies c = -\frac{\int_0^1 (g(u)-u)du}{\int_{-\infty}^{\infty} (u')2}.$$

In particular from Lemma 4.2, we know that $\int_0^1 (g(u) - u)du = 0$. Note additionally that the denominator is positive, since $(u')^2 \geq 0$ and $u' \not\equiv 0$. Hence we have $c = 0$, as desired. □

Now we know that our traveling wave solutions satisfies the ODE

$$U'' + f(U) = 0.$$

Again, this means that the travelling wave solution is in fact a steady state solution to equation (4.1).

Let

$$W(u) = \left| \int_0^u f(v)\, dv \right| = \begin{cases} -\int_0^u f(v)dv & v \in \left(0, \frac{1}{2}\right) \\ -\int_0^{1-u} f(v)dv & v \in \left(\frac{1}{2}, 1\right) \end{cases}$$

we have $W'(u) = -f(u)$ and also

$$\frac{d}{dx}\left(u'^2 - 2W(u)\right) = 2u'u'' - 2W'(u)\, u' = 0.$$

This implies $u'^2 - 2W(u) = \lambda$, and we have that $\lambda = 0$ by plugging in the boundary condition for $U(-\infty) = 0$, $U'(-\infty) = 0$ since $W(0) = 0$.

We have that $u' = \sqrt{2W(u)}$. From Theorem 4.15 of [Fif79], we know that taking the boundary conditions $\lim_{x \to -\infty} W(x) = 0$, $\lim_{x \to +\infty} W(x) = 1$ produces a unique traveling wave up to translation. WLOG, we may choose the translation of our travelling wave to be the one such that $U(0) = \frac{1}{2}$. Then, separating variables and integrating, we have

$$\int_{1/2}^u \frac{dv}{\sqrt{2W(v)}} = \int_0^x 1 dy \implies \int_{1/2}^u \frac{dv}{\sqrt{2W(v)}} = x.$$

Now, we will set $F(u) = \int_{1/2}^u \frac{dv}{\sqrt{2W(v)}} = x$. Notice that $F$ is an increasing function since it is the integral of a positive quantity. Therefore, to show that $U(x) \geq 1 - \varepsilon^k$ for all $x \geq c_1(u)|\log \varepsilon|$, it suffices to show that

$$F(U(x)) = x \geq F(1 - \varepsilon^k).$$

for all $x \geq c_1(u)|\log \varepsilon|$ Therefore, it suffices to show that $c_1(k)|\log \varepsilon| \geq F(1 - \varepsilon^k)$. To show this comparison, we introduce a lemma.

**Lemma 4.3.**

$$W(v) = \left| \int_0^v f(w)dw \right| > \left| \int_0^v f_{AC}(w)dw \right| = W_{AC}(v),$$

*where $f_{AC}(w) = 2w^3 - 3w^2 - w$ denotes the nonlinearity of the Allen-Cahn equation.*

*Proof of Lemma 4.3.* If $v \in \left(0, \frac{1}{2}\right)$ we have that:

$$W(v) = \int_0^v -f(w)dw = \int_0^v f(1-w)dw = \int_{1-v}^1 f(z)dz. \tag{8}$$

If we denote by $f_{2m+1}(u) = g_{2m+1}(u) - u$, the nonlinearity corresponding to majority voting with $2m+1$ branchings, then if $u \in \left(\frac{1}{2}, 1\right)$ we have $g_{2m+1}(u) \geq g_{2m-1}(u)$ from Condorcet's Jury Theorem 3.2, and so in particular $f_{2m+1}(u) \geq f_3(u) = f_{AC}(u)$. Now, since $1 - v \in \left(\frac{1}{2}, 1\right)$ in Equation (8), we get that $W(v) = \int_{1-v}^v f_{2m+1}(z)dz \geq \int_{1-v}^v f_{AC}(z)dz = W_{AC}(v)$. □

We may compute that

$$W_{AC}(v) = \int_0^v (2w^3 - 3w^2 + w)dw = \frac{v^4}{2} - v^3 + \frac{v^2}{2},$$

for $v \in \left(0, \frac{1}{2}\right)$ and analagously for $v \in \left(\frac{1}{2}, 1\right)$ we get

$$W_{AC}(v) = \int_0^{1-v} (2w^3 - 3w^2 + w)dw = \frac{(1-v)^4}{2} - (1-v)^3 + \frac{(1-v)^2}{2}.$$

