Topics in Lattice Gauge Theory: Behavior of Wilson Loops in the Thermodynamic Limit

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Abstract

This manuscript present a mathematically self-contained exposition of the charge confinement problem in lattice gauge theory. After introducing the physical foundations of gauge theory, we motivate a lattice discretization and relate confinement to the behavior of Wilson loop operators on the lattice.

The bulk of the manuscript organizes proofs of classical results on lattice gauge theory, with special focus to proofs relying on combinatorial expansions of the partition function, and duality (Fourier theory) arguments. In particular, we discuss the existence of the high temperature confining phase and the 4-D U(1) de-confinement phase transition. We conclude with a theorem on the relation between abelian and non-abelian gauge theories.

Contents

1 Introduction

Beginning in the 1960s, work amongst physicists and mathematical physicists in the Constructive Quantum Field Theory (CQFT) program aimed to establish heuristic results of quantum field theorists as rigorous mathematical theory. As described in [GJ87], a primary goal of the program is to establish both axiom schemes for continuum field theories, and to produce examples of "non-trivial" field theories in all space-time dimensions satisfying the axioms. A detailed survey of relevant axiom schemes, and explicit constructions of simple field theories, may be found in [GJ87].

In this paper we review a particularly fruitful approach to the construction of quantum gauge theories, and more precisely, quantum Yang–Mills theories [1]. The latter are a class of field theories forming the basis for the successful Standard Model of Particle physics, a unified description of the electromagnetic, weak nuclear, and strong nuclear forces. Unfortunately, the structure of the non-Gaussian components of the gauge theory action renders their explicit construction especially difficult, and one of the Clay Millenium prizes concerns the construction of such Yang–Mills theories in 4-D, with proof of the desired properties [JW06].

The approach taken in this paper is that of lattice Euclidean gauge theory, an approach taken originally by Wilson [Wil75] in his study of quantum chromodynamics, the SU(3) gauge theory characterizing the strong nuclear force. The combination of a discrete lattice, and Euclidean structure (as opposed to the Lorentzian structure native to quantum field theories) allows rigorous interpretation of the lattice gauge theories as probability measures, and the problem of analyzing such theories becomes one of statistical mechanics. Borrowing intuition from statistical mechanics models, one expects scaling limits (and thus the continuum gauge theory) to exist around critical points of the lattice theory. Thus one is naturally interested in the phase structure of lattice gauge theories, a unifying theme for the work of this document.

In this paper we are primarily interested in classical proofs of charge confinement/de-confinement in lattice theories. Originally observed in quantum chromodynamics, charge confinement is the property that particles transforming non-trivially under a gauge group (said to carry the "charge" of the corresponding gauge theory) are observed only in configurations of zero net charge, i.e. the presence of charge is "confined." Charge confinement in QCD amounted to a negative observation of quarks, the hypothesized fundamental particles of the strong interaction, and thus a proof from first principles of such a confinement criterion is strongly desired. Unfortunately, even on the lattice, a proof of confinement in 4-D SU(3) theory is lacking for values of couplings near suspected phase transitions. One goal of this manuscript is to examine successful proofs in the abelian gauge theory setting, with hopes of either generalizing the given argument, or identifying a relationship between abelian and non-abelian theory (as is hypothesized, e.g. between a G-gauge theory and $Z(G)$-gauge theory).

First, we review relevant mathematical background in sections 2.1 and 2.2, discussing Lie group analysis and discrete exterior calculus respectively. Then we turn to a sketch of the structure of quantum gauge theory in 2.3, and the lattice discretization in 2.4.

In the remaining sections, we discuss classical results concerning the infinite volume limit of lattice gauge theories, with attention to results on confinement. We begin with a classical theorem of Elitzur on the preservation of gauge invariance in 3.2. In 3.3 we prove properties of the high coupling region of the phase diagram, and show charge confinement. Then we turn to the proof of U(1) deconfinement in 4-D for sufficiently large inverse coupling in 3.4. Finally, we discuss in 3.5 a useful theorem relating the confinement problem in non-abelian gauge theories, to that in abelian theories. In particular, the latter allows us to conclude 3-D U(n) theory is confining for all values of n, and all couplings.

We do not consider here the point of scaling limits of lattice gauge theories, crucial for the eventual goal of proving confinement/de-confinement results for continuum gauge theories.

[1] In this paper, we use "gauge theory" and "Yang–Mills theory" interchangeably.

2 Background

2.1 Mathematical Background I: Analysis on Lie Groups

In this section we review some group-theoretic and representation-theoretic tools that will be of use throughout the remainder of this document. We begin with a review of Lie-theoretic terminology, and turn to properties of Haar measure and character theory of Lie groups. Most facts are drawn from [BtD85].

We begin by recalling some useful definitions.

Definition 1. *A **Lie Group** G is a C^∞ manifold, endowed with a smooth group structure. Thus the multiplication $\times : G \times G \to G$ and inversion $\cdot^{-1} : G \to G$ are smooth diffeomorphisms of G.*

Remark 1. *The commonly occurring Lie Groups in gauge theory are matrix Lie groups, i.e. subgroups $G \subset \mathrm{GL}_n(\mathbb{R})$ or $\subset \mathrm{GL}_n(\mathbb{C})$. We will restrict our attention to this specific case, working with compact subgroups of GL_n, e.g. $\mathrm{O}(n)$ and $\mathrm{SU}(n)$. Note we implicitly endow GL_n with the Euclidean topology given by the isomorphism $\mathrm{GL}_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ (similarly in the $\mathbb{C}$ case).*

The compactness assumption is particularly useful for analytic methods, since we may define a finite measure on such G, compatible with the group structure:

Definition 2. *Let G be a compact Lie group, and $C^0(G)$ the space of continuous functions on G. Then there exists a left-invariant measure dg on G - termed **Haar measure** - satisfying the following properties:*

1. *$\int : C^0(G) \to \mathbb{R}$ (or $\mathbb{C}$) is linear, monotone, and is volume 1, i.e. $\int dg = 1$*
2. *(Left Invariance) For any $h \in G$ fixed, $\int f(g)dg = \int f(hg)dg$.*

It will be useful to record the following analog of Fubini's theorem in the setting of Haar integration, a formula that will be useful when one has control on the integrand on a subgroup $H \subset G$. One may show that H a closed subgroup implies the quotient G/H has a well-defined Lie group structure, and thus a well-defined notion of Haar integration.

Theorem 2.1. *Let G be a compact Lie group, and $H \subset G$ a closed subgroup. Let $dg, d(gH), dH$ be the Haar measures on G, G/H, H respectively. Then for any $F \in C^0(G)$,*

$$\int_G f(g)dg = \int_{G/H} \left(\int_H f(gh)dH \right) d(gH) \tag{1}$$

If the subgroup is contained in $Z(G)$, the center of G, the above formula takes an especially convenient form.

Theorem 2.2. *With the assumptions of the previous theorem, and $H \subset Z(G)$ a closed subgroup of the center, we get for any $f \in C^0(G)$:*

$$\int_G f(g)dg = \int_G \left(\int_H f(gh)dH \right) dG \tag{2}$$

The Haar measure is particularly useful for discussing the representation theory of compact Lie groups, allowing the development of a theory in direct parallel to the representation theory of finite groups. As in the setting of finite groups, we define representations, and their associated characters:

Definition 3. *For G a compact Lie group, and V a real, finite-dimensional vector space, a **real representation** of G on V is a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$. The associated **character** is the map $\chi : G \to \mathbb{R}$ given by $\chi(g) = \mathrm{Tr}(\rho(g))$. One may similarly define complex representations, and associated characters.*

We say the representation $\rho : G \to \mathrm{GL}(V)$ is **irreducible** if no proper, non-trivial subspace $B \subsetneq V$ satisfies $\rho(g)B \subset B$ for all $g \in G$, i.e. there are no proper subspaces of the vector space fixed under the action of G.

One important observation, following from the cyclicity of trace, is the property that for any $h, g \in G$ $\chi(hgh^{-1}) = \chi(g)$. Thus one says $\chi(g)$ is a **class function**. We will see in the Peter-Weyl theorem that characters of representations occupy a privileged space in the set of such continuous class functions.

Before proceeding to the Peter-Weyl theorem and its corollaries, we recall the notion of isomorphism classes of representations. This notion is critical, as we wish to view irreducible representations as building blocks for the general representation theory, so we first must define what we mean by "different" representations.

Definition 4. *Two representations $\rho : G \to \mathrm{GL}(V)$, $\psi : G \to \mathrm{GL}(W)$ are **isomorphic** if there exists a linear isomorphism of vector spaces $T : V \to W$ such that for all $g \in G$, $v \in V$*

$$T(\rho(g)v) = \psi(g)T(v).$$

Proposition 2.1. *Two representations $\rho : G \to \mathrm{GL}(V)$, $\psi : G \to \mathrm{GL}(W)$ are isomorphic if and only if their characters are equal.*

The following theorems illustrates the utility of the representation-theoretic notions introduced above to general problems of analysis on Lie groups. These theorems and their corollaries will be used throughout proofs of confinement and de-confinement.

Theorem 2.3 (Peter-Weyl)**.** *Let G be a compact Lie group. Then the characters χ_τ corresponding to isomorphism classes of irreducible representations $\tau : G \to \mathrm{GL}(V)$ form a dense subspace of the set of continuous class functions on G.*

Note by proposition 2.1, the character χ_τ is well-defined on a isomorphism class of representations. The utility of irreducible characters as a basis is made clear in the next theorem, establishing several useful relations among characters.

Theorem 2.4. *Let G be a compact Lie group, and χ_V, χ_W characters corresponding to (possibly complex) representations of G. Then the following hold:*

1. $\int_G \chi_V(g) = dim(G)$.
2. *If χ_V, χ_W are irreducible characters, then* $\displaystyle\int_G \bar{\chi}_V(g)\chi_W(g) = \begin{cases} 1 & V \simeq W \\ 0 & otherwise \end{cases}$.
3. *If G is in addition abelian, and χ_V a non-trivial character, then* $\int_G \chi_V(g) = 0$.
4. *If the representation V of G is faithful, then the corresponding character satisfies $|\chi(g)| \leq |\chi(1)|$ for all $g \in G$.*

2.2 Mathematical Background II: Lattice Constructions and Discrete Exterior Calculus

In this section, we develop the mathematical machinery for dealing with lattice systems. We begin by establishing notation, and discussing useful notions of "connectedness" on the lattice. We then review the construction of the dual lattice, and the basic results of discrete exterior calculus. The latter will be our primary language for the proofs in section 3.

2.2.1 Notation

In this paper, we define a **lattice** as any subset $\Lambda \subset a\mathbb{Z}^d$, where d is the space-time dimension and a the lattice spacing. Unless otherwise specified, we work with $a = 1$.

The combinatorial units of interest on the lattice are **n-cells**, which we define to be embedded unit n-hypercubes in the lattice. We assume the presence of an orientation on the set of n-cells, denoting the set of oriented, n-cells by $K^n(\Lambda)$.

For the cases $n = 0, 1, 2$, called **vertices**, **links/bonds**, and **plaquettes** respectively, we use special notation. We identify Λ with $K^0(\Lambda)$, and use the notation $B(\Lambda), P(\Lambda)$ for the set of oriented bonds and plaquettes respectively.

An additional word on orientation is warranted here. Let e_i, $i = 1, \cdots, d$ denote the set of canonical lattice unit vectors. Then we say an oriented bond $b \in B(\Lambda)$ with ordered endpoints $(x, y) \in \Lambda$ is positively oriented if $y = x + e_i$ for some i. Similarly, given a plaquette P with endpoints $x_1, \cdots, x_4$, we say P is positively oriented if there exists a cyclic permutation of the x_i such that $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ (all additions modulo 4) is of the form $a, a + e_i, a + e_j, a + e_i + e_j$ for some $a \in \Lambda$, $1 \leq i < j \leq d$. Since such a a is unique for any given plaquette, we may extend $\geq$ to an ordering on $P(\Lambda)$ by comparing this unique first vertex. Similarly, one may express explicit conditions for general n-cells to be positively oriented, and define an ordering $\geq$, which suffice for the lattices considered here.

Note also that this explicit notion of orientation gives a notion of the oriented boundary operator, denoted $\partial : K^n(\Lambda) \to K^{n-1}(\Lambda)$. The boundary of an n dimensional cell c_n, excluding orientation, is simply the union of $n-1$-cells that are faces of c_n. One choice of orientation of the boundary is as follows: the $n-1$ cells in ∂c_n come in pairs parallel to each other. One may assign a positive orientation to the $n-1$ cell in each pair larger with respect to the ordering $\geq$, and negative orientation to the other.

