A Counterexample to an Endpoint Mixed Norm Estimate of Calderón-Zygmund Operators

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Abstract

In this article, we give a counterexample to an endpoint mixed norm estimate of Calderón-Zygmund Operators.

1 Introduction

Definition 1.1. *A kernel* $K : \mathbb{R}^n - \{0\} \to \mathbb{C}$ *is called a Calderón-Zygmund kernel if there exists some constant* B *such that*

1. $|K(x)| < B|x|^{-n}$
2. $\int_{|x|>2|y|} |K(x) - K(x-y)| dx \le B$ *for all* $|y| > 0$
3. $\int_{r<|x|<s} K(x) dx = 0$ *for* $0 < r < s < \infty$

Definition 1.2. *For a Calderón-Zygmund kernel* K *and* $f \in \mathcal{S}(\mathbb{R}^n)$*, let*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y) f(y) dy$$

T *is called the Calderón-Zygmund operator with kernel* K*.*

Now we give two examples of Calderón-Zygmund operators.

Definition 1.3. *The Riesz transforms* R_j*'s are the Calderón-Zygmund operators given by kernels* $K_j(x) = \frac{x_j}{|x|^{n+1}}$

Definition 1.4. *The double Riesz transforms* R_{ij}*'s are the Calderón-Zygmund operators given by kernels*

$$K_{ij}(x) = \begin{cases} C_n \frac{x_i x_j}{|x|^{n+2}} & \text{if } i \neq j \\ C_n \frac{x_i^2 - n^{-1}|x|^2}{|x|^{n+2}} & \text{if } i = j \end{cases}$$

where C_n*'s are some dimensional constants.*

By computation, for Schwartz function u,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = R_{ij}(\Delta u)(x) + \frac{1}{n}\delta_{ij}\Delta u(x) \tag{1}$$

So, in particular, the L^p boundedness of R_{ij} is equivalent to the following L^p estimate:

$$\|\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\|_{L^p} \le C\|\Delta u(x)\|_{L^p}$$

A Calderón-Zygmund operator T can be extended to an bounded operator on $L^p(\mathbb{R}^n)$:

Theorem 1.5 (Calderón-Zygmund). *For T a Calderón-Zygmund operator and $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|Tf(x)\|_{L^p} \le C_p\|f(x)\|_{L^p}$$

for $1 < p < \infty$. Hence, T can be extended to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This fails when $p = 1$ or ∞

It is also known that the following mixed norm estimate holds for Calderón-Zygmund operators given by Calderón-Zygmund kernels with some additional regularity assumptions. [1, p.448, Theorem 1].

Theorem 1.6 (Mixed norm estimates). *For T, a Calderón-Zygmund operator given by kernel K defined on $\mathbb{R}^2 - \{0\}$ satisfying additional regularity conditions*

$$|\partial_2 K(x_1, x_2)| \le A(x_1^2 + x_2^2)^{-\frac{3}{2}} \quad |\partial_1\partial_2 K(x_1, x_2)| \le A(x_1^2 + x_2^2)^{-2}$$

we have

$$\|\,\|Tf(x_1, x_2)\|_{L^p_{x_1}}\|_{L^q_{x_2}} \le C\|\,\|f(x_1, x_2)\|_{L^p_{x_1}}\|_{L^q_{x_2}}$$

for $1 < p, q < \infty$

Remark. *The above theorem no longer holds when $q = 1$ or ∞. For example, consider the the double Riesz transform and a sequence of compactly supported functions with supports shrinking to a point.*

The above mixed estimate no longer holds when $q = \infty$. We are curious about what will happen if we make the right hand side of the above mixed estimate larger by changing the order of the two norms.

That is, for T, a Calderón-Zygmund operator given by a kernel K defined on $\mathbb{R}^2 - \{0\}$, can we always find a constant C_p such that $\|\,\|Tf(x, y)\|_{L^p_x}\|_{L^\infty_y} \le C_p\|\,\|f(x, y)\|_{L^\infty_y}\|_{L^p_x}$ for all Schwartz function f? In the next section we will see that the answer is negative. For example, we cannot have such estimate for the double Riesz transform R_{12} and $p \ge 2$. Indeed, it can be easily verified that the kernel corresponding to R_{12} satisfies the additional regularity conditions in Theorem 1.6: $|\partial_2 K_{12}(x_1, x_2)| \le A(x_1^2 + x_2^2)^{-\frac{3}{2}}$ and $|\partial_1\partial_2 K_{12}(x_1, x_2)| \le A(x_1^2 + x_2^2)^{-2}$ for some constant $A > 0$.

2 Main Results

We first give a counterexample to the estimate $\| \|Tf(x,y)\|_{L^p_x}\|_{L^\infty_y} \le C \| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x}$ for $T = R_{12}$, a double Riesz transform, and $p = 2$.