Now we can bound

$$F(1-\varepsilon^k) = \int_{1/2}^{1-\varepsilon^k} \frac{dv}{\sqrt{2W(v)}} \leq \int_{1/2}^{1-\varepsilon^k} \frac{dv}{\sqrt{2W_{AC}(v)}},$$

and so it suffices to show that

$$\int_{1/2}^{1-\varepsilon^k} \frac{dv}{\sqrt{2W_{AC}(v)}} \leq c_1(k)|\log \varepsilon|.$$

For $v \in \left(\frac{1}{2}, 1\right)$ we have

$$\int_{1/2}^{1-\varepsilon^k} \frac{dv}{\sqrt{2W_{AC}(v)}} = \int_{1/2}^{1-\varepsilon^k} \frac{dv}{\sqrt{(1-v)^2v^2}} = \int_{1/2}^{1-\varepsilon^k} \frac{dv}{v(1-v)} = \left[\log \frac{v}{1-v}\right]_{1/2}^{1-\varepsilon^k},$$

$$= \log(1-\varepsilon^k) - k \log \varepsilon \leq -k \log \varepsilon \leq B_1(k)|\log \varepsilon|,$$

from which we see it suffices to choose $B_1(k) = k$.

The last thing we want to do is show that we can choose our constants so that the subsolution proof works, and so that our constants are suitably small. We choose $q_0 = \varepsilon^k$ in the subsolution proof so that $\frac{1}{2} < 1 - q_0 < 1$. Then, we choose $z^*$ such that

$$U(-z^*) = q_0 = \varepsilon^k.$$

Now, we define

$$\phi(u, q) = \begin{cases} \frac{f(u-q)-f(u)}{q} & q > 0 \\ -f'(u) & q = 0 \end{cases}.$$

We have that

$$\phi(1, q) = \frac{f(1-q) - f(1)}{q} > 0,$$

for $q \in [0, q_0]$, since $f$ is decreasing for all $u \in [\alpha_1, 1]$, where $\alpha_1 = \frac{1}{2} + \frac{1}{2\sqrt{3}}$ is chosen appropriately as the threshold for $f'_{AC}(u)$ to be negative in $[\alpha_1, 1]$ (see Figure 3). Next, we see that

$$f'_{2m+1}(u) \leq f'_{2m-1}(u),$$

for all $u \in [\alpha_1, 1]$. To see this, we introduce another lemma.

**Lemma 4.4.** *Let $m \geq 1$. Then*

$$f'_{2m+1}(u) \leq f'_{2m-1}(u),$$

*for $u \in \left[\frac{1}{2} + \frac{1}{2\sqrt{3}}, 1\right]$.*

*Proof of Lemma 4.4.* From our computations in Lemma 3.1 we want to show that on this interval we have

$$(2m+1)\binom{2m}{m}(u-u^2)^m - 1 \leq (2m-1)\binom{2m-2}{m-1}(u-u^2)^{m-1} - 1,$$

$$\iff u - u^2 \leq \frac{2m-1}{2m+1} \cdot \frac{m^2}{2m(2m-1)} = \frac{m}{4m+2} = \frac{1}{4} - \frac{1}{8m+4}.$$

Since the right hand side is increasing in $m$, it suffices for $u - u^2 \leq \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ on this interval. Note that $u - u^2 = \frac{1}{6}$ has roots $\frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, and since it has negative end behavior, we have that $u - u^2 - \frac{1}{6}$ is nonpositive on the interval $\left[\frac{1}{2} + \frac{1}{2\sqrt{3}}, 1\right]$, and so our inequality holds. Therefore, $f'_{2m+1}(u) \leq f'_{AC}(u)$ for all $u \in [\alpha_1, 1]$. $\square$