This concludes the discussion of n-cells, orientation, and boundary. We next turn to definitions of connectedness on the lattice, a notion on which most of the combinatorial expansions of lattice theory will eventually rely. First, we define the meaning of two n-cells being connected as cells in Λ. We say two 0-cells a_1, a_2 are connected if $a_1, a_2 \in \Lambda$, $|a_1 - a_2| = 1$. For higher order n-cells a_1, a_2, we say they are connected as cells if there exists an $n-1$-cell b such that $b \subset a_1, a_2$. Thus one sees that two bonds are connected as cells if they share an endpoint, and two plaquettes if they share a bond.

Next, we aim to generalize connectedness to sets of n-cells, capturing when two plaquettes are perhaps disconnected, but there exists a sequence of connected plaqeuttes taking one to the other. So to any set $V \subset K^n(\Lambda)$, associate the **connectedness graph** $G(V)$ as follows. The vertex set of $G(V)$ is just V, and for $a, b \in V$, $G(V)$ has an edge between the two if and only if they are connected as n-cells.

With the associated graph $G(V)$, we may say that a set of n-cells V is *connected* if $G(V)$ is connected. Otherwise, we can decompose V into its *connected components*, determined again by those of $G(V)$. Similarly, we may say two sets A, B of n-cells are **connected** if the graph $G(A \cup B)$ is a connected graph.

The final connectedness relation between sets of n-cells A, B is denoted $A \to B$, indicating that for all connected components $A_i \subset A$, the sets A_i, B are connected in the sense of the previous paragraph.

2.2.2 The Dual Lattice

Given a lattice $\Lambda \subset \mathbb{Z}^d$, it will often prove useful to construct a "dual" lattice Λ^*, defined formally by

$$\Lambda^* = \Lambda + \frac{1}{2}\mathbb{Z}^d.$$

This expression indicates that, embedding $\mathbb{Z}^d$ in an ambient space $\mathbb{R}^d$, the dual lattice is the lattice arising from translating each vertex (and n-cell) of Λ by $\frac{1}{2}$ in each coordinate direction. Exploiting the Euclidean structure in $\mathbb{R}^d$, we see that this dual lattice has the following nice properties:

1. There exists a bijective map $\phi : K^n(\Lambda) \to K^{d-n}(\Lambda^*)$ for all $1 \leq n \leq d$. Given a n-cell c_n in Λ, ϕ associates the unique $d-n$ cell orthogonal to, and intersecting, c_n.
2. The infinite lattice is self-dual, i.e. $\mathbb{Z}^d \simeq (\mathbb{Z}^d)^*$. Note this is not in general true for finite sized lattices.

2.2.3 Discrete Exterior Calculus

In this section we develop a discrete analog of the theory of differential forms, which will prove a compact language for duality arguments later in the document. Our discussion mostly follows [FS82]. First, we define the relevant analogs of differential forms:

Definition 5. *Let K^n be the set of unit, oriented n-cells on $\mathbb{Z}^d$. An* **n-form** *is a map $\alpha : K^n \to F$, where $F = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$. If c_n, c_n^{-1} are identical cells differing only by orientation, we require the orientation condition $\alpha(c_n^{-1}) = -\alpha(c_n)$. The space of n-forms is denoted Λ^n.*

Often, we we will denote $\mathbb{Z}$ valued n-forms $\Lambda^n_{\mathbb{Z}}$, and similarly for other rings. Following as in the continuum case, we next define the natural **boundary operator** $\delta : \Lambda^n \to \Lambda^{n-1}$, and **co-boundary operator** $d : \Lambda^n \to \Lambda^{n+1}$.

Definition 6. *Given $\alpha \in \Lambda^n$, there exist $n+1$ forms $d\alpha$ and $n-1$ forms $\delta\alpha$ defined as follows: For all c_{n+1} and c_{n-1},*

$$d\alpha(c_{n+1}) \equiv \sum_{c_n : c_n \subset \partial c_{n+1}} \alpha(c_n).$$

And similarly,

$$\delta\alpha(c_{n-1}) \equiv \sum_{c_n : c_{n-1} \subset \partial c_n} \alpha(c_n).$$

There is a natural inner product on Λ^n, defined on the subspace of square summable n-forms. Given two square summable n-forms α, β, define

$$(\alpha, \beta)_{\mathbb{Z}^d} = \sum_{c_n \in K_n^+} \bar{\alpha}(c_n)\beta(c_n),$$

where K_n^+ is the set of *positively oriented* n-cells.

The final operation that will be of use is the discrete **Hodge dual** operation, which maps between forms on $\mathbb{Z}^d$ to forms on the dual lattice. So let $(\Lambda^n)^*$ denote the set of n forms on the dual lattice $(\mathbb{Z}^d)^*$. Recall the dual lattice provides a bijection $K_n \leftrightarrow (K_{d-n})^*$. Thus, given a n-form α, define a $d-n$-form $*\alpha$ on $(\Lambda)^*$ by the action:

$$(*\alpha)(c^*_{d-k}) \equiv \alpha(c_k),$$

for all $c^*_{d-k} \in (K_{d-n})^*$, and associated $c_k \in K_d$.

The following proposition establishes properties of the boundary and co-boundary operators, including their relations under the inner product and Hodge dual.

Proposition 2.2. *Let α, β be arbitrary n-forms. Then:*

1. *$\delta\delta\alpha = dd\alpha = 0$. This fact justifies the identification of δ, d with boundary and co-boundary operators.*
2. *The boundary and co-boundary operators are adjoint with respect to $(\cdot,\cdot)_{\mathbb{Z}^d}$, i.e. $(\alpha, d\beta)_{\mathbb{Z}^d} = (\delta\alpha, \beta)_{\mathbb{Z}^d}$*

3. *(Poincaré Lemma) The homology groups*

$$H_n(\mathbb{Z}^d) \equiv \{\alpha \in \Lambda^k : \delta\alpha = 0\}/\{\alpha \in \Lambda^k : \exists \beta \in \Lambda^{k+1} \ s.t. \ \alpha = \delta\beta\}$$

are trivial for all n. Thus given a n-form α with $\delta\alpha = 0$, there exists a $n+1$ form β such that $\alpha = \delta\beta$. Moreover, if Ω is the smallest hypercube such that $\operatorname{supp}(\alpha) \subset \Omega$, then one can choose β to have support contained in Ω. We also have the following bound on β:

$$\max_{c_{n+1}\in\Lambda^{n+1}} |\beta(c_{n+1})| \leq \sum_{\substack{c_n \in \operatorname{supp}(\alpha) \\ c_n \in K_n^+}} |\alpha(c_n)|.$$

Finally, we note an exactly analogous statement holds for the co-boundary operation.

4. *(Compatibility under Duality) The following are equal as forms:*

$$*d*\alpha = \delta\alpha.$$

With the exception of the Poincaré Lemma, the result above are the results of short computations: refer to [FS82] for more details.

2.3 Physical Gauge Theory

This section provides an introduction to the formal structure of gauge theory, first introduced in the pioneering work of Yang and Mills [YM54]. Gauge theory itself is a rich subject in geometric analysis and theoretical physics, the details of which we do not address here. Readers may consult [BM94] for a rigorous mathematical and physical introduction to the subject.

A key physical insight of gauge theory is the presence of a symmetry space, parameterized by a (compact) Lie group G, at each point x of the space-time manifold M. In this section we consider only $M = \mathbb{R}^d$ for some space-time dimension d, and G a subgroup of GL_n for some n.

In pure gauge theory, the relevant structure is a G-principal bundle over M, on which one defines a gauge field A via the data of a connection 1-form. For our purposes it is sufficient to imagine such a bundle E as a space locally diffeomorphic to $\mathbb{R}^d \times G$, but perhaps with non-trivial global structure. Locally, a **gauge field** ϕ on $\mathbb{R}^d$ is a smooth map

$$\phi : \mathbb{R}^d \to \mathfrak{g}^d, \tag{3}$$

with $\mathfrak{g}$ the Lie algebra of G. Equivalently, it is useful to represent ϕ via a differential form A, a **connection 1-form**. If $\phi(x) = (A_1(x), \cdots, A_d(x))$, then the connection 1-form is locally just

$$A = \sum_{i=1}^{d} A_i dx^i. \tag{4}$$

The language of differential forms provides a coordinate-independent way of discussing the gauge field. In a physical setting, A is called the Yang–Mills vector potential, generalizing the vector potential of Maxwell's equations for electromagnetism. Associated with A is the **curvature 2-form**

$$F = dA + A \wedge A. \tag{5}$$

At a space-time coordinate x, F is a $d \times d$ matrix of elements of $\mathfrak{g}$, with jk entry

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$

As a 1-form taking values in $\mathfrak{g}$, A supplies a mode of lifting closed space-time curves $\gamma : I \to M$ to curves $\tilde{\gamma}(t) : I \to E$, with $I = [0, T]$ a closed interval. We require $\tilde{\gamma}(t)$ to always lie over $\gamma(t)$, i.e. $\tilde{\gamma}(t) = (\gamma(t), g(t))$ locally, with $g(t) \in G$. We lift the curve by the following ODE:

$$\frac{d}{dt} g(t) = -\left[A\left(\gamma(t)\right)\left(\frac{d}{dt} g\right)\right] g(t) \tag{6}$$

with the following solution:

$$g(t) = H(A, \gamma, t) g(0). \tag{7}$$

By construction of the space E, without the notion of A there is no canonical choice of lifting curves in M, and thus of comparing different points $(x_1, g_1), (x_2, g_2) \in E$. It is precisely the existence of such an operator $H(A, \gamma, t)$ that allows such a comparison, and furnishes a notion of differentiation on E at a point $(x, g) \in E$, in the direction $v \in \mathfrak{g}$. Moreover, observe that while $(\gamma(0), H(A, \gamma, T)g(0))$ and $(\gamma(0), g(0))$ both lie over $\gamma(0)$, in general they are not equal. So define the linear map $H(A, \gamma, T) : G \to G$ on the set of points lying over $\gamma(0)$ (the fiber over $\gamma(0)$), which we call the **holonomy operator**. When $H(A, \gamma, T) \not\equiv I$, we say the space E has non-trivial curvature. One can show [Sei82] that if $S \subset \mathbb{R}^d$ is a surface with boundary γ, then to leading order in $|S|$, the surface area of S,

$$e^{\int_S F} \approx H(A, \gamma, T), \tag{8}$$

justifying our interpretation of F as a measure of the local curvature of E.

These remarks conclude our general overview of the geometric structures in gauge theory. We now turn to development of quantum gauge theory, which (unlike the above) lacks a purely satisfactory mathematical structure. Thus the following constructions are formal in nature.

To quantize the gauge fields, which we identified above with the connection 1-forms A on the bundle E, we consider the space $\mathcal{A}$ of all such 1-forms, and introduce the physical action, a map $S_{\mathrm{YM}} : \mathcal{A} \to \mathbb{C}$:

$$S_{\mathrm{YM}}(A) = -\frac{1}{2g^2} \int \mathrm{Tr}(F \wedge *F), \tag{9}$$

where F is the associated curvature 2-form, and $*F$ denotes the Hodge star operator, mapping the space k forms to that of $(d-k)$ forms. Moreover, g is a constant, referred to as the theory's coupling strength. The quantized gauge theory with Yang–Mills action, which we simply call Yang–Mills theory, is formally defined by the complex measure on $\mathcal{A}$

$$d\mu(A) = \frac{1}{Z} e^{-iS_{\mathrm{YM}}(A)} \prod_{j=1}^{d} \prod_{x \in \mathbb{R}^d} dA_j(x), \tag{10}$$

where $dA_j(x)$ is a Lebesgue measure on the vector space $\mathfrak{g}$. However, such a definition cannot in general yield a finite normalizing constant Z, implying that equation (10) is not a reasonable explicit definition. Thus one must be more careful to define the measure $dA = \prod_{j=1}^{d} \prod_{x \in \mathbb{R}^d} dA_j(x)$, such that a subset of the nice properties of Lebesgue measure are retained.

The turn to lattice gauge theory is motivated by two refinements of equation (10). First, one may introduce a lattice regularization as follows: formally, one imagines imposing the integer lattice $\mathbb{Z}^d$ on space-time, and integrating out all spatial degrees of freedom associated with distances less than a lattice spacing. One then views the gauge field as taking values in G, and connecting nearest neighbor points of $\mathbb{Z}^d$. This procedure is called an ultraviolet cutoff by physicists (one has removed the high frequency = ultraviolet components of the field), and is accompanied by a restriction of the lattice to a finite subset $\Lambda \subset \mathbb{Z}^d$. The latter procedure is called an infrared cutoff. One hopes that with suitable estimates, uniform in the lattice extent and spacing, a continuum gauge theory may eventually may be recovered.

The introduction of cutoffs renders the measure well-defined, but as a second refinement one goes further, replacing $-i \to 1$ in equation (10). The substitution is associated with a turn to "imaginary time," rendering the underlying metric Euclidean, rather than Lorentzian. Termed a Wick rotation in physics, this procedure (mathematically, an analytic continuation) is justified on the basis of work by Osterwalder and Schrader, who showed that it is sufficient to construct field theories in this Euclidean setting, as all relevant quantities of interest are analytic in this time coordinate, and thus one may analytically continue "back" to a Lorentzian field theory. For a rigorous discussion of this point, see [FFS92].