Proposition 2.1 (A Counterexample). *Let R_{12} be the double Riesz transform given by the kernel $K(x_1,x_2) = \frac{x_1x_2}{2\pi(x_1^2+x_2^2)^2}$. Then there exists a sequence of Schwartz functions g_n such that the sequence $\{\| \|R_{12}g_n(x_1,x_2)\|_{L^2_{x_1}}\|_{L^\infty_{x_2}}\}$ goes to infinity, while the the sequence $\{\| \|g_n(x_1,x_2)\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}\}$ stays bounded.*

Proof. We first construct a sequence of Schwartz functions f_j's as follows. Define $\chi_1^{(j)}$ to be a smooth bump function supported in $[2^{-j}, 2^{-j+1}]$ with maximum value 1. Let $\chi_2(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. Define χ to a smooth bump function supported in $[0, A]$ for $3 \le A \le 100$ such that $\chi(x) = 1$ for $x \in [\frac{1}{4}, A - \frac{1}{4}]$. Define $\chi_3^{(j)} = \chi(x - 2^j) + \chi(-x + 2^j)$. Now we set $f_j(x_1,x_2) = \chi_1^{(j)}(x_2)\chi_2(x_1)\mathcal{F}^{-1}(\chi_3^{(j)})(x_1)$. Now, for $n > 100$, define $g_n = \sum_{j=100}^n f_j$.

Next, we show that $\{\| \|g_n(x_1,x_2)\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}\}$ is bounded by some fixed constant.

$$\begin{aligned}|\mathcal{F}^{-1}(\chi_3^{(j)})(x)| &\le |\mathcal{F}^{-1}(\chi(\xi - 2^j))(x)| + |\mathcal{F}^{-1}(\chi(-\xi + 2^j))(x)| \\ &= |\int e^{2\pi i x\xi}\chi(\xi - 2^j)d\xi| + |\int e^{2\pi i x\xi}\chi(-\xi + 2^j)d\xi| \\ &\le \sup |\mathcal{F}^{-1}(\chi)| + \sup |\mathcal{F}(\chi)|\end{aligned}$$

Let $D = \sup |\mathcal{F}^{-1}(\chi)| + \sup |\mathcal{F}(\chi)|$. Thus,

$$|f_j(x_1,x_2)| \le |\frac{1}{\sqrt{\pi}}e^{-x^2}\mathcal{F}^{-1}(\chi_3^{(j)})(x_1)| \le \frac{D}{\sqrt{\pi}}e^{-x^2}.$$

Since f_j's have disjoint supports in x_2, we have

$$\|e^{x_1^2}g_n(x_1,x_2)\|_{L^\infty_{x_2}} \le \sup_{100\le j\le n} e^{x_1^2}|\frac{1}{\sqrt{\pi}}e^{-x^2}\mathcal{F}^{-1}(\chi_3^{(j)})(x_1)| \le \frac{D}{\sqrt{\pi}}$$

Hence,

$$\| \|g_n(x_1,x_2)\|_{L^\infty_{x_2}}\|_{L^2_{x_1}} \le \|e^{-x^2}\|_{L^2_x}\|e^{x_1^2}g_n(x_1,x_2)\|_{L^\infty_{x_2}} \le \frac{D}{\sqrt{\pi}}\|e^{-x^2}\|_{L^2_x}$$

Therefore, $\{\| \|g_n(x_1,x_2)\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}\}$ is bounded.

In order to show that $\{\| \|R_{12}g_n(x_1,x_2)\|_{L^2_{x_1}}\|_{L^\infty_{x_2}}\}$ goes to infinity, it suffices to show that $\{\|R_{12}g_n(x_1,0)\|_{L^2_{x_1}}\}$ goes to infinity.

$$\begin{aligned}\|R_{12}g_n(x_1,0)\|^2_{L^2_{x_1}} &= \int \left|\sum_{i=100}^n R_{12}f_i(x_1,0)\right|^2 dx_1 = \int \left|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)\right|^2 d\xi_1 \\ &\ge \sum_{j=100}^n \int_{2^j+1}^{2^j+A-1} \left|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)\right|^2 d\xi_1\end{aligned}$$

We begin with computing $|\mathcal{F}_1(R_{12}f)(\xi_1,x_2)|$ as follows, where $\mathcal{F}_i$ is the fourier transform with respect to the i^{th} coordinate.

$$|\mathcal{F}_1(R_{12}f)(\xi_1,0)| = \left|\mathcal{F}_2^{-1}\left(\frac{\xi_1\xi_2}{|\xi|^2}\mathcal{F}f\right)(\xi_1,0)\right| = \left|\int \mathcal{F}_1(f)(\xi_1,z)K(\xi_1,-z)dz\right|$$

where

$$K(\xi_1, z) = \int e^{2\pi z\xi_2}\frac{\xi_1\xi_2}{|\xi|^2}d\xi_2 = \int e^{2\pi\xi_1 z\eta}\frac{\eta}{1+\eta^2}\xi_1 d\eta = \xi_1\mathcal{F}_\eta^{-1}(\frac{\eta}{1+\eta^2})(\xi_1 z)$$