By the Mean Value Theorem, there is a real number $\xi_1$ so that

$$\frac{f(1-q) - f(1)}{q} = -f'(\xi_1),$$

for $\xi_1 \in (1-q, 1)$. But by Lemma 4.4, $-f'(\xi_1) \geq -f_{AC}(\xi_1)$ for $\xi_1 \in [\alpha_1, 1]$ and so it suffices that

$$1 - q > \alpha_1.$$

In other words, we need $q_0 = \varepsilon^k < 1 - \alpha_1 = \frac{1}{2} - \frac{1}{2\sqrt{3}}$. Now, we look at $\mu$ such that

$$\phi(1, q) \geq 2\mu,$$

for all $q \in [0, q_0]$. We can choose $\delta > 0$ such that

$$\phi(u, q) \geq \mu,$$

for all $u \in [1-\delta, 1]$ and all $q \in [0, q_0]$. Recall that

$$\phi(u, q) = \frac{f(u-q) - f(u)}{q},$$

and so there exists $\xi_2 \in [u-q, u]$ such that

$$\frac{f(u-q) - f(u)}{q} = -f'(\xi_2).$$

We have $u \in [1-\delta, 1]$ so that $\xi_2 \in [1-\delta-q_0, 1]$. We choose $\delta$ such that $1 - \delta - q_0 > \alpha_1$ or equivalently that $0 < \delta < \frac{1}{2} - \frac{1}{2\sqrt{3}} - \varepsilon^k$. It will suffice to choose $\delta = \sqrt[4]{\varepsilon^k}$ as long as

$$\sqrt[4]{\varepsilon^k} + \varepsilon^k < \frac{1}{2} - \frac{1}{2\sqrt{3}}.$$

Let $y = \sqrt[4]{\varepsilon^k}$. Then it suffices for

$$y + y^4 \leq 2y \leq \frac{1}{2} - \frac{1}{2\sqrt{3}}.$$

In other words, $y \leq \frac{1}{4} - \frac{1}{4\sqrt{3}} \implies \varepsilon^k \leq \left(\frac{1}{4} - \frac{1}{4\sqrt{3}}\right)^4$ suffices.

Now $\phi(u, q) \geq \min\left(-f'_{AC}(\xi_2)\right) := \mu$ for all $\xi_2 \in [1 - \delta - \varepsilon^k, 1]$ since then $\xi_2 \in [a_1, 1]$ by Lemma 13. But since $-f'_{AC}(u) = 6u^2 - 6u + 1$, is increasing on $[1 - \delta - \varepsilon^k, 1]$, we have that the minimum occurs at $1 - \delta - \varepsilon^k$. That is, we choose $\mu$ so that

$$\mu = -f'_{AC}(1 - \delta - \varepsilon^k) = 6(1 - \delta - \varepsilon^k)^2 - 6(1 - \delta - \varepsilon^k) + 1 > 0.$$

Also, we want $\mu$ to satisfy that

$$\phi(u, q) \geq \mu,$$

for $u \in [0, \delta]$. Setting $u - q = v$ we need $v \geq 0$ so that

$$\frac{f(v) - f(v + q)}{q} \geq \mu,$$

for $q \in [0, q_0]$. Applying the Mean Value Theorem again, there is $\xi_3 \in [v, v + q]$ so that

$$\frac{f(v) - f(v + q)}{q} = -f'(\xi_3),$$

for $\xi_3 \in [0, \delta]$. From Lemma 4.4 we have that $-f'(\xi_3) \geq -f'_{AC}(\xi_3)$ for $\xi_3 \in [a_1, 1]$. We also have that

$$f(u) = -f(1 - u) \qquad \text{and} \qquad f'(u) = f'(1 - u) \implies f'(\xi_3) = f'(1 - \xi_3),$$

and $1 - \xi_3 \in \left[\frac{1}{2} + \frac{1}{2\sqrt{3}}, 1\right]$ and $1 - \xi_3 \in [1 - \delta, 1]$. Therefore, $-f'_{AC}(\xi_3) = -f'_{AC}(1 - \xi_3) \geq \mu$, and so $\mu$ still holds as the desired lower bound for $\phi(u, q)$ with $u \in [0, \delta], u \geq q$ for our $\delta$.
Finally, for the values in the interval $[\delta, 1 - \delta]$, we have to choose $\beta$ such that $U'(z) \geq \beta > 0$ for $U \in [\delta, 1 - \delta]$ and $k$ such that $f(u) - f(u - q) \leq kq$.