Combined, we will see that these two refinements lead naturally to the lattice gauge theory introduced in the following section. The latter theory defines a finite-dimensional probability measure, setting our analysis firmly in the realm of equilibrium statistical mechanics.

2.4 Lattice Gauge Theories and Confinement

2.4.1 Pure Yang–Mills Theory

In this section we introduce a discretization of continuum gauge theories, originally introduced in the setting of Yang–Mills theory by Wilson [Wil75]. This discretization excludes the addition of matter fields (e.g. fermions, Higgs fields), and is called "Pure Yang–Mills" lattice gauge theory. In the next subsection, we expand our definition to include these more general objects.

The following discretization is motivated by a desire to produce a finite-dimensional analog of the continuum path integral for the gauge theory.

To specify a lattice gauge theory on Λ, we introduce the following data:

1. A Compact Lie Group G, with Lie algebra $\mathfrak{g}$. For simplicity, one can imagine $G \subset \mathrm{GL}_n(\mathbb{C})$ as a closed subgroup of the set of $n \times n$ complex matrices, for $n \in \mathbb{Z}_{\geq 0}$

2. A finite-dimensional faithful representation $U(g)$ of G, with character $\chi(g) = \mathrm{Tr}(U(g))$. We use the same notation for G and its representation. We will often use the upper bound for faithful characters. Physically, this representation characterizes the transformation properties of the gauge field.

3. An additional finite-dimensional faithful representation $U'(g)$ of G, with character χ'. Physically, this representation characterizes the transformation of the quark field.

4. The inverse coupling strength, $\beta \in \mathbb{R}_{\geq 0}$. The coupling is the continuous parameter of the system, which defines the phase structure of the theory.

In general, given a gauge group $G \subset \mathrm{GL}_n$, we will take $\chi(g) = \chi'(g) = Tr(g)$. But the generality is useful to separate the (physically distinct) notions of gauge field from that of quarks.

Define a **configuration** to be a map $g : B(\Lambda) \to G$ (called "configurations") with the property $g((y,x)) = g((x,y))^{-1}\ \forall\{x,y\} \in \Lambda$. Denote $g((x,y)) \equiv g_{xy}$, and the set of all configurations $G(\Lambda)$. Note that associated to a configuration g and path $\mathcal{P}$, there is a naturally induced map $W_g : \mathcal{P}(\Lambda) \to G$ given by

$$W_g\mathcal{P} = \prod_{i=0}^{|\mathcal{P}|} g_{(x_i,x_{i+1})}. \tag{11}$$

Note for non-abelian G, the orientation and starting points of the path affect the output of this operator, called the **Wilson operator**. We will often drop the subscript g, when the configuration is clear.

To draw the analogy with the path continuum path integral representation of gauge theories, we next introduce a discretization of the action. This discretization is not unique, and may only be formally motivated by showing formal convergence to the continuum action in the $a \to 0$ limit. One is motivated by the continuum dependence on the local curvature (a 2-form) to introduce actions composed of discrete 2-forms, i.e. with plaquette variables only. Given such a 2-form ϕ_β, a general discretized action will take the form

$$S_\phi(g) \equiv \sum_{P\in P(\Lambda)} \phi_\beta(P), \tag{12}$$

for all $g \in G(\Lambda)$. The most common choice of ϕ defines the **Wilson action**, namely

$$\phi_\beta^W(P) \equiv -\beta\,\mathrm{Re}(\chi(W_g(P))). \tag{13}$$

The use of χ indicates the Wilson action is a map on configurations of the *gauge field*. A formal justification of the convergence of the Wilson action (with the fundamental representation of G, for G a matrix group) to the continuum Yang–Mills action may be found in [Cha18].

In the more restricted setting of U(1) abelian gauge theory, an alternative form of the Wilson action exists, called the **Villain action**. The form of the action follows from the following approximate identity, holding for β small:

$$e^{\beta(\cos(x)-1)} \approx \sum_{n\in\mathbb{Z}} e^{-\frac{\beta}{2}(x-2\pi n)^2}. \tag{14}$$

By representing U(1) elements by the angular form

$$g_{xy} = e^{i\theta_{xy}}, \theta_{xy} \in [-\pi, \pi),$$

we recognize the left hand side of (14) as the exponentiated Wilson action for the fundamental representation of U(1), up to an irrelevant constant. So if we view $\theta : B(\Lambda) \to \mathbb{R}$ as a 1-form (depending on the configuration g), we define the Villain action via the following choice of ϕ_β:

$$\phi^V_\beta(P) = -\log\left(\sum_{n\in\mathbb{Z}} e^{-\frac{\beta}{2}(d\theta_p+2\pi n)^2}\right). \tag{15}$$

For a proof that the Villain action has the same formal continuum limit as the Wilson action, see [Kno05].

With this preparation, we can now define a lattice gauge theory $d\mu_\Lambda$ as follows:

Definition 7 (Pure Y-M Lattice Gauge Theory). *Let $\langle \Lambda, \beta, G, U(g), U'(g)\rangle$ be given. Consider the (finite) product Haar measure $d\sigma$ on the set of configurations $G(\Lambda)$. Then the* ***lattice gauge theory on*** *Λ, with data $\langle \Lambda, \beta, G, U(g), U'(g)\rangle$ and choice of Wilson or Villain action (denoted simply $S(g)$) is the probability measure on $G(\Lambda)$*

$$d\mu_\Lambda(g) = \frac{1}{Z} e^{-S(g)} d\sigma,$$

where Z normalizes the measure to unit total mass. Denote expectation with respect to this measure as $\langle \cdot \rangle_\Lambda$.

We will be primarily interested in the values of functions $F(\{g_{xy}\})$ on the set of configurations, taking values in G, or in a field $\mathbb{C}$ or $\mathbb{R}$. We define the support of a function F to be the set of bonds $(x, y) \in B(\Lambda)$ appearing in the definition of F.

So far, we have defined gauge theory on finite lattices, for which our explicit construction of the gauge theory measure makes sense. Physically, finite lattice size corresponds to microscopic physics; however, one is often interested in scaling effects that arise in the macroscopic, or thermodynamic limit. Mathematically, this corresponds to the limit $\Lambda \nearrow \mathbb{Z}^d$. The definition of the infinite volume limit (alternatively called the thermodynamic limit) of a lattice gauge theory is given below:

Definition 8 (Infinite Volume Limit). *In space-time dimension d, define $\Lambda_n = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \mathbb{Z}_n$ to be a d dimensional hypercube (here with periodic boundary conditions) of side length n. Given a collection $\langle \Lambda_n, \beta, G, U(g), U'(g)\rangle$, let $d\mu_n$ be the associated lattice gauge theory. Then the* ***infinite volume limit*** *is the weak limit of measures $d\mu_\infty \equiv \lim_{n\to\infty} d\mu_{\Lambda_n}$, i.e. a measure on configurations $G(\mathbb{Z}^d)$ such that for all $F : B(\mathbb{Z}^d) \to \mathbb{R}$ of finite support, given n_0 such that $\operatorname{supp}(F) \subset B(\Lambda_{n_0})$, we get the following limit:*

$$\lim_{n\to\infty} \langle F\rangle_{\Lambda_n} = \langle F\rangle_\infty,$$

where $\langle \cdot \rangle_\infty$ is expectation in the infinite volume theory. Alternatively, we often just write $\langle \cdot \rangle$ for the infinite volume expectation.

The existence and uniqueness of this limit requires justification, and the latter is unknown for certain gauge theories. Note one may also define the infinite volume limit through the general formalism of Gibbs states, the language of which we will often adopt in our proofs. However, it is generally easier to construct a limit explicitly, given uniform bounds on correlation functions.

In our definition of gauge theories, β is called the inverse coupling strength of this theory, and we are primarily interested in the existence/uniqueness of the infinite volume limit, as well as the analyticity of the functions $\langle \cdot \rangle_\infty(\beta)$ as β varies. Points of non-analyticity for a given expectation $\langle F\rangle_\infty(\beta)$ are called **critical points** of the theory (and they are said to indicate **phase transitions**), and signal qualitative transitions in the theory's physical behavior. In particular, Wilson [Wil75] showed that the relevant critical points in gauge theories can be located by studying the behavior of $\langle W(\mathcal{L})\rangle_\infty(\beta)$, i.e. the Wilson loop expectations. To these critical points, he showed that qualitative changes in the asymptotics of these Wilson loops correspond precisely to transitions between confining and de-confining phases. The precise statement of this change is discussed in a later section.

To conclude this section, we remark on the general philosophy of approaching a continuum gauge theory via lattice discretization. As described in [FFS92] and [GJ87], and as is familiar from the study of scaling limits of statistical mechanics systems, the presence of phase transitions is crucial for defining a continuum limit of the lattice theory. In particular, around (a particular class of) critical points, the correlation length ξ defining the exponential fall off of correlations approaches ∞. Thus, by an appropriate rescaling of the lattice spacing a as $\beta \to \beta_c$, one can equate the following two limits:

1. ξ fixed, $a \to 0$.
2. $\xi \to \infty$, a fixed.

The first limit has the chance of being a non-trivial continuum field theory, explaining why the study of a lattice theory's behavior in the second limit (i.e. around critical regions) is particularly important.

2.4.2 Gauge Invariance

Both the Wilson and Villain actions are functions only of products of group elements around plaquettes. This property is responsible for the invariance of gauge theory measures $d\mu_\Lambda$ under a wide class of transformations, called **gauge transformations**.

Given a map $h : \Lambda \to G$, the associated gauge transformation is the map on configurations $H_h : G(\Lambda) \to G(\Lambda)$ given by

$$(H_h g)_{xy} = h(x) g_{xy} h(y)^{-1}.$$

Gauge transformations offer the freedom to "fix" a gauge in the course of a proof, by strategically selecting a gauge transformation.

Example 2.1. *Givem a path $\mathcal{P} = (x_1, \cdots, x_n), x_i \in \Lambda$ containing no loops, and a configuration g_{xy}, the following gauge transformation h is easily seen to set all links in $\mathcal{P}$ to the identity link:*

$$h(x_1) = 1, h(x_2) = g^{-1}_{x_1 x_2}, \cdots, h(x_j) = g^{-1}_{x_{j-1} x_j}, \cdots.$$

More generally, let $T \subset B(\Lambda)$ be any forest of bonds, i.e. directed graph with no cycles (not necessarily connected). Then the above construction gives a gauge transformation h for any configuration, setting all links in T to the identity.

Gauge invariance singles out a class of functions $F(\{g_{xy}\})$ that are invariant under gauge transformations, i.e. $F(H_h g) = F(g)$ for all H_h, g. These are called gauge-invariant functions (alternatively, physical observables), and a natural example is the character evaluated on a Wilson operator for closed loops:

Proposition 2.3. *Given an oriented loop $\mathcal{L}$ with vertices $(x_1, x_2, \cdots, x_n)$, interpret the Wilson loop operation $W_g(\mathcal{L})$ as a map $G(\Lambda) \to G$. Then given a gauge transformation H_h,*

$$\chi'(W_{H_h g}(\mathcal{L})) = \chi'(W_g(\mathcal{L})).$$

Proof. Follows immediately from the definition of gauge transformations, and from χ' being a class function. $\square$

It follows from 2.3 that the lattice gauge theory measure is similarly invariant under gauge transformations for finite Λ.

The above proposition proves part of the following, showing the overall lattice gauge measure is gauge-invariant.

Even for a non gauge-invariant operator $F(\{g_{xy}\})$ there is a gauge-averaged version of the operator, denoted $\overline{F}$. Given a gauge transformation H_h, define $F^h(\{g_{xy}\}) = F(\{H_h g_{xy}\})$. Moreover, denote the space of maps $h : \Lambda \to G$ as $H(\Lambda)$. Then the gauge averaged operator is defined to be

$$\overline{F}(\{g_{xy}\}) \equiv \int_{H(\Lambda)} F^h(\{g_{xy}\}) \prod_{x \in \Lambda} \sigma_x,$$

where σ_x is the Haar measure on G. It is not difficult to show $\overline{F}$ is gauge-invariant, and that the gauge average of a gauge-invariant observable is just the original observable.