Thus,

$$\begin{aligned}
|\mathcal{F}_1(R_{12}f)(\xi_1,0)| &= \left|\int \mathcal{F}_1(f)(\xi_1,z)\xi_1\mathcal{F}_\eta^{-1}(\frac{\eta}{1+\eta^2})(-\xi_1 z)dz\right| \\
&= \sqrt{\frac{\pi}{2}}\left|\int \mathcal{F}_1(f)(\xi_1,z)\xi_1 e^{-|\xi_1||z|}dz\right| \\
&= \sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(j)}\right)(\xi_1)\xi_1\int \chi_1(z)e^{-|\xi_1||z|}dz\right| \\
&= \sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(j)}\right)(\xi_1)\left(e^{-|\xi_1|2^{-j}} - e^{-|\xi_1|2^{-j+1}}\right)\right|
\end{aligned}$$

Now we estimate $\int_{2^j+1}^{2^j+A-1}\left|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)\right|^2 d\xi_1$. To this end, we estimate $|\mathcal{F}_1(R_{12}f_j)(\xi_1,0)|$ from below for $\xi_1 \in [2^j+1, 2^j+A-1]$.

$$\begin{aligned}
|\mathcal{F}_1(R_{12}f_j)(\xi_1,0)| &= \sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(j)}\right)(\xi_1)\left(e^{-|\xi_1|2^{-j}} - e^{-|\xi_1|2^{-j+1}}\right)\right| \\
&\geq \sqrt{\frac{\pi}{2}}(e^{-2}-e^{-4})\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(j)}\right)(\xi_1)\right| \\
&\geq \sqrt{\frac{\pi}{2}}(e^{-2}-e^{-4})\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-\pi^2\xi^2}d\xi
\end{aligned}$$

Set $E = \sqrt{\frac{\pi}{2}}(e^{-2}-e^{-4})\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-\pi^2\xi^2}d\xi > 0$. Observe that $e^{-\pi^2x^2} \leq \frac{1}{e\pi^2}x^{-2}$ for $x > 0$.

Next, we estimate $|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)|$ from above for $\xi_1 \in [2^j+1, 2^j+A-1]$ and $i < j$.

$$\begin{aligned}
|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)| &= \sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(i)}\right)(\xi_1)\left(e^{-|\xi|2^{-i}} - e^{-|\xi|2^{-i+1}}\right)\right| \\
&\leq \frac{1}{4}\sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2)*\chi_3^{(i)}\right)(\xi_1)\right| \\
&\leq \frac{1}{4}\sqrt{\frac{\pi}{2}}\int_{2^i}^{2^i+A} e^{-\pi^2(\xi_1-\xi)^2}d\xi + \frac{1}{4}\sqrt{\frac{\pi}{2}}\int_{-2^i-A}^{-2^i} e^{-\pi^2(\xi_1-\xi)^2}d\xi \\
&\leq \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\int_{2^i}^{2^i+A}(\xi_1-\xi)^{-2}d\xi + \leq \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\int_{-2^i-A}^{-2^i}(\xi_1-\xi)^{-2}d\xi \\
&= \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\xi_1-2^i-A} - \frac{1}{\xi_1-2^i} + \frac{1}{\xi_1+2^i} - \frac{1}{\xi_1+2^i+A}\right) \\
&\leq \frac{1}{2e\pi^2}\sqrt{\frac{\pi}{2}}\frac{1}{\xi_1-2^i-A} \leq \frac{1}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-j}
\end{aligned}$$

Next, we estimate $|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)|$ from above for $\xi_1 \in [2^j+1, 2^j+A-1]$ and $i > j$.

$$
\begin{aligned}
|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)| &= \sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2) * \chi_3^{(i)}\right)(\xi_1)\left(e^{-|\xi|2^{-i}} - e^{-|\xi|2^{-i+1}}\right)\right| \\
&\le \frac{1}{4}\sqrt{\frac{\pi}{2}}\left|\left(\mathcal{F}(\chi_2) * \chi_3^{(i)}\right)(\xi_1)\right| \\
&\le \frac{1}{4}\sqrt{\frac{\pi}{2}}\int_{2^i}^{2^i+A} e^{-\pi^2(\xi_1-\xi)^2}d\xi + \frac{1}{4}\sqrt{\frac{\pi}{2}}\int_{-2^i-A}^{-2^i} e^{-\pi^2(\xi_1-\xi)^2}d\xi \\
&\le \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\int_{2^i}^{2^i+A}(\xi_1-\xi)^{-2}d\xi + \le \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\int_{-2^i-A}^{-2^i}(\xi_1-\xi)^{-2}d\xi \\
&= \frac{1}{4e\pi^2}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\xi_1-2^i-A} - \frac{1}{\xi_1-2^i} + \frac{1}{\xi_1+2^i} - \frac{1}{\xi_1+2^i+A}\right) \\
&\le \frac{1}{2e\pi^2}\sqrt{\frac{\pi}{2}}\frac{1}{2^i-\xi_1} \le \frac{1}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-i}
\end{aligned}
$$