We have that $U' = \sqrt{2W(u)} \geq \sqrt{2W_{AC}(u)}$ and so we can choose $\beta$ such that

$$\sqrt{2W_{AC}(u)} \geq \beta,$$

for $u \in [\delta, 1 - \delta]$. We have

$$2W_{AC}(u) \geq \beta^2 \iff \left|\int_0^u f_{AC}(w)dw\right| \geq \frac{\beta^2}{2},$$

so $2\left|u^3 - \frac{u^4}{2} - \frac{u^2}{2}\right| \geq \beta^2$ for $u \in [\delta, 1 - \delta]$. Since this is just $u - u^2 \geq \beta$, we have that $\beta = \delta - \delta^2$ is our minimum.

Now we show that $k$ can be chosen to be $f'\left(\frac{1}{2}\right)$. That is, we need

$$\frac{f(u) - f(u - q)}{q} \leq k,$$

for all $u \in (\delta, 1-\delta)$, $u \geq q$. Indeed, notice that

$$f'(x) \leq f'\left(\frac{1}{2}\right),$$

since

$$f'(u) = (2m+1)\binom{2m}{m}(u-u^2)^m - 1 = (2m+1)\binom{2m}{m}\left(\frac{1}{4} - \left(u - \frac{1}{2}\right)^2\right)^m - 1,$$

which, by the Trivial Inequality, clearly achieves its maximum at $u = \frac{1}{2}$ for $u$ in the interval $[0,1]$. Thus we can choose $k = f'(\frac{1}{2})$. Finally, for the subsolution proof, we have then that

$$u(t,x) \geq U(x-z_1) - q_0 e^{-\mu t} \geq U(x-z_1) - \varepsilon^k,$$

and we have shown that $U(x-z_1) \geq 1-\varepsilon^k$ as long as $x - z_1 > B_1(k)|\log\varepsilon|$. Now we will choose $c_1(k)$ so that if $x > c_1(k)|\log\varepsilon|$ then $x > z_1 + B_1(k)|\log\varepsilon|$. It suffices to control $z_1$ where

$$z_1 = z^* + q_0\left(\frac{1}{\beta} + \frac{k}{\mu\beta}\right).$$

Note that

$$\frac{q_0}{\beta}\left(1 + \frac{k}{\mu}\right) = \frac{\varepsilon^k}{\sqrt{(\delta-\delta^2)^2}}\left(1 + \frac{f'\left(\frac{1}{2}\right)}{6(1-\delta-\varepsilon^k)^2 - 6(1-\delta-\varepsilon^k) + 1}\right).$$

For $\delta = \sqrt[4]{\varepsilon^k}$, we have

$$\frac{q_0}{\beta}\left(1 + \frac{k}{\mu}\right) = \frac{\sqrt[4]{\varepsilon^{3k}}}{1-\sqrt[4]{\varepsilon^k}}\left(1 + \frac{f'\left(\frac{1}{2}\right)}{6(1-\sqrt[4]{\varepsilon^k}-\varepsilon^k)^2 - 6(1-\sqrt[4]{\varepsilon^k}-\varepsilon^k) + 1}\right),$$

and so this is a term of order $\mathcal{O}(\varepsilon^{3k/4})$ and can thus be bounded above by $B_2(k)|\log\varepsilon|$ for some $B_2(k)$. Now, for $z^*$, we chose it so that

$$U(-z^*) = \varepsilon^k,$$

or equivalently that

$$F(U(-z^*)) = -z^* = F(\varepsilon^k) = \int_{1/2}^{\varepsilon^k} \frac{1}{\sqrt{2W(v)}}dv \implies z^* = \int_{\varepsilon^k}^{1/2} \frac{1}{\sqrt{2W(v)}}dv.$$

Again, we bound $\sqrt{2W(v)} \geq \sqrt{2W_{AC}(v)}$ to get that

$$z^* \leq \int_{\varepsilon^k}^{1/2} \frac{1}{\sqrt{2W_{AC}(v)}}dv = \int_{\varepsilon^k}^{1/2} \frac{1}{v(1-v)}dv = \left[\log\frac{v}{1-v}\right]_{\varepsilon^k}^{1/2} = -\log\left(\frac{\varepsilon^k}{1-\varepsilon^k}\right),$$