Gauge invariance allows us to reduce expectations of gauge-invariant observables in lattice theory to a conditional expectation, where the latter is conditioned on a subset $s \in B(\Lambda)$ of links in a forest fixed to prescribed values g'. Given a gauge-invariant $F(\{g_{xy}\})$, let

$$F'(\{g_{xy}\}) = F(\{g_{xy}\}_{xy \notin s}, \{g'\}),$$

$$d\mu'_\Lambda(g) = d\mu_\Lambda|_{xy \notin s},$$

where we "freeze" all links in the measure in the set s' to their prescribed values, and integrate only over remaining links. Then the conditional expectation of F is defined to be

$$\langle F \rangle'_\Lambda = \int F'(\{g_{xy}\}) d\mu'_\Lambda(g).$$

Gauge invariance guarantees that this conditional expectation is equal to the global expectation:

Proposition 2.4. *For F gauge-invariant, $s \subset B(\Lambda)$ a forest with prescribed values $g' \subset G$, and $\langle \cdot \rangle'_\Lambda$ the conditional expectation constructed above, we have*

$$\langle F \rangle_\Lambda = \langle F \rangle'_\Lambda.$$

Proof. By definition of the expectation $\langle F \rangle_\Lambda$, we can write

$$\langle F \rangle_\Lambda = \int \int F(\{g_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s}) e^{-S(\{g_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s})} \prod_{xy \in s} d\sigma_{xy} \prod_{uv \notin s} d\sigma_{uv}.$$

The generalization alluded to in example 2.1 implies there exists a gauge transformation h_g, depending on the configuration links g restricted to s, such that $H_h g$ has the prescribed values g' on the set of links s. Thus, using invariance of the Haar measure under left multiplication, and gauge invariance of the action and F, we can apply the gauge transformation, yelding

$$\begin{aligned}
\langle F \rangle_\Lambda &= \int \int F(\{g'_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s}) e^{-S(\{g'_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s})} \prod_{xy \in s} d\sigma_{xy} \prod_{uv \notin s} d\sigma_{uv} \\
&= \int F'(\{g_{xy}\}) \prod_{xy \in s} d\sigma_{xy} \prod_{uv \notin s} d\sigma_{uv} \\
&= \int \langle F \rangle' \prod_{xy \in s} d\sigma_{xy} \\
&= \langle F \rangle'.
\end{aligned}$$

□

2.4.3 Confinement in Pure Yang–Mills

Wilson [Wil75] argued that color confinement in lattice gauge theories is not restricted to the theory of QCD, but is a general feature of the phase structure of pure Yang–Mills. An unproven, but physically reasonable, assumption is that the setting of pure Yang–Mills theory — with gluons, but not dynamic matter particles (those that experience the confinement) — is sufficient for the study of confinement. The following statement expresses the Wilson characterization of color confinement — see [Wil75] and [Kno05] for physical motivation.

Statement 1 (Confinement Characterization)**.** *Let $\mathcal{L}$ be (for simplicity) a planar loop in Λ with dimensions R, T in fixed lattice directions. Given an infinite volume limit $d\mu \equiv d\mu_\infty$ of a lattice gauge theory, we say particles in the theory are **confined** on the lattice at inverse coupling strength β of the following "area law" bound holds for large R, T:*

$$\langle W(\mathcal{L}) \rangle_\infty \leq C(\beta) e^{-c(\beta) RT}, \tag{16}$$

*for constants C, c depending on β and the gauge group. Particles are said to **unconfined** if the following "perimeter law" bound holds:*

$$\langle W(\mathcal{L}) \rangle_\infty \geq C(\beta) e^{-c(\beta)(R+T)}. \tag{17}$$

A discussion of the physical motivations for these laws may be found in [GL10].

We probe the existence of phase transitions in lattice gauge theories by considering transitions between these two qualitatively different behaviors of loops on the lattice. Having framed the problem as one of locating phase transitions in a spin model, we are able to apply expansion techniques from the general theory of spin systems (in particular, Ising model approaches) to prove the bounds in Statement 1.

2.4.4 Addition of Matter Fields

In this section, we extend the definition of a pure Yang–Mills gauge theory to include matter fields, including both Higgs and fermion fields. This expanded theory allows for complete analysis of Standard Model physics, including the interactions of quarks, other fermion fields, and the symmetry breaking Higgs field. For a description of the underlying physics of these fields, see [Sei82]. Although we introduce additional gauge theory measures/expectations in this section, we reserve the symbols $d\mu_\Lambda, \langle\,\cdot\,\rangle_\Lambda$ for the pure Y-M setting.

To introduce a lattice Higgs field, we introduce a finite-dimensional (real/complex), normed vector space, on which there is a (orthogonal/unitary) representation U_H of the gauge group G. Denote the vector space V_H, with norm $\|\cdot\|_H$. Moreover, assume we are given an even polynomial $V(x)$ of degree ≥ 4, with positive leading term. Then a **lattice Higgs field** is a map

$$\phi : \Lambda \to V_H,$$

to which we associate an action that couples the field to the gauge field, as well as to itself:

$$S_H(\{\phi_x, g_{xy}\}) = -\frac{\lambda}{2} \sum_{(x,y)\in B(\Lambda)} (\phi_x, U_H(g_{xy})\phi_y) + \sum_{x\in\Lambda} V(\|x\|_H).$$

Here $\lambda \in \mathbb{R}$ is the coupling strength of the Higgs field.

Fermion fields require an additional anti-commutative structure, as reflected in their common representation via Grassmann numbers in the continuum path integral representation. Suppose, in the discrete setting, we are given a vector space V_S, which carries a representation of the Clifford algebra, i.e. we have Hermitian operators $\gamma_i \in \mathcal{L}(V_S), i = 0, 1, \cdots, d-1$ such that

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

This is the spin vector space, carrying internal degrees of freedom of the fermion.

In addition, suppose there is a complex vector space V_G, the gauge space, carrying a representation U_F of the group G. The fermion vector space is then the tensor product $V_F = V_S \otimes V_G$.

A **lattice fermion field** is a map

$$\psi : \Lambda \to \{\psi_{\alpha a}(x)\}\ \alpha = 1, \cdots, \dim(V_S)\ a = 1, \cdots, \dim(V_G),$$

where $\{\psi_{\alpha a}(x)\}$ is an orthonormal frame of V_F for each x. We also require that each $\{\psi_{\alpha a}(x)\}$ decomposes as a tensor product

$$\psi_{\alpha a}(x) = u_\alpha(x) \otimes v_a(x), e_\alpha \in V_S, f_a \in V_G.$$

To construct the fermionic action, we introduce the conjugate vector space $\overline{V}_G$, carrying the conjugate representation to U_F. Defining $\overline{V}_F = V_S \otimes \overline{V}_G$, we get a complementary fermion field

$$\overline{\psi} : \Lambda \to \{\overline{\psi}_{\alpha a}(x)\}.$$

We introduce the Grassmann algebra $\mathcal{A}$ generated by the fermionic fields $\{\psi_{\alpha,a}(x), \overline{\psi}_{\alpha,a}(x)\}_{\alpha,a,x}$, i.e. $\mathcal{A}$ is generated by linear combinations (over $\mathbb{C}$) of wedge products of fermion field values for different values of α, a, x.

The fermion action is then given by:

$$S_F(\{g_{xy}\}, \phi(x), \psi) = \sum_{x\in\Lambda} [m\psi(x)\overline{\psi}(x) - \frac{\kappa}{2}\psi(x)\gamma_\mu\big(U(g_{x,x-e_\mu})\overline{\psi}(x-e_\mu) - U(g_{x,x+e_\mu})\overline{\psi}(x+e_\mu)\big)],$$

in which there are implied sums over the internal α, a indices in each term, and κ is a coupling strength, m a fermion mass.

Given the combined action $S_{TOT} = S_F + S_H + S_{YM}$, we now proceed to define the complete lattice gauge theory measure $d\mu_\Lambda^M$, the M indicating the addition of matter fields. We first define the relevant observables of the theory to be elements of the Grassmann algebra $\mathcal{A}$, but with the added freedom of coefficients taking values in the set of bounded functions of $\phi(x), g_{xy}$. To define numerical expectations against the measure, it remains to introduce an evaluation mapping $⨏ : \mathcal{A} \to \mathbb{C}$. We proceed as in [Sei82], first fixing the value of the map on monomials in the algebra with the following two relations:

$$⨏ \bigwedge_{\alpha,a} \psi_{\alpha,a}(x) \wedge \overline{\psi}_{\alpha,a}(x) = 1$$

$$⨏ (\text{monomial of less than full degree in } \alpha, a) = 0.$$

Linearity then defines the map on the remainder of $\mathcal{A}$. Armed with this definition, we define the gauge theory measure to be

$$d\mu_\Lambda^M(g_{xy}, \phi(x), \psi(x)) = \frac{1}{Z_\Lambda^M} e^{S_{TOT}} d\sigma,$$

and expectation of functions F as

$$\langle F \rangle_\Lambda^M = \frac{1}{Z_\Lambda^M} \int d\sigma ⨏ F e^{S_{TOT}}.$$

This section is meant only as an introduction to the complete language of lattice gauge theories, and an indication that the questions we pose in this document about the pure Y-M phase structure should be additionally explored in this more general setting. However, since our main goal is the study of confinement, and according to the Wilson criterion 1 it is sufficient to consider pure Y-M for this purpose, we will seldom return to this most general setting.

3 Classical Results

3.1 Technical Overview

In the following sections, we repeat selected classical proofs of the infinite volume behavior of lattice gauge theory, with focus on the pure Yang–Mills case. The results, while not an exhaustive representation of the technical results in the field, are selected to illustrate the primary techniques for probing different parameter ranges in a lattice gauge theory, and proving results about Wilson loop behavior. This introductory section aims to motivate the various upcoming techniques and results.

1. In section 3.2, we prove Elitzur's theorem for general lattice gauge theories, illustrating the qualitative difference between classical spin systems (e.g. the Ising model) and gauge theories. The latter having a *local* symmetry group, we show that the existence of gauge transformations is sufficient to show all expectations of local order parameters, e.g. $\left\langle \sigma_{xy} \right\rangle_{\Lambda}$, vanish in the infinite volume limit. While not technically involved, the result is a foundational one in situating gauge theories among the general framework of statistical mechanics.

2. Next in 3.3, we show that lattice gauge theories have a well-defined low β phase, in which confinement obtains. Key technical inputs are perturbative expansions of the Wilson action around $\beta = 0$, which converge uniformly in $|\Lambda|$ for β small. This expansion factors naturally into an expansion in terms of connected paths on the lattice—this expansion is an example of a cluster expansion, and is borrowed directly from the analysis of spin systems.

3. Section 3.4 establishes the existence of a Kosterlitz-Thouless topological phase transition in the 4-D U(1) gauge theory. This argument is the first to rely explicitly on dimension-dependent duality arguments relating expectations in different ("dual") statistical theories. Key technical inputs are results in Fourier theory and lattice exterior calculus, which allow for a rigorous treatment of duality.

4. Finally, in section 3.5 we review some results related to non-abelian lattice gauge theory. In particular, results on the relation between gauge theories with groups G and $Z(G)$ highlight Lie-theoretic aspects of the problem of confinement, which are not as visible in the abelian settings considered earlier. Additionally, this section illustrates the utility of correlation inequalities, a technically useful tool for relation expectations in different theories via global "convexity" properties of theories.

It is useful to note that several other methods have been successful in proving properties about the phase diagrams of lattice gauge theories. Some methods which we do not address here include reflection positivity techniques, infrared bounds, chessboard estimates, diamagnetic inequalities, and dimensional reduction.

3.2 Gauge Invariance of the Infinite Volume Limit

In this section we prove a result of Elitzur, roughly stating that gauge symmetries cannot be "broken" in the infinite volume limit. To motivate the statement in a gauge theory setting, we first review the analogous notion of "symmetry breaking" in the Ising model in a magnetic field. The construction of such a model is likely familiar to readers, but we recall the essential definition below:

Definition 9. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice. Let a state on the lattice be a map $\sigma : \Lambda \to \mathbb{Z}_2 = \{\pm 1\}$, with associated energy*

$$H(\sigma) = -\beta \sum_{xy \in B(\Lambda)} \sigma(x)\sigma(y) - h \sum_{x \in \Lambda} \sigma(x). \tag{18}$$

Denote the set of states as $G(\Lambda)$. The ***Ising model*** *on Λ, with coupling β, and magnetic field strength h, is the probability measure $d\mu_\Lambda^I$ on the set $G(\Lambda)$, assigning to σ a probability*

$$d\mu(\{\sigma\}) = \frac{1}{Z^I} e^{-H(\sigma)}. \tag{19}$$

We define the infinite volume measure as in the gauge theory case, and note [GJ87] that convexity properties of the model guarantee the existence of the limit. Examining (18) more carefully, we observe that the map $s : G(\Lambda) \to G(\Lambda)$ defined by

$$(s(\sigma))(x) \equiv -\sigma(x),$$

leaves $H(\sigma)$ invariant when $h = 0$, but has a non-vanishing effect whenever $h \neq 0$. Identifying s with an action of $\mathbb{Z}_2$ on the Ising model measure, we say that the $h = 0$ Ising model has a global symmetry group $\mathbb{Z}_2$ arising from the flipping of *all* spins in the model, and that non-vanishing h *explicitly breaks* the symmetry.