Now we estimate $|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)|$ from below for $\xi_1 \in [2^j+1, 2^j+A-1]$.

$$
\begin{aligned}
\left|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)\right| &\ge |\mathcal{F}_1(R_{12}f_j)(\xi_1,0)| - \sum_{100\le i<j}|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)| - \sum_{j<i\le N}|\mathcal{F}_1(R_{12}f_i)(\xi_1,0)| \\
&\ge E - (j-100)\frac{1}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-j} - \frac{1}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-j} \\
&\ge E - \frac{j}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-j} \ge E - \frac{100}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-100} > 0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|R_{12}g_n(x_1,0)\|_{L^2_{x_1}}^2 &\ge \sum_{j=100}^n \int_{2^j+1}^{2^j+A-1}\left|\sum_{i=100}^n \mathcal{F}_1(R_{12}f_i)(\xi_1,0)\right|^2 d\xi_1 \\
&\ge \sum_{j=100}^n \int_{2^j+1}^{2^j+A-1}(E - \frac{j}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-j})^2 d\xi_1 \\
&\ge \sum_{j=100}^n (A-2)(E - \frac{100}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-100})^2 \\
&\ge (n-100)(A-2)(E - \frac{100}{e\pi^2}\sqrt{\frac{\pi}{2}}2^{-100})^2
\end{aligned}
$$

Therefore, $\{\|R_{12}g_n(x_1,0)\|_{L^2_{x_1}}\}$ goes to infinity and so does $\{\|\|R_{12}g_n(x_1,x_2)\|_{L^2_{x_1}}\|_{L^\infty_{x_2}}\}$. □

To better understand when we will or will not have the estimate $\|\|Tf(x,y)\|_{L^p_x}\|_{L^\infty_y} \le C_p\|\|f(x,y)\|_{L^\infty_y}\|_{L^p_x}$ for Calderón-Zygmund operator T, we prove the following interpolation theorem and the following mixed weak L^1 estimate.

Proposition 2.2 (An Interpolation theorem). *Let T be a sublinear operator from $L^{p_0}(\mathbb{R}, L^\infty(\mathbb{R})) + L^{p_1}(\mathbb{R}, L^\infty(\mathbb{R}))$ to the space of measurable functions. Assume that for $p_0 < p_1 < \infty$ we have*

$$\|\|Tf(x,y)\|_{L^{p_0,\infty}_x}\|_{L^\infty_y} \le A_0\|\|f(x,y)\|_{L^\infty_y}\|_{L^{p_0}_x}$$

$$\|\|Tf(x,y)\|_{L^{p_1,\infty}_x}\|_{L^\infty_y} \le A_1\|\|f(x,y)\|_{L^\infty_y}\|_{L^{p_1}_x}$$

Then for $p_0 < p < p_1$, we have

$$\| \|Tf(x,y)\|_{L^p_x}\|_{L^\infty_y} \leq C(p_0,p_1,p,A_0,A_1)\| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x}$$

where $C(p_0,p_1,p,A_0,A_1)$ is a constant depending on p_0,p_1,p,A_0,A_1.

Proof. Pick any $p_0 < p < p_1$. For any $f \in L^p(\mathbb{R}, L^\infty(\mathbb{R}))$, we have $\| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x} < \infty$. Now we decompose f into a sume of two functions defined as follows:

$$f_0^\alpha(x,y) = \begin{cases} f(x,y) & \text{for } |f(x,y)| > \alpha \\ 0 & \text{for } |f(x,y)| \leq \alpha \end{cases}$$

$$f_1^\alpha(x,y) = \begin{cases} f(x,y) & \text{for } |f(x,y)| \leq \alpha \\ 0 & \text{for } |f(x,y)| > \alpha \end{cases}$$

By our definition, $f = f_0^\alpha + f_1^\alpha$.