$$= -k\log\varepsilon + \log(1-\varepsilon^k) \leq -k\log\varepsilon.$$

Thus, we have bound $z^*$ above by $-k\log\varepsilon$ so that $z^* \leq B_3(k)|\log\varepsilon|$ with $B_3(k) = k$. Finally, taking $c_1(k) = B_1(k) + B_2(k) + B_3(k)$, we have that if $z > c_1(k)|\log\varepsilon|$ then $z > z_1 + B_1(k)|\log\varepsilon|$ and thus

$$u(t,x) > 1 - 2\varepsilon^k.$$

We will note that this works for $\varepsilon^k \in (0, \varepsilon_1(k))$ where $\varepsilon_1(k) = \left(\frac{1}{4} - \frac{1}{4\sqrt{3}}\right)^4$. We may even choose $\varepsilon'$ such that $2\varepsilon^k = \varepsilon'^k$ and get that this proof works for $\varepsilon'^k \in \left(0, \frac{\varepsilon_1(k)}{2}\right)$.

The result on the upper bound on $u(t, x)$ follows similarly, using the supersolution.

**Remark 4.1.** *The above computation is very simple for the case $n = 3$, i.e. the case of ternary branching, since we have an explicit expression for the steady state solution when our nonlinearity is $f(u) = u(1-u)(2u-1)$. In that case we have the traveling wave $U(x) = \frac{1}{2}(1+\tanh(\frac{x}{\sqrt{2}}))$ and* $\tanh x = 1 + \mathcal{O}(e^{-2x})$*, so we can easily see that $U(x) \geq 1 - \varepsilon^k$ if $x \geq d(k)|\log \varepsilon|$.*

# 5 Simulations

PDEs which we cannot explicitly solve are generally very difficult to understand visually. Usually, we can at best say things about the solution's long-term behavior and convergence. In order to get a better handle on how the solutions to our PDEs behaved, we created simulations. This work allowed us to confirm surprising results we had proved and make conjectures about other behaviors of our solutions.

## 5.1 Graphing Branching Brownian Motion

We begin with the graphing of the standard Brownian Motion of a single particle. There are many ways in which mathematicians have discussed simulating Brownian Motion. We have used a very popular option: sampling many times from a normal distribution.

Suppose we want to graph the Brownian Motion path of a single particle starting from $x$ on a time interval of length $t$. We know that the particle's final location $y$ will be normally distributed by $y \sim \mathcal{N}(x, t)$. The problem we have is that rather than simulating a full Brownian Motion path, we have only the particle's starting and ending points. We can rectify this by splitting this time interval into many smaller intervals and sample from the normal distribution over each interval. If a time interval of length $t$ is split into $10,000$ smaller intervals, then we can sample from the normal distribution $\mathcal{N}\left(0, \frac{t}{10{,}000}\right)$ for each of these intervals and take the cumulative sum to depict our Brownian Motion path. It is well-known that a sum of $10,000$ normal distributions that follow $\mathcal{N}\left(0, \frac{t}{10{,}000}\right)$ results in a normal distribution of $\mathcal{N}(0, t)$, which is the same as our original distribution. The resulting graph is shown as Figure 4.

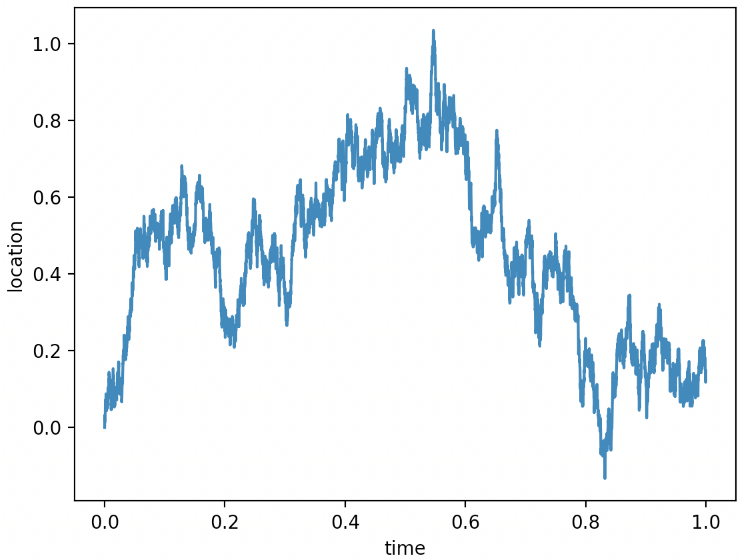


Figure 4: Graph of a particle undergoing Brownian Motion.