However, we are ultimately interested in what physicists refer to as *spontaneously broken* symmetries, which we define in the following way. Label the Ising model expectation $\langle \cdot \rangle_\Lambda^{I,h}$, and construct the infinite volume expectation

$$\langle \cdot \rangle^{I,h} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \cdot \rangle_\Lambda^{I,h},$$

manifestly a function of the couplings β and h. Thus the phase diagram of the Ising model constructed this way is two dimensional. Suppose $0 \in \Lambda$, and consider $\langle \sigma(0) \rangle^{I,h}$. For $h = 0$, it is clear by symmetry arguments that $\langle \sigma(0) \rangle^{I,0} = 0$, a reflection of the symmetry under the map s. However, Peierls arguments show (in 2-D) that $\langle \sigma(0) \rangle^{I,h} \neq 0$ for $h \neq 0$, suggesting the following problem: if we "remove" the symmetry-breaking term by taking the $h \to 0$ limit, does the expectation converge to 0, its $h = 0$ value? If so, the symmetry is restored continuously, and one says the $\mathbb{Z}_2$ symmetry is not spontaneously broken (i.e. it may only be broken explicitly, by setting $h \neq 0$). The remarkable feature of the 2-D Ising model is that the symmetry breaking is spontaneous in the low temperature (low β) region:

Theorem 3.1. *In the 2-D Ising model, there exists β_C such that for $\beta < \beta_C$,*

$$\lim_{h \to 0^+} \langle \sigma(0) \rangle^{I,h} = - \lim_{h \to 0^-} \langle \sigma(0) \rangle^{I,h} \neq 0. \tag{20}$$

Theorem 3.1 is just another way of expressing the existence of multiple phases in the low temperature region of the Ising model. In the language of symmetry, we see that spin models with *global* symmetries may have spontaneously broken symmetries in regions of parameter space.

The discussion thus far has been a preamble to the setting of gauge theories, for which the symmetry group is much larger than in the Ising model. In particular, the measure is manifestly invariant under local gauge transformations, which are defined pointwise at the sites of the lattice. We wish to show that the addition of local symmetry implies the above spontaneous symmetry breaking cannot happen. More precisely, given arbitrary gauge data, let $\langle \cdot \rangle_\Lambda^h$ denote the expectation in the theory, with an added symmetry breaking term $-h \sum_{xy \in B(\Lambda)} \chi(g_{xy})$ to the Wilson action. Note this action is no longer invariant under local gauge transformations. For simplicity, we will restrict attention to matrix Lie groups $G \subset \mathrm{GL}_n$ for some n, with character χ simply the trace of the corresponding matrix.

Theorem 3.2 (Elitzur)**.** *Consider, if it exists, the infinite volume expectation* $\langle\,\cdot\,\rangle^h$ *of a lattice gauge theory with* $G \subset \mathrm{GL}_n$ *for some* n*, and with added symmetry breaking term. Then for* $ab \in B(\mathbb{Z}^d)$ *fixed,*

$$\lim_{h\to 0}\big\langle \chi(g_{ab})\big\rangle^h$$

exists for all β*, and is equal to* 0*.*

Proof. Let $\{\Lambda_n\}$ be a sequence of finite lattices in $\mathbb{Z}^d$, with the property that the gauge measures $d\mu_{\Lambda_n}$ converge weakly to an infinite volume limit as $n \to \infty$. Recall that the notation $ab \in B(\Lambda)$ indicates $a, b \in \Lambda$ are nearest neighbors, and that the bond ab is directed from a to b. So consider the expectation

$$\big\langle \chi'(g_{ab})\big\rangle^h_{\Lambda_n} = \frac{1}{Z_\Lambda}\int \prod_{xy\in B(\Lambda_n)} d\sigma_{xy}\big(\chi(g_{ab})e^{\beta\sum_P \chi(A_P)+h\sum_{xy}\chi(g_{xy})}\big), \tag{21}$$

where we write $A_P = \mathrm{Re}(\chi(W_g(p)))$. Now perform a variable gauge transformation at a, sending $a \to -I \in \mathrm{GL}_n$. Given any configuration $\{g_{xy}\}$, the result of this gauge transformation is denoted $\{g'_{xy}\}$, and is given by:

$$g'_{xy} = \begin{cases} -g_{xy}, & x \text{ or } y = a \\ g_{xy}, & \text{otherwise.}\end{cases} \tag{22}$$

Similarly, under the change of variables $g_{xy} \to g'_{xy}$, the terms of the form A_P are invariant, but $\chi(g_{xy}) \to \chi(g'_{xy}) - \chi(\delta g_{xy})$, where

$$\delta g_{xy} = \begin{cases} -2g_{xy}, & x \text{ or } y = a \\ 0, & \text{otherwise.}\end{cases} \tag{23}$$

Inserting this change of variables into (21), we arrive at the expression

$$\begin{aligned}\big\langle \chi(g_{ab})\big\rangle^h_{\Lambda_n} &= \frac{1}{Z_\Lambda}\int \prod_{xy\in B(\Lambda_n)} d\sigma_{xy}\big(-\chi(g'_{ab})e^{\beta\sum_P \chi(A_P)+h\sum_{xy}\chi(g_{xy})-h\sum_{xy}\chi(\delta g_{xy})}\big),\\ &= \big\langle -\chi(g_{ab})e^{-h\sum_{xy\in B_a}\chi(\delta g_{xy})}\big\rangle^h_{\Lambda_n},\end{aligned} \tag{24}$$

where by B_a we denote all bonds leaving a. Thus we may bound the following quantity:

$$\begin{aligned}\big|\big\langle \chi(g_{ab})\big\rangle^h_{\Lambda_n} - \big\langle -\chi(g_{ab})\big\rangle^h_{\Lambda_n}\big| &= \big|\big\langle -\chi(g_{ab})\big\{e^{-h\sum_{xy\in B_a}\chi(\delta g_{xy})}-1\big\}\big\rangle\big| &(25)\\ &\leq |e^{c(d)h}-1|\big|\big\langle \chi(g_{ab})\big\rangle^h_{\Lambda_n}\big|, &(26)\end{aligned}$$

where $c(d)$ is a dimensional constant. The final step requires the boundedness of the character, which is easily checked for the trace character on matrix groups. By assumption, the $\Lambda_n \nearrow \mathbb{Z}^d$ limit in (26) exists, giving the same inequality for the expectations in the infinite volume setting. Taking $h \to 0$, the righthand side of (26) vanishes, implying $\big\langle \chi(g_{ab})\big\rangle^h = 0$ as desired. □

3.3 High Temperature Phase

In this section, we follow the argument in the seminal 1977 work of Osterwalder and Seiler [OS78], in which the authors use convergent low β expansions to construct the infinite volume limit of a pure Yang–Mills lattice gauge theory, with arbitrary data. In the first subsection, we construct this low β, or "high temperature", expansion for expectations of observables, and prove uniform convergence for a range of couplings. In the following section, we apply the cluster expansion to prove existence and uniqueness of the infinite volume Gibbs state, as well as exponential clustering of correlations. Afterwards, we prove quark confinement according to Wilson's criterion. These proofs are standard applications of high temperature expansions, unique in their applicability to all gauge-theoretic data. Throughout this section, we use the Wilson action.

The discussion in [OS78] also extends the cluster expansion to prove existence/uniqueness properties of the lattice theory with Higgs field coupling. Certain technical conditions on the Higgs potential are required, but we do not discuss this proof here.

Cluster Expansion

For gauge data $\langle \Lambda, \beta, G, U(g), U'(g) \rangle$, write the lattice gauge measure as

$$d\mu_\Lambda(g) = \frac{1}{Z_\Lambda} e^{-\beta \sum_{P \in P(\Lambda)} A_P} d_\sigma = \frac{1}{Z_\Lambda} \prod_{P \in P(\Lambda)} e^{-\beta A_P} d\sigma,$$

where recall $A_P = Re(\chi(W_g(p)))$. Recalling that $U(g)$ is a faithful representation, we have the boundedness property

$$|A_P| \leq \chi(1) \equiv D,$$

with D the dimension of the representation. It will be convenient for the expansion (for positivity reasons) to add to the action a constant $-\beta D$, adjusting the partition function accordingly. We are free to do this, as the associated scaling of the partition function leaves the measure invariant. Thus in this section we write

$$d\mu_\Lambda(g) = \frac{1}{Z_\Lambda} e^{-\beta \sum_{P \in P(\Lambda)} (A_P + D)} d\sigma = \frac{1}{Z_\Lambda} \prod_{P \in P(\Lambda)} e^{-\beta (A_P + D)} d\sigma, \tag{27}$$

where

$$Z_\Lambda = \int e^{-\beta \sum_{P \in P(\Lambda)} (A_P + D)} d\sigma.$$

Observe that in equation (27), for large β the mass of the measure increasingly centers on configurations with small $|A_P|$, motivating the following expansion of the measure about unity:

$$\begin{aligned} \prod_{P \in P(\Lambda)} e^{-\beta(A_P + D)} &= \prod_{P \in P(\Lambda)} (1 + [e^{-\beta(A_P+D)} - 1]) \\ &= \sum_{Q \subset P(\Lambda)} \prod_{P \in Q} [e^{-\beta(A_p + D)} - 1] \\ &\equiv \sum_{Q \subset P(\Lambda)} \prod_{P \in Q} \rho_P, \end{aligned}$$

with ρ having the evident definition, and the sum taken over all subsets of plaquettes $P(\Lambda)$. The boundedness of A_P, and the added constant to the action, thus guarantee the very useful bounds

$$0 \leq \rho_P \leq C(\beta), \tag{28}$$

for $C(\beta)$ a constant depending only on β (and D, which is fixed throughout the discussion).

With an expansion of the measure in terms of *subsets* of the set of plaquettes, we hope to be able to control expectations $\langle F \rangle$ for large β by bounding the number of contributing subsets. More precisely, let $F : G(\Lambda) \to S$ for some set S (either the group G, or a field $\mathbb{R}, \mathbb{C}$, with support containing only finite many bonds of Λ. Denote $Q_0 \equiv \operatorname{supp}(F)$. By definition of gauge theory expectations, we have

$$\begin{aligned}\langle F\rangle_\Lambda &= \frac{1}{Z_\Lambda}\int F(\{g_{xy}\})\prod_{P\in P(\Lambda)} e^{-\beta(A_p+D)}d\sigma \\ &= \sum_{Q\subset P(\Lambda)}\int F(\{g_{xy}\})\prod_{P\in Q}\rho_p d\sigma. \qquad (29)\end{aligned}$$

Now, we recall a graph-theoretic notion of plaquette connectedness from section 2.2. Given a set of plaquettes $Q\subset P(\Lambda)$, we define the associated graph $G(Q)$ with vertex set Q, and edges between $p,q\in Q$ if and only if the two plaquettes share a bond. We say the set Q is "connected" if $G(Q)$ is connected. This gives a natural notion of connected components of an set $Q\subset P(\Lambda)$. We also say that two plaquettes have non-trivial overlap if they share at least one bond.

With this definition, given a subset $Q\subset P(\Lambda)$, there exists a unique decomposition $Q=Q_1\cup Q_2$, where Q_1 is the union of all connected components of Q containing a plaquette with non-trivial overlap with a plaquette in Q_0. Then, $Q_2 = Q\backslash Q_1$. Since F only depends on plaquette variables represented in Q_1, we may split the integration variables in (29), with $d\sigma = d\sigma_1(Q)d\sigma_2(Q)$ and $d\sigma_1(Q)$ the product measure over bonds in Q_1, and similarly for $d\sigma_2(Q)$. The result is:

$$\langle F\rangle_\Lambda = \frac{1}{Z_\Lambda}\sum_{Q\subset P(\Lambda)}\left(\int F(\{g_{xy}\})\prod_{P\in Q_1}\rho_p d\sigma_1(Q)\right)\left(\int\prod_{P\in Q_2}\rho_p d\sigma_2(Q)\right) \qquad (30)$$

Now, for fixed Q_1 (and fixed Q_0), the above sum over Q amounts to a sum over all $Q_2\subset P(\Lambda)$ with trivial overlap with $Q_1\cup Q_0$. In summing over Q_2, note that

$$\sum_{\substack{Q_2\subset P(\Lambda)\\ Q_2\cap(Q_0\cup Q_1)=\varnothing}}\int\prod_{P\in Q_2}\rho_P d\sigma_2(Q) = Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}},$$

where $Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}$ refers to the associated lattice theory on Λ, excluding the vertices contained in plaquettes in $Q_0\cup Q_1$. In contrast, we will denote $Z_{\Lambda\backslash\{P\}}$ without the overbar, for P a plaquette, to mean the gauge theory excluding the plaquette from the action, but retaining all vertices (and thus all adjacent plaquettes).