Next, we note that $f_0^\alpha(\cdot,y) \in L^{p_0}(\mathbb{R})$ and $f_1^\alpha(\cdot,y) \in L^{p_1}(\mathbb{R})$ for $a.e. \, y$. Indeed, for $a.e. \, y$, we have

$$\begin{aligned} \|f_0^\alpha(\cdot,y)\|^{p_0}_{L^{p_0}_x} &= \int_{\{x:|f(x,y)|>\alpha\}} |f(x,y)|^{p_0} dx = \int_{\{x:|f(x,y)|>\alpha\}} |f(x,y)|^p |f(x,y)|^{p_0-p} dx \\ &\leq (\alpha)^{p_0-p} \int_{\{x:|f(x,y)|>\alpha\}} \|f(x,\cdot)\|^p_{L^\infty_y} dx \leq (\alpha)^{p_0-p} \| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x} \end{aligned}$$

$$\begin{aligned} \|f_1^\alpha(\cdot,y)\|^{p_1}_{L^{p_1}_x} &= \int_{\{x:|f(x,y)|\leq\alpha\}} |f(x,y)|^{p_1} dx = \int_{\{x:|f(x,y)|\leq\alpha\}} |f(x,y)|^p |f(x,y)|^{p_1-p} dx \\ &\leq (\alpha)^{p_1-p} \int_{\{x:|f(x,y)|\leq\alpha\}} \|f(x,\cdot)\|^p_{L^\infty_y} dx \leq (\alpha)^{p_1-p} \| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x} \end{aligned}$$

Since T is sublinear, $|T(f)| \leq |T(f_0^\alpha)| + |T(f_1^\alpha)|$. Hence, $d_{Tf(\cdot,y)}(\alpha) \leq d_{Tf_0^\alpha(\cdot,y)}(\frac{\alpha}{2}) + d_{Tf_1^\alpha(\cdot,y)}(\frac{\alpha}{2})$ for $a.e. \, y$. Therefore, for $a.e. \, y$, we have

$$\begin{aligned} d_{Tf(\cdot,y)}(\alpha) &\leq d_{Tf_0^\alpha(\cdot,y)}(\frac{\alpha}{2}) + d_{Tf_1^\alpha(\cdot,y)}(\frac{\alpha}{2}) \\ &= (\frac{\alpha}{2})^{-p_0} \left((\frac{\alpha}{2}) \left(d_{Tf_0^\alpha(\cdot,y)}(\frac{\alpha}{2}) \right)^{\frac{1}{p_0}} \right)^{p_0} + (\frac{\alpha}{2})^{-p_1} \left((\frac{\alpha}{2}) \left(d_{Tf_1^\alpha(\cdot,y)}(\frac{\alpha}{2}) \right)^{\frac{1}{p_1}} \right)^{p_1} \\ &\leq (\frac{\alpha}{2})^{-p_0} \|Tf_0^\alpha(\cdot,y)\|^{p_0}_{L^{p_0,\infty}_x} + (\frac{\alpha}{2})^{-p_1} \|Tf_1^\alpha(\cdot,y)\|^{p_1}_{L^{p_1,\infty}_x} \\ &\leq (\frac{\alpha}{2})^{-p_0} \| \|Tf_0^\alpha(x,y)\|_{L^{p_0,\infty}_x}\|^{p_0}_{L^\infty_y} + (\frac{\alpha}{2})^{-p_1} \| \|Tf_1^\alpha(x,y)\|_{L^{p_1,\infty}_x}\|^{p_1}_{L^\infty_y} \\ &\leq A_0^{p_0} (\frac{\alpha}{2})^{-p_0} \| \|f_0^\alpha(x,y)\|_{L^\infty_y}\|^{p_0}_{L^{p_0}_x} + A_1^{p_1} (\frac{\alpha}{2})^{-p_1} \| \|f_1^\alpha(x,y)\|_{L^\infty_y}\|^{p_1}_{L^{p_1}_x} \end{aligned}$$

Now we will compute $\| \|f_0^\alpha(x,y)\|_{L^\infty_y}\|^{p_0}_{L^{p_0}_x}$ and $\| \|f_1^\alpha(x,y)\|_{L^\infty_y}\|^{p_1}_{L^{p_1}_x}$ in terms of f.

First, we observe that

$$\begin{aligned} \|f(x,y)\|_{L^\infty_y} \leq \alpha &\Rightarrow \text{ for } a.e. \, y \quad |f(x,y)| \leq \alpha \\ &\Rightarrow \text{ for } a.e. \, y \quad f_0^\alpha(x,y) = 0 \\ &\Rightarrow \|f_0^\alpha(x,y)\|_{L^\infty_y} = 0 \end{aligned}$$

and

$$\begin{aligned}\|f(x,y)\|_{L_y^\infty} > \alpha &\Rightarrow \text{ for } a.e.\, y \quad |f(x,y)| > \alpha \\ &\Rightarrow \text{ for } a.e.\, y \quad f_0^\alpha(x,y) = f(x,y) \\ &\Rightarrow \|f_0^\alpha(x,y)\|_{L_y^\infty} = \|f(x,y)\|_{L_y^\infty}\end{aligned}$$