From this process of graphing a path of Brownian Motion, we can introduce the branching mechanism to graph Branching Brownian Motion. To graph the ternary BBM discussed in earlier sections, we start with a single particle and sample from the exponential distribution to determine the length of our time interval. After graphing the Brownian Motion along this interval, we restart the process using this endpoint point as the starting point for 3 new particles, each of which follows its own path of Brownian Motion. By plotting this on a graph, we get a nice visual representation of the process as seen in Figure 5.

### 5.2 Monte Carlo Trials

To exhibit our probabilistic model for the solution to the Allen-Cahn, we simulate the actual process of Branching Brownian Motion and use this data to propagate the votes of each individual particle back to the original starting particle and determine the original particle's vote. We run this simulation many times for various starting points to estimate the probability of voting 1 conditioned on the original particle's starting location.

To actually simulate the ternary branching voting model, we utilized a ternary tree data structure. We use individual nodes to represent particles, and each node has four data points associated with it: three nodes representing the three children into which the particle splits, and one variable representing that particle's position in space.

We first construct the entire tree starting with one particle with location $x$. We then sample from the exponential distribution to determine the time $t$ at which the particle branches. We then sample from the normal distribution $\mathcal{N}(x, t)$ to determine the particle's location at the end of this time interval. We then create three new children nodes representing the three new branching particles; each child has a starting location equal to the ending location of the parent.

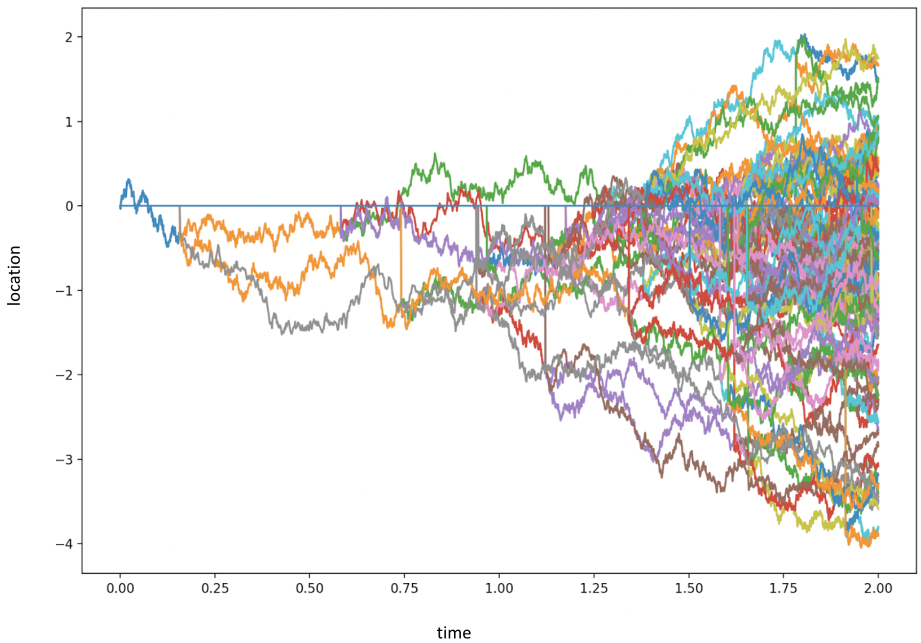


Figure 5: Ternary Branching Brownian Motion graph.

We then repeat this procedure for each of the children, and the process continues as more and more particles are born until we reach a stopping time established at the beginning.

Once we have constructed the tree, we can use vote propagation to determine the vote of the original particle. For each particle in the most recent generation (the "leaf" nodes), we determine their vote based on the initial condition, $\mathbb{1}_{(x \geq 0)}$. Then, the parent node will vote the same as the majority of its children. By repeating this process, the votes of the current generation propagate back to the original particle.

This process can be run many times for a specific starting location to experimentally determine the original particle's probability of voting 1 in that location. In our case, we ran this process with a stopping time of 5 seconds and an exponential rate parameter of 1. We used 600 equally spaced data points on the interval $[-3, 3]$ and ran the simulation 500 times for each data point. For each point, we recorded the number of times that the particle voted 1 and divided it by 500 to calculate the desired probability. Then, we plotted the data points against their estimated probability and achieved the graph shown in Figure 6.