Finally, we get the representation

$$\langle F\rangle_\Lambda = \sum_{\substack{Q_1\to Q_0\\ Q_1\subset P(\Lambda)}}\int F(\{g_{xy}\})\prod_{P\in Q_1}\rho_P d\sigma\frac{Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}}{Z_\Lambda}, \qquad (31)$$

which is the **cluster expansion** for finitely supported observables. By the relation $A\to B$ between plaquette sets, we mean that for all plaquettes $P\in A$, the connected component of P in A (in the sense of the connectedness graph of section 2.2) shares a bond in common with a plaquette of B. We say A is "connected to B." As written, the cluster expansion is valid for any β and lattice Λ, but its primary utility lies in the following theorem. Here, we prove absolute convergence of the cluster expansion *uniformly* in the lattice size, which requires β sufficiently small. For this reason, we often call the cluster expansion used here a "high temperature" expansion, identifying β with inverse temperature. We will prove convergence for $F\in L^\infty(C(\Lambda),d\mu)$ of finite support, where $\operatorname{supp}(F)$ is the set of bonds in $C(\Lambda)$ on which F depends, and the infinity norm is

$$\|F\|_\infty = \inf_t\{t\in\mathbb{R}:\mu_\Lambda(\{|F(\{g_{xy}\})|>t\})=0\}.$$

Theorem 3.3. *Let $F\in L^\infty(C(\Lambda),d\sigma)$ have finite support. For β sufficiently small, there exist constants $a=a(F,d,G,\chi), b=b(d,G,\chi)$ such that*

$$\sum_{\substack{Q\to Q_0\\ Q\subset P(\Lambda)\\ |Q|\geq K}}\left|\int F(\{g_{xy}\})\prod_{P\in Q_1}\rho_P d\sigma\frac{Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}}{Z_\Lambda}\right|\leq a(b\beta)^K \qquad (32)$$

Following the discussion in [OS78], we break the proof of the above into several lemmas, some of strictly combinatorial nature, while others utilizing explicit properties of the measure and cluster expansion. We begin with a combinatorial result:

Lemma 3.4. *Given a finite lattice $\Lambda \subset \mathbb{Z}^d$ and $Q \subset P(\Lambda)$, let $N(K)$ be the number of subsets $Q_1 \subset P(\Lambda)$ such that $|Q_1| = K$ and $Q_1 \to Q$ holds. Then*

$$N(K) \leq c_1 c_2^K,$$

for c_1 a function of $|Q|, d$, c_2 a function of the dimension d.

Proof. Note that any connected component of Q_1 must intersect at least one bond $b \subset P$ for a plaquette $P \in Q$. So first, suppose $S \subset Q_1$ is a connected set of plaquettes, with $0 \leq |S| \equiv t \leq K$, and let a bond b be given.

We claim first that the number of connected sets of size t containing bond b is upper bounded by $D^{|S|}$ for a $D = D(d)$ a constant. To see this, recall a set of plaquettes S is connected if and only if $G(S)$, the associated graph, is connected. By the Königsberg Bridge Lemma, the connected graph $G(S)$ may be covered by a walk beginning at the plaquette $P_b \in Q$ containing bond b, of length $2t$, with jumps restricted to adjacent plaquettes. For D_1 the number of neighbors of each plaquette (a function of only d), the number of such walks is upper bounded by $D_1^{2t} \equiv D^t$.

Thus given K plaquettes, a partition $(K_1, K_2, \cdots, K_n)$ with $\sum_i |K_i| = K$, and a set of bonds $(b_1, \cdots, b_n)$ of Q, the number of connected sets T_i of plaquettes with $|T_i| = K_i$, T_i containing b_i is upper bounded by

$$\prod_{i=1}^{n} D^{K_i} = D^K.$$

Thus it remains to determine the number of such partitions of the above form. If $|B(Q)|$ is the number of bonds in Q, then the number of partitions is upper bounded by the number of arrangements of K plaquettes in $|B(Q)|$ urns, i.e.

$$\binom{|B(Q)| + K - 1}{K} \leq 2^{|B(Q)|+K-1}.$$

Combining with the above estimate gives $N(K) \leq c_1(|Q|) c_2(d)^K$, as desired. $\square$

Next, we turn to a simple estimate on the integral appearing in equation (32), relying on the boundedness of ρ_P.

Lemma 3.5. *There is a constant $c_3 = c_3(g, \chi)$ such that for all subsets $Q \subset P(\Lambda)$,*

$$\left| \int F(\{g_{xy}\}) \prod_{P \in Q} \rho_P d\sigma \right| \leq \|F\|_\infty (c_3 \beta)^{|Q|}. \tag{33}$$

Proof. This bound follows immediately from the bound (28) on ρ_P, and the assumption that $F \in L^\infty(C(\Lambda), d\sigma)$. $\square$

The final lemma we will need is a bound on the ratio of partition functions, in terms of the size of the excluded set. Here we will utilize the freedom to choose β small. More precisely:

Lemma 3.6. *For β sufficiently small, and $Q \subset P(\Lambda)$ an arbitrary subset of plaquettes, we have*

$$2^{-|Q|} \leq \left| \frac{Z_{\Lambda \backslash Q}}{Z_\Lambda} \right| \leq 2^{|Q|},$$

where we give the same interpretation to $\Lambda \backslash Q$ as above (i.e. we remove all vertices contained in the plaquette set Q).

Proof. First, suppose we have the result for $|Q| = 1$, i.e. the statement

$$\frac{1}{2} \leq \left| \frac{Z_{\Lambda \backslash \{R\}}}{Z_\Lambda} \right| \leq 2, \tag{34}$$

for $Q = \{R\}$ a single plaquette set. Since any finite $Q = \{P_1, \cdots, P_{|Q|}\}$ may be written as $Q = \bigcup_{i=1}^{|Q|} \bigcup_{j=1}^{i} P_j \equiv \bigcup_{i=1}^{|Q|} T_i$, with $T_j = T_{j-1} \cup P_j$, then equation (34) gives

$$\frac{1}{2} \leq \left| \frac{Z_{T_j \backslash T_{j-1}}}{Z_{T_j}} \right| \leq 2,$$

for all $j = 2, \cdots |Q|$, so the product of these inequalities gives the statement of the lemma. So it suffices to show (34). So let $R \in P(\Lambda)$ be a fixed plaquette, and consider the difference

$$Z_\Lambda - Z_{\Lambda \backslash \{R\}} = \sum_{\substack{Q \subset P(\Lambda) \\ R \in Q}} \int \prod_{P \in Q} \rho_P d\sigma,$$

where the contribution of subsets not containing R vanish in the difference. As in the derivation of the cluster expansion, we can decompose any plaquette subset Q with $R \in Q$ into a subset $Q_1 \to R$, and $Q_2 \subset \Lambda \backslash \overline{(R \cup Q_1)}$. Summing over Q_2 yields a reduced partition function, and inserting into the above line gives

$$Z_\Lambda - Z_{\Lambda \backslash \{R\}} = \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} \int \prod_{P \in Q_1} \rho_P d\sigma Z_{\Lambda \backslash \overline{Q_1 \cup \{R\}}}.$$

We now proceed via induction. Suppose equation (34) has been shown for $|\Lambda| = N$; we will extend to $|\Lambda| = N + 1$. Consider

$$\left| 1 - \frac{Z_\Lambda}{Z_{\Lambda \backslash \{R\}}} \right| \leq \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} \int \prod_{P \in Q_1} \rho_P d\sigma \frac{Z_{\Lambda \backslash \overline{Q_1 \cup \{R\}}}}{Z_{\Lambda \backslash \{R\}}}.$$

By lemma 3.5 and the inductive hypothesis, we get the above line is bounded above as follows:

$$\leq \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} (c_3 \beta)^{|Q|} 2^{|\overline{Q_1 \cup R}| + 1}$$

$$\leq \sum_{K=1}^{\infty} c_4 (c_3 \beta)^K c_5^K,$$

where we applied lemma 3.3 to bound the number of connected subsets. But by picking β sufficiently small, the final line may be made to be less than $\frac{1}{2}$, giving

$$\left| 1 - \frac{Z_\Lambda}{Z_{\Lambda \backslash \{R\}}} \right| \leq \frac{1}{2},$$

which gives equation (34).

□

Proof of Theorem. Applying the lemmas, we may bound (for β sufficiently small)

$$\sum_{\substack{Q \to Q_0 \\ Q \subset P(\Lambda) \\ |Q| \geq K}} \left| \int F(\{g_{xy}\}) \prod_{P \in Q_1} \rho_P d\sigma \frac{Z_{\Lambda \backslash \overline{(Q_0 \cup Q_1)}}}{Z_\Lambda} \right| \leq \sum_{\substack{Q \to Q_0 \\ Q \subset P(\Lambda) \\ |Q| \geq K}} \|F\|_\infty (c_3 \beta)^{|Q|} 2^{|Q|}$$

$$\leq \sum_{|Q|=K}^{\infty} N(|Q|) \|F\|_\infty (c_3 \beta)^{|Q|} 2^{|Q|}$$

$$\leq \sum_{|Q|=K}^{\infty} \|F\|_\infty (2 c_3 c_1 \beta)^{|Q|}$$

$$\leq \|F\|_\infty \frac{(2 c_3 c_1 \beta)^K}{1 - 2 c_3 c_1 \beta}$$

$$\leq \|F\|_\infty (2c_3c_1\beta)^K,$$

as desired.

□

Construction of High T Phase and Properties

In this section, we apply the cluster expansion to prove existence of the infinite volume limit in a limited parameter region (the "high temperature phase"), as well as some key properties of the limit. That all the properties of this section follow from the simple cluster expansion is a testament to the power of the latter - similar tools are unavailable in other regions of a theory's phase diagram.

We begin with the existence and uniqueness statement.

Theorem 3.7. *Let $\Lambda_n \equiv \mathbb{Z}_n \times \cdots \mathbb{Z}_n$ be a d dimensional integer lattice of side length n. Let $d\mu_n \equiv d\mu_{\Lambda_n}$ be the associated lattice gauge theory. Then for all β sufficiently small, the weak limit $d\mu_n \to d\mu$ exists, and is unique.*

Proof. We begin with existence, and follow a typical proof strategy for studying the infinite volume limits of lattice systems. First, we will show that for any function F on the infinite volume lattice (denoted Λ_∞) of finite support, the limit of $\langle F \rangle_n$ exists as $n \to \infty$. Here we use the cluster expansion. In the second step, we apply a functional analytic argument to define a positive, bounded linear functional on the space $C^0(M)$, where $M = C(\Lambda_\infty)$ is the (compact) metric space of configurations on the infinite volume lattice. The Riesz-Markov theorem then gives the desired infinite volume measure.

So to begin, let F be a continuous, bounded function of finite support on Λ_∞, the set of which we denote $C^0_F(M)$. Let N be such that $\mathrm{supp}(F) \subset B(\Lambda_N)$. For $m, n \geq N$ $m \geq n$, we consider the cluster expansion applied to the difference $|\langle F \rangle_m - \langle F \rangle_n|$, noting that in (31), all terms vanish expect for those arising from $Q \subset P(\Lambda), Q \to \mathrm{supp}(F), Q \cap (\Lambda_m \backslash \Lambda_n) \neq \varnothing$. But this implies that all contributing subsets must have $|Q| \geq \mathrm{dist}(\mathrm{supp}(F), \Lambda_m \backslash \Lambda_n)$, where $\mathrm{dist}(A, B)$ for $A, B \subset B(\Lambda)$ is the length of the shortest path of adjacent bonds needed to connect the two subsets. So theorem 3.3 implies

$$\left|\langle F \rangle_m - \langle F \rangle_n\right| \leq a(b\beta)^{\mathrm{dist}(\mathrm{supp}(F), \Lambda_m \backslash \Lambda_n)} \to 0 \ as \ n, m \to \infty.$$

Thus the sequence $\{\langle F \rangle_m\}_{m=1}^\infty$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$, giving the existence of the limit. Next, let $F \in C^0_F(M)$ be given. Define the linear functional $l : C^0_F(M) \to \mathbb{R}$ (or $\mathbb{C}$) by

$$l(F) = \lim_{n \to \infty} \langle F \rangle_n.$$

This satisfies $l(1) = 1$, $l(f) \geq 0$ for $f \geq 0$, and is bounded by the following:

$$|l(F)| \leq \|F\|_\infty.$$

Now, following the discussion in [Kup14] for the Ising model, we recall the following topological facts, which allow us to uniquely extend l to a positive linear functional $\bar{l} : C(M) \to \mathbb{R}$ (or $\mathbb{C}$). Since G is a compact Lie group, by Tychonoff's theorem $C(\Lambda_\infty) = \prod_{i=1}^\infty G$ is a compact space with topology determined by the metric on $C(\Lambda)$

$$d'(g, g') = \sum_{(x,y) \in B(\Lambda_\infty)} 2^{|(x,y)|} d''(g_{xy}, g'_{xy}),$$

where $|(x, y)| = \max\{\mathrm{dist}(x, 0), \mathrm{dist}(y, 0)\}$ is a measure of the Euclidean distance between the bond and the origin, and d'' is a metric on the Lie group. An appeal to the Stone-Weierstrass Theorem shows that $C^0_F(M) \subset C^0(M)$ is in fact dense, implying there is a unique extension of l to the desired positive, bounded, linear functional $\bar{l} : C(M) \to \mathbb{R}$ (or $\mathbb{C}$). But Riesz-Markov then gives the existence of a $d\mu$ on $C(\Lambda_\infty)$ such that for all $F \in C^0_F(M)$,

$$\lim_{n \to \infty} \langle F \rangle_n = \int F d\mu,$$

which is just the desired infinite volume limit.