Hence,

$$\|\|f_0^\alpha(x,y)\|_{L_y^\infty}\|_{L_x^{p_0}}^{p_0} = \int_{\mathbb{R}} \|f_0^\alpha(x,y)\|_{L_y^\infty}^{p_0} dx = \int_{\{x:\|f(x,y)\|_{L_y^\infty} > \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_0} dx$$

Similarly, we get

$$\|\|f_1^\alpha(x,y)\|_{L_y^\infty}\|_{L_x^{p_1}}^{p_1} = \int_{\{x:\|f(x,y)\|_{L_y^\infty} \le \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_1} dx$$

With $\|\|f_0^\alpha(x,y)\|_{L_y^\infty}\|_{L_x^{p_0}}^{p_0}$ and $\|\|f_1^\alpha(x,y)\|_{L_y^\infty}\|_{L_x^{p_1}}^{p_1}$ computed, we can now estimate $d_{Tf(\cdot,y)}(\alpha)$ by $\|f(x,y)\|_{L_y^\infty}$.

$$\begin{aligned}d_{Tf(\cdot,y)}(\alpha) &\le (\frac{\alpha}{2})^{-p_0} {A_0}^{p_0} \|\|Tf_0^\alpha(x,y)\|_{L_x^{p_0,\infty}}\|_{L_y^\infty}^{p_0} + (\frac{\alpha}{2})^{-p_1} {A_1}^{p_1} \|\|Tf_1^\alpha(x,y)\|_{L_x^{p_1,\infty}}\|_{L_y^\infty}^{p_1} \\ &\le (\frac{\alpha}{2})^{-p_0} {A_0}^{p_0} \int_{\{x:\|f(x,y)\|_{L_y^\infty} > \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_0} dx \\ &+ (\frac{\alpha}{2})^{-p_1} {A_1}^{p_1} \int_{\{x:\|f(x,y)\|_{L_y^\infty} \le \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_1} dx\end{aligned}$$

Finally, we estimate $\|Tf(\cdot,y)\|_{L_x^p}^p$ as follows:

$$\begin{aligned}\|Tf(\cdot,y)\|_{L_x^p}^p &= p\int_0^\infty \alpha^{p-1} d_{Tf(\cdot,y)}(\alpha) d\alpha \\ &\le p(2A_0)^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_{\{x:\|f(x,y)\|_{L_y^\infty} > \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_0} dx\, d\alpha \\ &+ p(2A_1)^{p_1} \int_0^\infty \alpha^{p-p_1-1} \int_{\{x:\|f(x,y)\|_{L_y^\infty} \le \alpha\}} \|f(x,y)\|_{L_y^\infty}^{p_1} dx\, d\alpha \\ &= p(2A_0)^{p_0} \int_{\mathbb{R}} \|f(x,y)\|_{L_y^\infty}^{p_0} \int_0^{\|f(x,y)\|_{L_y^\infty}} \alpha^{p-p_0-1} d\alpha\, dx \\ &+ p(2A_1)^{p_1} \int_{\mathbb{R}} \|f(x,y)\|_{L_y^\infty}^{p_1} \int_{\|f(x,y)\|_{L_y^\infty}}^\infty \alpha^{p-p_1-1} d\alpha\, dx \\ &= p\left(\frac{(2A_0)^{p_0}}{p-p_0} + \frac{(2A_1)^{p_1}}{p_1-p}\right) \|\|f(x,y)\|_{L_y^\infty}\|_{L_x^p}^p\end{aligned}$$

for $a.e.\, y$.

Therefore, we conclude that

$$\|\|Tf(x,y)\|_{L_x^p}\|_{L_y^\infty} \le C(p_0,p_1,p,A_0,A_1) \|\|f(x,y)\|_{L_y^\infty}\|_{L_x^p}$$

□

Proposition 2.3. *For T, a Calderón-Zygmund operator given by a kernel K defined on $\mathbb{R}^2 - \{0\}$ such that*

$$\| \|Tf(x,y)\|_{L^p_x}\|_{L^\infty_y} \le C\| \|f(x,y)\|_{L^\infty_y}\|_{L^p_x}$$

we have

$$\| \|Tf(x,y)\|_{L^{1,\infty}_x}\|_{L^\infty_y} \le D\| \|f(x,y)\|_{L^\infty_y}\|_{L^1_x}$$

Proof. It suffices to prove the inequality for $f \in \mathcal{S}(\mathbb{R}^2)$. Fix any $\alpha > 0$. We apply Calderón-Zygmund decomposition to $\|f(x,y)\|_{L^\infty_y}$, i.e. we can find a collection of intervals $\mathcal{B} = \{Q\}$ such that

$$|\bigcup_{\mathcal{B}} Q| \le \frac{1}{\alpha}\| \|f(x,y)\|_{L^\infty_y}\|_{L^1_x} \quad \alpha < \frac{1}{|Q|}\int_Q \|f(x,y)\|_{L^\infty_y} dx \le 2\alpha$$
$$\|f(x,y)\|_{L^\infty_y} = \tilde{g}(x) + \tilde{b}(x)$$

where $\tilde{b}(x) = \sum_{\mathcal{B}} \chi_Q \|f(x,y)\|_{L^\infty_y}$ and $|\tilde{g}(x)| \le \alpha$