We can observe the following properties of our graph:

- the probability of voting 1 given that we start from the origin is approximately 0.5, which we would expect.
- As $x$ becomes more positive, the probability that the original particle votes 1 decreases to 0, and as $x$ becomes more negative, it increases to 1.

Note that the solution to Allen-Cahn is monotonically decreasing. However, our graph is not. While the overall shape of the graph is that of a decreasing curve, restricting our curve to smaller

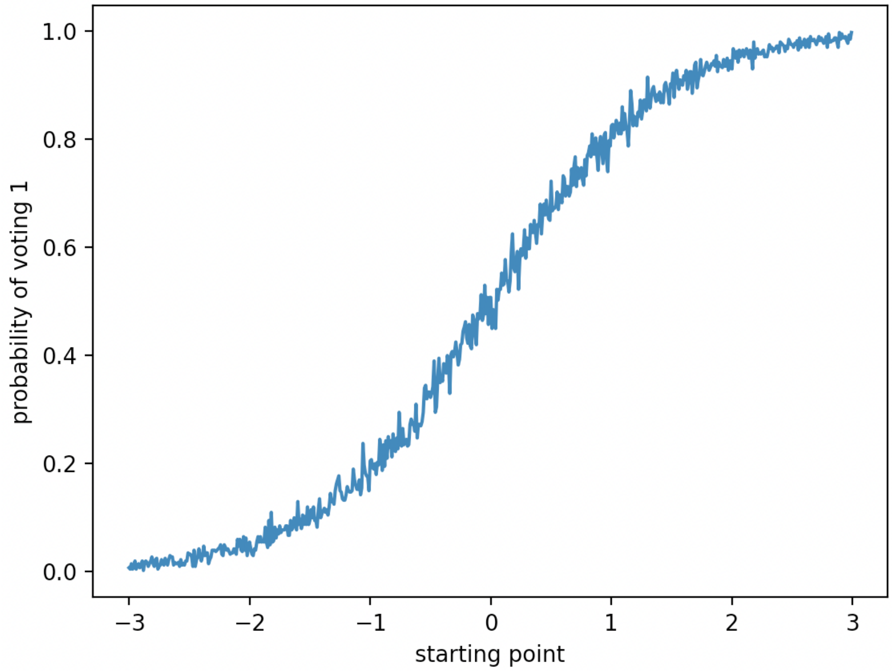


Figure 6: Probabilistic graph of Allen-Cahn Solution with the initial condition $\mathbb{1}_{(x \geq 0)}$, with Monte-Carlo trials.

intervals reveals a much more jagged structure. This is due to the natural, random variation that comes with sampling from exponential and normal distributions. Had we run the simulation more than 500 times for each data point or increased the stopping time, we should expect to see a smoother curve, as the experimental probability converges to its theoretic value.

### 5.3 Allen-Cahn PDE and Results

While the probabilistic graph gives us the general shape of the solution to Allen-Cahn,it is much less accurate than if we were to numerically simulate the PDE itself.

To estimate more accurate values for the PDE and to graph it, we refer to Euler's method, utilizing Laplacian intervals to slowly converge towards the actual solution. The resulting graph when using Euler's method as well as the initial condition $\mathbb{1}_{(x \geq 0)}$, is much more accurate and evidently monotonic than the Monte-Carlo simulated graph appeared to be.

Using this simulation for the Allen-Cahn PDE, we are able to find explicit values for $c_1(k)$ as in Theorem 3.1. We denote by $c_1(k)$ the constants for the $\geq 1 - \varepsilon^k$ case and $c_2(k)$ the constant for the $\leq \varepsilon^k$ case. As expected, we found the $c_1(k)$ and $c_2(k)$ values to be identical. In Figure 8 below, we depict an array of values for the case $\varepsilon = 0.1$.