To discuss uniqueness, we first note that thus far, we have been using free boundary conditions, i.e. we include only plaquettes strictly contained in Λ, setting all other bonds outside Λ to be zero.

The above existence proof works identically with other boundary conditions, but for uniqueness we must consider all possible choices of limiting sequences, i.e. limits with different $\Lambda \nearrow \mathbb{Z}^d$ and different boundary conditions. Independence on the sequence of finite lattices follows immediately from the above cluster expansion computation, as no direct use of the definition of Λ_n was used, other than its limiting properties. Similarly, independence of boundary conditions follows from a cluster expansion argument: writing $\langle F\rangle'_n, \langle F\rangle''_n$ for the expectations taken with different boundary conditions, an appeal to equation (32) shows all terms vanishing except those with a connected path from $\mathrm{supp}(F)$ to $\partial\Lambda_n$, the boundary of the lattice. But the distance $d(\mathrm{supp}(F), \partial\Lambda_n) \to \infty$ as $n \to \infty$, showing $|\langle F\rangle'_n - \langle F\rangle''_n| \to 0$. Since we have shown both sequences converge identically, we get that the resulting linear functional (and thus the infinite volume measure) are independent of boundary conditions. This completes the proof.

□

Next, we turn to exponential clustering of the correlation functions. In a field-theoretic setting, recall we identify the inverse correlation length ξ^{-1} with the lightest mass m of the theory. The existence of a positive correlation length, finite correlation length is called the "mass gap" property of the theory.

Theorem 3.8. *Let $A, B : C(\Lambda) \to \mathbb{R}$ (or $\mathbb{C}$) be functions on the lattice with finite, disjoint supports Q_1, Q_2. Let $\mathrm{dist}(A, B)$ be the length of the smallest path of bonds connecting the supports. Then for β sufficiently small, there exists $0 < \xi < \infty$, and a constant $c = c(A, B)$ such that for all Λ, the exponential clustering property holds:*

$$|\langle AB\rangle_\Lambda - \langle A\rangle_\Lambda \langle B\rangle_\Lambda| \leq c e^{-\frac{\mathrm{dist}(A,B)}{\xi}} \tag{35}$$

Proof. As in the proof of the infinite volume limit, our goal is to suitably manipulate the cluster expansion for the two point function, showing that terms in equation (32) vanish for subsets $Q \subset P(\Lambda)$ with $Q < \mathrm{dist}(A, B)$. To do this, we adapt a technique from [Kup14], in which we consider a copy $\bar\Lambda$ of the lattice Λ, embedded in $\mathbb{Z}^d$ alongside Λ such that the two lattices have non-overlapping plaquette sets. Thus we may consider the gauge theory $d\mu_{\Lambda\cup\bar\Lambda}$ on $\Lambda \cup \bar\Lambda$, with its own copies of the variables $\bar A \equiv A, \bar B \equiv B$. Denote expectation against the union measure $d\mu_{\Lambda\cup\bar\Lambda}$ as $\langle \cdot \rangle_U$.

By construction, the union lattice measure factors as $d\mu_{\Lambda\cup\bar\Lambda} = d\mu_\Lambda d\mu_{\bar\Lambda}$, and the variable pairs $(A, \bar A)$, $(B, \bar B)$ are independent, i.e. $\langle A\bar A\rangle_U = \langle A\rangle_U \langle \bar A\rangle_U$.

Thus we get the following concise representation of the two point function:

$$|\langle AB\rangle_\Lambda - \langle A\rangle_\Lambda \langle B\rangle_\Lambda| = \langle (A - \bar A)(B - \bar B)\rangle_U \tag{36}$$

Let $\Omega = \mathrm{supp}((A - \bar A)(B - \bar B))$. We may repeat the construction of the cluster expansion for the union measure, giving

$$\begin{aligned}\langle (A - \bar A)(B - \bar B)\rangle_U &= \frac{1}{2Z_{\Lambda\cup\bar\Lambda}} \sum_{\Gamma\subset\Lambda\cup\bar\Lambda} \int (A - \bar A)(B - \bar B) \prod_{P\in\Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar\Lambda} \\ &= \frac{1}{2Z_{\Lambda\cup\bar\Lambda}} \sum_{\substack{\Gamma\subset\Lambda\cup\bar\Lambda \\ \Gamma\to\Omega}} \int (A - \bar A)(B - \bar B) \prod_{P\in\Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar\Lambda} Z_{\Lambda\cup\bar\Lambda\setminus\overline{\Gamma\cup\Omega}}.\end{aligned}$$

Observe that if $|\Gamma| < \mathrm{dist}(A, B)$, then no connected component of Γ is connected to the support of *both* $\mathrm{supp}(A) \cup \mathrm{supp}(\bar A)$ and $\mathrm{supp}(B) \cup \mathrm{supp}(\bar B)$. Thus in this case, we may uniquely decompose $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \to \mathrm{supp}(A) \cup \mathrm{supp}(\bar A)$, $\Gamma_2 \to \mathrm{supp}(B) \cup \mathrm{supp}(\bar B)$. Thus the integration will factor into integration over the links in $A_1 = \Gamma_1 \cup \mathrm{supp}(A) \cup \mathrm{supp}(\bar A)$, and those in $A_2 = \Gamma_2 \cup \mathrm{supp}(B) \cup \mathrm{supp}(\bar B)$.

Thus the overall contribution of a term with $|\Gamma| < \mathrm{dist}(A, B)$ factors as

$$\begin{aligned}\int (A - \bar A)(B - \bar B) \prod_{P\in\Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar\Lambda} Z_{\Lambda\cup\bar\Lambda\setminus\overline{\Gamma\cup\Omega}} = \left(\int (A - \bar A) \prod_{P\in\Gamma_1} \rho_P \prod_{xy\in A_1} d\sigma_{xy}\right) \times \\ \left(\int (B - \bar B) \prod_{R\in\Gamma_2} \rho_R \prod_{xy\in A_2} d\sigma_{xy}\right) Z_{\Lambda\cup\bar\Lambda\setminus\overline{\Gamma\cup\Omega}}\end{aligned} \tag{37}$$

Here, we may use the symmetry of the two lattices to observe that the expectation is invariant under $A \leftrightarrow \bar{A}$, *independently* of the $B, \bar{B}$. Our freedom to interchange $A, \bar{A}$ stems from the factorization property of the integral - no connected component of the sets Γ connects the supports of the $A, \bar{A}$ functions with their B counterparts. But this symmetry under interchange forces the integral to be zero, showing all contributions to the cluster expansion with $|\Gamma| < \mathrm{dist}(A, B)$ vanish. Thus equation (32) and the convergence of the cluster expansion give the desired correlation bound. □

Proof of Quark Confinement

Here we include a proof that in the region of β small, for which the convergence result in theorem 3.3 holds, we have the following confinement bound:

Theorem 3.9.

$$\left|\left\langle\chi(C)\right\rangle_\Lambda\right| \leq a(\beta)e^{b(\beta)\,\mathrm{Area}(C)},$$

where C is a rectangular loop of size $R \times T$ contained entirely in a lattice plane, and $\mathrm{Area}(C)$ *is the number of plaquettes in the interior of C.*

Note the bound holds independently of the lattice size, indicating confinement in the infinite volume limit. The proof will rely on key inputs from character theory and the Peter-Weyl theorem, described in section 2.1. Throughout this proof, $W(C)$ denotes the Wilson operator $\chi(\prod_{xy\in C} g_{xy})$.

Proof. Consider again the statement of theorem 3.3, which bounds the tail of the cluster expansion by an exponential in the number of excluded terms. Thus, to show the bound in theorem 3.9, it suffices to show that all subsets $Q \subset P(\Lambda)$, $Q \to Q_0 \equiv \mathrm{supp}(W(C))$ with $|Q| < \mathrm{Area}(C)$ have vanishing contribution to the cluster expansion. More precisely, the bound follows from the following lemma. □

Lemma 3.10. *Let $Q \subset P(\Lambda)$, $Q \to Q_0$, with $|Q| < \mathrm{Area}(C)$. Moreover, suppose the character χ_τ in the definition of the gauge theory is non-trivial on the center $Z(G)$ of G. Then*

$$\int W(C) \prod_{P\in Q} \rho_P d\sigma = 0.$$

The following proof is adapted from [Sei82].

Proof. Recall the following integration formula for Haar measure:

$$\int_G F(g)d\sigma = \int_G \int_{Z(G)} F(\omega g)d\omega d\sigma,$$

where $d\sigma, d\omega$ are the Haar measures on G, $Z(G)$ respectively. So the lemma in fact follows from showing that for each $g \in G$ fixed, the integral over the center vanishes:

$$\int W(\omega C) \prod_{P\in Q} \rho_{\omega P} \prod_{(x,y)\in B(\Lambda)} d\omega_{xy} = 0, \tag{38}$$

where we have introduced the notation $\omega P, \omega C$ to indicate that in computing the values of the ρ function and Wilson loop operator, one first pre-multiplies the group element g_{xy} by the element ω_{xy} in the center (of course, multiplicative ordering is actually irrelevant).

Next, we observe that $T(\{\omega_{xy}\}) \equiv \prod_{P\in Q} \rho_{\omega P}$ is a function on the group $Z(G^{|Q|})$, and moreover is a *class function* by commutativity of the group. Denoting $\omega_{\partial P} = \prod_{(x,y)\in\partial P} \omega_{xy}$, we can apply Peter-Weyl to get the expansion

$$T(\{\omega_{xy}\}) = \sum_\gamma a_\gamma \chi_\gamma(\{\omega_{xy}\}) = \sum_\gamma a_\gamma \prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}),$$

where the sum is taken over the characters for all inequivalent, finite-dimensional, irreducible representations of the group $Z(G^{|Q|})$. We have also used the fact that irreducible representations of abelian groups are one-dimensional, allowing us to factor the character on $Z(G)^{|Q|}$ to a product of characters on the plaquettes comprising the set Q. We explicitly write the Wilson loop as a (non-trivial) character χ_τ on the center, which decomposes into a product of characters acting on the bonds contained in C. Inserting the expansion into (38), we see it is sufficient to show each term in the product vanishes, i.e. for all γ,

$$\int \prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}) \prod_{(x,y)\in C} \chi_\tau(\omega_{xy}) \prod_{(x,y)\in B(\Lambda)} d\omega_{xy} = 0. \tag{39}$$

Recall from 2.4 that if any of the above links $(x,y)\in B(\Lambda)$ contribute non-trivial representations of the center, then the integral in (39) immediately vanishes.

We now isolate these non-trivial links, by defining a set $\bar{Q}$ to be the $(x,y)\in B(\Lambda)$ such that

$$\prod_{\substack{P\in Q\\(x,y)\in P}} \chi_\gamma^P(\omega) \not\equiv 1.$$

We may then factor as follows:

$$\prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}) = \prod_{(x,y)\in\bar{Q}} \Bigg(\prod_{\substack{P\in Q\\(x,y)\in P}} \chi_\gamma^P(\omega_{xy})\Bigg)$$

Since χ_τ is a non-trivial character on the center $Z(G)$, whenever $C \not\subset \bar{Q}$, (39) holds. But of course, $|Q| < \text{Area}(C)$ necessarily implies $C\not\subset\bar{Q}$, proving the claim.

□

3.4 Existence of $\mathrm{U}(1)$ 4D Phase Transition

In this section, we review the argument in the 1982 paper of Fröhlich and Spencer [FS82], in which the authors prove the existence of a de-confining phase transition in 4-D U(1) lattice gauge theory, with the Villain action. In light of the existence of the high-temperature confining phase for small β, we see the existence of a phase transition follows from a proof of the perimeter law for Wilson loop expectations, for sufficiently large β. In the next section we outline the key steps of the proof, which revolves around a duality argument. In particular, we derive the dual model to 4-D U(1) gauge theory, which we will use in the remainder of the argument. In the following sections we study the dual model further, and establish the desired phase transition.