Now we define

$$b(x,y) := \sum_{Q\in\mathcal{B}} \chi_Q(x) f(x,y)$$

$$g(x,y) := f(x,y) - b(x,y)$$

$$f_1(\cdot,y) := g(\cdot,y) + \sum_{Q\in\mathcal{B}} \chi_Q \left(\frac{1}{|Q|}\int_Q f(x,y)dx\right)$$

$$f_2(\cdot,y) := b(\cdot,y) - \sum_{Q\in\mathcal{B}} \chi_Q \left(\frac{1}{|Q|}\int_Q f(x,y)dx\right)$$

Hence we have $f = f_1 + f_2$. Define $f_Q(x,y) = \chi_Q\left(f(x,y) - \frac{1}{|Q|}\int_Q f(x,y)dx\right)$. Thus, $f_2(x,y) = \sum_{\mathcal{B}} f_Q(x,y)$ and $\int f_Q(x,y)dx = 0$. Also, we have

$$\|f_1(x,y)\|_{L^\infty_y} \le \|g(x,y)\|_{L^\infty_y} + \sum_{\mathcal{B}} \chi_Q \frac{1}{|Q|}\int_Q \|f(x,y)\|_{L^\infty_y} dx$$
$$\le \tilde{g}(x) + 2\alpha \sum_{\mathcal{B}} \chi_Q$$

Thus, $\| \|f_1(x,y)\|_{L^\infty_y}\|_{L^\infty_x} \le 2\alpha$. Moreover,

$$\| \|f_1(x,y)\|_{L^\infty_y}\|_{L^1_x} \le \int |\tilde{g}(x)|dx + \sum_{\mathcal{B}} \int_Q \|f(x,y)\|_{L^\infty_y} dx$$
$$\le \| \|f(x,y)\|_{L^\infty_y}\|_{L^1_x}$$

Now we estimate $d_{Tf(\cdot,y)}(\alpha)$ as follows:

$$d_{Tf(\cdot,y)}(\alpha) = |\{x \in \mathbb{R} : |Tf(x,y)| > \alpha\}|$$
$$\le |\{x \in \mathbb{R} : |Tf_1(x,y)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R} : |Tf_2(x,y)| > \frac{\alpha}{2}\}|$$

We first estimate $|\{x\in\mathbb{R}:|Tf_1(x,y)|>\frac{\alpha}{2}\}|$ using Chebyshev's Inequality, what we observed above, and our assumption on T:

$$\begin{aligned}|\{x\in\mathbb{R}:|Tf_1(x,y)|>\frac{\alpha}{2}\}| &\le \frac{2^p}{\alpha^p}\|Tf_1(\cdot,y)\|^p_{L^p_x}\\ &\le \frac{2^p}{\alpha^p}\|\,\|Tf_1(x,,y)\|_{L^p_x}\|^p_{L^\infty_y}\\ &\le \frac{2^pC^p}{\alpha^p}\|\,\|f_1(x,y)\|_{L^\infty_y}\|^p_{L^p_x}\\ &= \frac{2^pC^p}{\alpha^p}\|\,\|f_1(x,y)\|_{L^\infty_y}\|^{p-1}_{L^\infty_x}\,\|\,\|f_1(x,y)\|_{L^\infty_y}\|_{L^1_x}\\ &\le \frac{2^{2p-1}C^p}{\alpha}\|\,\|f(x,y)\|_{L^\infty_y}\|_{L^1_x}\end{aligned}$$

Next, we estimate $|\{x\in\mathbb{R}:|Tf_2(x,y)|>\frac{\alpha}{2}\}|$. For each Q, let Q^* be the interval such that Q and Q^* have the same center, and $|Q^*|=2|Q|$.

$$\begin{aligned}|\{x\in\mathbb{R}:|Tf_2(x,y)|>\frac{\alpha}{2}\}| &\le |\cup_{\mathcal{B}}Q^*|+|\{x\in\mathbb{R}-\cup_{\mathcal{B}}Q^*:|Tf_2(x,y)|>\frac{\alpha}{2}\}|\\ &\le 2\sum_{\mathcal{B}}|Q|+\frac{2}{\alpha}\int_{\mathbb{R}-\cup_{\mathcal{B}}Q^*}|Tf_2(x,y)|dx\\ &\le \frac{2}{\alpha}\|\,\|f(x,y)\|_{L^\infty_y}\|_{L^1_x}+\frac{2}{\alpha}\sum_{\mathcal{B}}\int_{\mathbb{R}-Q^*}|Tf_Q(x,y)|dx\end{aligned}$$

We now estimate $\int_{\mathbb{R}-Q^*}|Tf_Q(x,y)|dx$. First, from $\int f_Q(x,y)dx=0$ we conclude that for $x\in\mathbb{R}-Q^*$,

$$\begin{aligned}Tf_Q(x,y) &= \int_{\mathbb{R}}\int_Q K(x-s,y-t)f_Q(s,t)ds\,dt\\ &= \int_{\mathbb{R}}\int_Q\left(K(x-s,y-t)-K(x-x_Q,y-t)\right)f_Q(s,t)ds\,dt\end{aligned}$$

where x_Q is the center of interval Q.