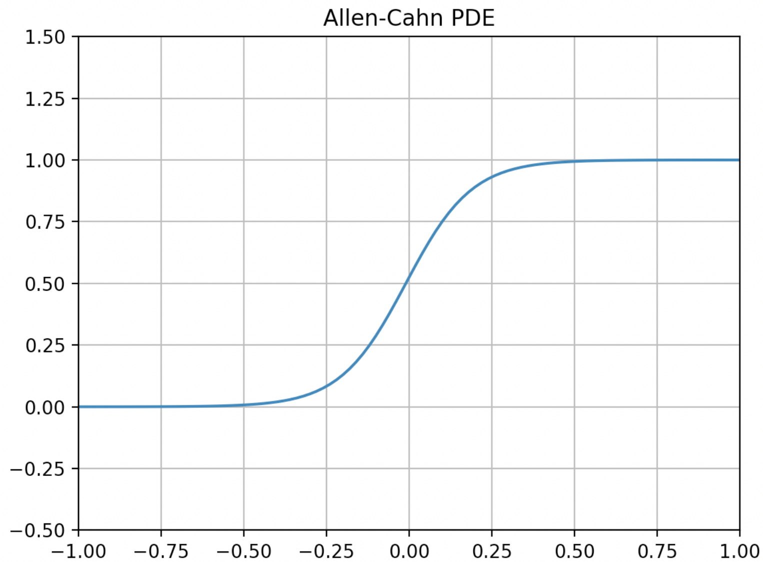


Figure 7: The graph illustrates the behavior of the solution under the initial condition $\mathbb{1}_{(x \geq 0)}$, and we can see that it resembles the traveling wave solution of form $U(x) = \frac{1}{2}\left(1 + \tanh\left(\frac{x}{\sqrt{2}}\right)\right)$.

| $k$ | $c_1(k)$ | $c_2(k)$ |
|---|---|---|
| 1 | 1.04230676 | 1.04230676 |
| 2 | 2.08461351 | 2.08461351 |
| 3 | 3.12692027 | 3.12692027 |
| $\vdots$ | $\vdots$ | $\vdots$ |

Figure 8: Table of values for $\varepsilon = 0.1$

# 6 Contributions

Yunchu Dai proposed the idea of generalizing Theorem 2.5 of [Ali16] and worked primarily on the PDE proof of the theorem. Bradley Moon worked primarily on the probability proof of Theorem 2.5. Overall, the two proofs were a result of collaborative efforts by Bradley Moon and Yunchu Dai with advice and guidance from Alexandra Stavrianidi. Simulations were designed and implemented by Taran Kota.

# References


[McK75] Henry McKean. "Application of Brownian Motion to the Equation of Kolmogrov - Petrocskii - Piskunov". In: *Communications on Pure and Applied Mathematics* 28 (1975), pp. 323–331. URL: https://onlinelibrary.wiley.com/doi/10.1002/cpa.3160280302.

[Fif79] Paul Fife. *Mathematical Aspects of Reacting and Diffusing Systems*. Springer-Verlag, 1979. ISBN: 3540091173.

[Dan03] Luba Sapir Daniel Berend. "Monotonicity in Condorcet Jury Theorem". In: (2003). URL: https://www.jstor.org/stable/41106652.

[Roq11] Jimmy Garnier Thomas Giletti Francois Hamel Lionel Roques. "Inside dynamics of pulled and pushed fronts". In: (2011). URL: https://arxiv.org/pdf/1110.1761.pdf.

[Bov13] Anton Bovier. "From spin glasses to branching Brownian motion - and back?" In: (2013). URL: https://www.math.ucla.edu/~biskup/Prague-school-2013/figs/Anton-notes.pdf.

[Ber14] Julien Berestycki. "Topics on Branching Brownian motion". In: (2014). URL: https://www.stats.ox.ac.uk/~berestyc/Articles/EBP18_v2.pdf.

[Per15] Benoît Perthame. *Parabolic equations in biology: Growth, reaction, movement and diffusion*. Springer, 2015. URL: https://hal.sorbonne-universite.fr/hal-01423552.

[Ali16] Sarah Penington Alison Etheridge Nic Freeman. "Branching Brownian Motion, mean curvature flow and the motion of hybrid zones". In: (2016). URL: https://doi.org/10.48550/arXiv.1607.07563.

[Ryz21] Lenya Ryzhik. "Stanford PDE mini-course, Fall 2021". In: (2021). URL: http://math.stanford.edu/~ryzhik/stanf-minicourse-21.pdf.