Recall that the Calerón-Zygmund kernal K has the following property: there is a constant $B>0$ such that

$$\int_{\{|x|>2|y|\}}|K(x)-K(x-y)|dx\le B \text{ for all } y\ne 0$$

Thus, by our selection of Q^*,

$$\begin{aligned}&\int_{\mathbb{R}-Q^*}|Tf_Q(x,y)|dx\\ &\le \int_{\mathbb{R}-Q^*}\int_{\mathbb{R}}\int_Q|K(x-s,y-t)-K(x-x_Q,y-t)|\,|f_Q(s,t)|ds\,dt\,dx\\ &\le \int_Q\|f_Q(s,y)\|_{L^\infty_y}\left(\int_{\mathbb{R}-Q^*}\int_{\mathbb{R}}|K(x-s,y-t)-K(x-x_Q,y-t)|dt\,dx\right)ds\\ &\le B\int_Q\|f_Q(s,y)\|_{L^\infty_y}ds\\ &\le 2B\int_Q\|f(x,y)\|_{L^\infty_y}dx\end{aligned}$$

Hence,

$$
\begin{aligned}
|\{x \in \mathbb{R} : |Tf_2(x,y)| > \frac{\alpha}{2}\}| &\leq \frac{2}{\alpha} \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^1} + \frac{2}{\alpha}\sum_{\mathcal{B}^y} \int_{\mathbb{R}-Q^*} |Tf_Q(x,y)|dx \\
&\leq \frac{2}{\alpha} \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^1} + \frac{4B}{\alpha}\sum_{\mathcal{B}^y} \int_{Q} \|f(x,\cdot)\|_{L_y^\infty} dx \\
&\leq \frac{2+4B}{\alpha} \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^1}
\end{aligned}
$$

Therefore, combining what we have proved now, for some $D > 0$ and $a.e.\, y$,

$$
\begin{aligned}
d_{Tf(\cdot,y)}(\alpha) &= |\{x \in \mathbb{R} : |Tf(x,y)| > \alpha\}| \\
&\leq |\{x \in \mathbb{R} : |Tf_1(x,y)| > \frac{\alpha}{2}\}| + |\{x \in \mathbb{R} : |Tf_2(x,y)| > \frac{\alpha}{2}\}| \\
&\leq \frac{D}{\alpha} \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^1}
\end{aligned}
$$

Thus, we conclude that

$$
\| \|Tf(x,y)\|_{L_x^{1,\infty}}\|_{L_y^\infty} \leq D \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^1}
$$

□

Corollary 2.4. *For Calderón-Zygmund operator* T*, if we have the estimate* $\| \|Tf(x,y)\|_{L_x^q}\|_{L_y^\infty} \leq C_q \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^q}$*, then we have the estimate* $\| \|Tf(x,y)\|_{L_x^p}\|_{L_y^\infty} \leq C_p \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^p}$ *for all* $1 < p \leq q$*.*

Proof. This follows from Proposition 2.2 and Proposition 2.3 above.

□

Theorem 2.5. *For* $p > 2$*, there exists no* C_p *such that* $\| \|R_{12}f(x,y)\|_{L_x^p}\|_{L_y^\infty} \leq C_p \| \|f(x,y)\|_{L_y^\infty}\|_{L_x^p}$ *for all Schwartz function* f*.*

Proof. Assume, for the sake of contradiction, we have such estimate for R_{12} and $p > 2$, then by the previous corollary, we would have such estimate for R_{12} and $p = 2$, which contradicts Proposition 2.1 above.

□

Corollary 2.6. *For* $p \geq 2$*, there exists no* C_p *such that* $\| \|\partial_x \partial_y u(x,y)\|_{L_x^p}\|_{L_y^\infty} \leq C_p \| \|\Delta u(x,y)\|_{L_y^\infty}\|_{L_x^p}$ *for all Schwartz function* u*.*

Proof. This proposition follows directly from Proposition 2.5 and equation (1).

□

3 Conclusion

In conclusion, we have shown that the double Riesz transform is a counterexample to this specific endpoint mixed norm estimate of Calderón-Zygmund operators for $p \geq 2$.

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