Summary of Result and Proof Sketch

In this section, we work with a lattice $\Lambda \subset \mathbb{Z}^4$, and the gauge group U(1). Identifying U(1) with the complex numbers $e^{i\theta}$ of unit norm, we recall here the definition of the Villain action:

$$S_V(\{g_{xy}\}) = \sum_{P \in P(\Lambda)} \log \left(\sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}(d\theta_P + 2\pi n)^2} \right),$$

where $d\theta_P \equiv \sum_{xy \in P} \theta_{xy}$ is the sum of phases of group elements along edges of the plaquette P, and $\{g_{xy}\}$ a configuration. The associated lattice gauge measure is then:

$$d\mu_\Lambda(\{g_{xy}\}) = \frac{1}{Z_\Lambda} e^{S_V(\{g_{xy}\})} d\sigma \equiv \frac{1}{\widehat{Z}_\Lambda} \prod_{P \in P(\Lambda)} \varphi_\beta(d\theta_P) \prod_{xy \in B(\Lambda)} d\theta_{xy}. \tag{40}$$

In the second equality we have identified Haar integration on $U(1)$ with integration over the phase angle, requiring a modified partition function

$$\widehat{Z}_\Lambda = \int \prod_{P \in P(\Lambda)} \varphi_\beta(d\theta_P) \prod_{xy \in B(\Lambda)} d\theta_{xy}. \tag{41}$$

We have also isolated the expression in the argument of the logarithm in the Villain action, giving the Villain function

$$\varphi_\beta(\theta) = \sum_{n \in \mathbb{Z}} \big[-\frac{\beta}{2}(\theta + 2\pi n)^2 \big].$$

As in our previous studies of confinement, we are interested in the behavior of Wilson loop expectations

$$\big\langle W(\mathcal{L}) \big\rangle(\beta) \equiv \lim_{\Lambda \nearrow \mathbb{Z}^4} \big\langle W(\mathcal{L}) \big\rangle_\Lambda(\beta)$$

in the infinite volume limit. It follows from the high temperature existence proof that $\big\langle W(\mathcal{L}) \big\rangle(\beta)$ satisfies an area law upper bound for sufficiently small β. In order to prove the existence of phase transition, it suffices to show the following qualitatively different behavior for large β:

Theorem 3.11. *Consider the* 4*-D* U(1) *lattice gauge theory* $d\mu_\Lambda$ *on finite* $\Lambda \in \mathbb{Z}^4$*, with expectation* $\big\langle \cdot \big\rangle_\Lambda$*. Let* $\mathcal{L}$ *be a rectangular loop of dimensions* $L \times T$ *contained in a single lattice plane.*

Consider the infinite volume limit of 4*-D* U(1) *lattice gauge theory, with expectation* $\big\langle \cdot \big\rangle$*. Then there is a constant* $d = d(\beta) > 0$ *independent of* Λ*, such that for sufficiently large* β *the perimeter law holds:*

$$\big\langle W(\mathcal{L}) \big\rangle_\Lambda(\beta) \geq e^{-d(L+T)} \tag{42}$$

The proof of theorem 3.11 proceeds in the following steps:

1. Fourier transformation of the measure $d\mu_\Lambda$, converting a Wilson loop expectation at coupling β into the expectation of a "dual operator" D at coupling $\frac{1}{\beta}$, with respect to the transformed measure. The resulting theory is roughly Gaussian.

2. Following the authors' work in [FS81], reformulating the measure in Sine-Gordon representation. This process transforms expectations in the dual theory to convex combinations of expectations in theories of "low activity".

3. Variable transformations give positivity of the low activity measures for sufficiently large β, allowing a use of Jensen's inequality for the eventual lower bound.

The duality argument is presented in Part 1, in which the precise nature of the dual operator D is considered. Details for remaining steps are given in the remaining parts.

Proof 1: Proof of Duality

Now we turn to the duality transformation of the U(1) lattice measure (40) and the associated Wilson loop expectations. This amounts to an exercise in Fourier series, and in the language of the exterior calculus discussed in section 2.2.3. We begin by inserting the fourier series expansion

$$\varphi_\beta(d\theta_P) = \sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P)e^{in_p d\theta_P}$$

in the expression (41) for the partition function:

$$\widehat{Z}_\Lambda = \int \prod_{P\in P(\Lambda)} \{ \sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P)e^{in_p d\theta_P}\} \prod_{xy\in B(\Lambda)} d\theta_{xy}. \tag{43}$$

We have suggestively labeled the integers in the sum n_P, recognizing that in the expansion of the first product in (43), each summand may be labeled uniquely by a map $n_P : P \to \mathbb{Z}$. Recognizing these maps as 2-forms on Λ, we expand the above, giving

$$\begin{aligned}\widehat{Z}_\Lambda &= \sum_{n\in\Lambda^2} \prod_{P\in P(\Lambda)} \hat{\varphi}_\beta(n_P) \int e^{i(n,d\theta)} \prod_{xy\in B(\Lambda)} d\theta_{xy} \\ &= \sum_{n\in\Lambda^2} \prod_{P\in P(\Lambda)} \hat{\varphi}_\beta(n_P) \int e^{i(\delta n,\theta)} \prod_{xy\in B(\Lambda)} d\theta_{xy}.\end{aligned}$$

Observe that if $\delta n_{xy} \neq 0$ for any $xy \in B(\Lambda)$, the exponential integral is 0. So we can restrict attention to n satisfying $\delta n = 0$, for which the integral is simply $(2\pi)^{|B(\Lambda)|}$. The exponent here is just the total number of oriented bonds in the lattice. Also note that for our choice of the Villain function, the Fourier coefficient has a simple closed form:

$$\hat{\varphi}_\beta(n) = ce^{-\frac{1}{2\beta}n^2}.$$

Collecting the above remarks, we get for the Villain action partition function the representation

$$\widehat{Z}_\Lambda = (2\pi)^{|B(\Lambda)|}c^{|P(\Lambda)|}Z_\Lambda,$$

with

$$Z_\Lambda = \sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in P(\Lambda)} e^{-\frac{1}{2\beta}n_P^2}. \tag{44}$$

With the choice of Villain action, we may further simplify equation (3.4). First, recall from proposition 2.2 that $\delta n = 0$ implies the existence of a 3-form m satisfying

$$n = \partial m, \mathrm{supp}(m) \subset \Lambda.$$

But duality gives the existence of a 1-form α on the dual lattice Λ^* with $m = *\alpha$, giving

$$n = *d\alpha.$$

Observe however that the set of such 1-forms α is not determined by α, with $\alpha' = \alpha + d\gamma$ satisfying $n = *d\alpha$ for any function $\gamma : (\mathbb{Z}^4)^* \to \mathbb{Z}$. So instead, define the equivalence class of 1-forms:

$$[\alpha] = \{\beta \in (\Lambda^*)^1_{\mathbb{Z}} : \beta = \alpha + d\gamma, \gamma : (\mathbb{Z}^4)^* \to \mathbb{Z}, \mathrm{supp}(\gamma) \subset (\Lambda)^*\}$$

in which we restrict attention to α taking values in $\mathbb{Z}$.

In order to write the partition function as a sum over such equivalence classes, we need to express terms involving n in terms of α. A short computation gives:

$$\sum_{P\in P(\Lambda)} n_P^2 = \sum_{P\in P(\Lambda)} (*d\alpha)_P^2 = \sum_{P\in P(\Lambda^*)} (d\alpha)_P^2 = (d\alpha, d\alpha)_{\Lambda^*}.$$

Thus

$$Z_\Lambda = \sum_{n:\delta n=0} e^{-\frac{1}{2\beta}(n,n)} = \sum_{\substack{[\alpha]\\ \alpha\in(\Lambda^*)^2_{\mathbb{Z}}}} e^{-\frac{1}{2\beta}(d\alpha,d\alpha)}. \tag{45}$$

Note we drop the subscript on the inner product, since it is evident α acts on the dual lattice. Moreover, $[\alpha]$ indicates that we choose one representative from each equivalence class in the sum.

In the remainder of the section, we apply the same Fourier technique to compute an alternative formula for the Wilson loop expectation. So let $\mathcal{L}$ be a rectangular loop in the (for simplicity) 0-1 dimension lattice plane of Λ, and let $\langle W(\mathcal{L})\rangle_\Lambda(\beta)$ the Wilson loop expectation in the measure (40).

We will need the following two inputs, which are easily verified:

- The n^{th} Fourier series coefficient of $\varphi_\beta(\theta)e^{i\theta}$ is just $\hat{\varphi}_\beta(n-1)$.
- Recall the definition of the Wilson loop: $W(\mathcal{L}) = \prod_{xy\in L} e^{i\theta_{xy}}$. Since $\mathcal{L}$ is rectangular, there is a connected surface of plaquettes Σ with $\partial\Sigma = \mathcal{L}$, with the natural notion of boundary. But the discretized Stoke's theorem then gives

$$W(\mathcal{L}) = \prod_{P\in\Sigma} e^{i(d\theta)_P}. \tag{46}$$

Now we proceed similarly as to the above computation, inserting the Fourier series expansion of $\varphi(\theta)W(\mathcal{L})$ into the Wilson loop expectation:

$$\begin{aligned}
\langle W(\mathcal{L})\rangle &= \frac{1}{\widehat{Z}} \int \prod_{P\in P(\Lambda)} \varphi(d\theta_P) \prod_{P\in\Sigma} e^{i(d\theta)_P} \prod_{xy\in B(\Lambda)} d\theta_{xy} \\
&= \frac{1}{\widehat{Z}_\Lambda} \int \prod_{P\in P(\Lambda)\backslash\Sigma} \{ \sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P) e^{in_P d\theta_P} \} \prod_{P\in\Sigma} \{ \sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P-1) e^{in_p d\theta_P} \} \prod_{xy\in B(\Lambda)} d\theta_{xy} \\
&= \frac{1}{\widehat{Z}_\Lambda} \sum_{n\in\Lambda^2} \prod_{P\in P(\Lambda)\backslash\Sigma} \hat{\varphi}_\beta(n_P) \prod_{P\in\Sigma} \hat{\varphi}_\beta(n_P-1) \int e^{i(d\theta)_P} \prod_{xy\in B(\Lambda)} d\theta_{xy} \\
&= \frac{1}{\widehat{Z}_\Lambda} (2\pi)^{L(\Lambda)} \{ \sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda\backslash\Sigma} \hat{\varphi}_\beta(n_P) \prod_{P\in\Sigma} \hat{\varphi}_\beta(n_P-1) \} \\
&= \frac{1}{\widehat{Z}_\Lambda} (2\pi)^{L(\Lambda)} c^{|P(\Lambda)|} \{ \sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda\backslash\Sigma} e^{-\frac{1}{2\beta}n_P^2} \prod_{P\in\Sigma} e^{-\frac{1}{2\beta}(n_P-1)^2} \} \\
&= \frac{1}{Z_\Lambda} \{ \sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda} e^{-\frac{1}{2\beta}n_P^2} \prod_{P\in\Sigma} e^{-\frac{1}{2\beta}n_P - \frac{1}{2\beta}} \},
\end{aligned}$$

where we applied the relationship (3.4). If we recall the construction of the equivalence class $[\alpha]$ of 1-forms from above, we see that the above sum is just

$$\langle W(\mathcal{L})\rangle = \frac{1}{Z_\Lambda} \sum_{\substack{[\alpha]\\ \alpha\in(\Lambda^*)^2_{\mathbb{Z}}}} e^{-\frac{1}{2\beta}(d\alpha,d\alpha)} \prod_{P\in\Sigma} e^{\frac{1}{\beta}(d\alpha)} e^{-\frac{1}{2\beta}}.$$

Thus we identify the dual of the Wilson loop expectation with the expectation of the operator

$$D_{\partial\Sigma}(\alpha) \equiv \prod_{P\in\Sigma} e^{\frac{1}{\beta}(d\alpha)} e^{-\frac{1}{2\beta}} \tag{47}$$

on the space of equivalence classes of integral 1-forms $\alpha \in (\Lambda^*)^2_{\mathbb{Z}}$. The expectation is taken against the measure

$$d\mu^*([\alpha]) = \frac{1}{Z_\Lambda} e^{-\frac{1}{2\beta}(d\alpha, d\alpha)}, \tag{48}$$

and is denoted $\big\langle \cdot \big\rangle^*_\Lambda(\beta)$. In summary, we have exchanged the problem of studying $\big\langle W(\mathcal{L})\big\rangle_\Lambda(\beta)$ for large β, for that of analyzing

$$\big\langle D_{\partial\Sigma}\big\rangle^*_\Lambda(\beta),$$

whose coupling carries an *inverse* dependence on β. Thus we expect the latter computation to be more tractable. We take up the problem of analyzing $D_{\partial\Sigma}$, called the **disorder operator**, in the next section.

Proof 2: Deriving the Sine-Gordon Representation

In this section we continue the proof of theorem 3.11, by proving the perimeter bound on the disorder operator defined above. The first step is a reformulation of the discrete measure $d\mu^*_\Lambda([\alpha])$ in terms of a continuous Gaussian measure on the space $(\mathbb{Z}^4)^{*1}$ of 1-forms on $(\mathbb{Z}^4)^*$. To begin, we define the Gaussian measure of interest, $d\mu^0_{\Lambda,\epsilon}(\alpha)$:

$$d\mu^0_{\Lambda,\epsilon}(\alpha) \equiv \frac{1}{N_{\Lambda,\epsilon}} e^{-\frac{1}{2\beta}\left\{(d\alpha,d\alpha)_{\Lambda^*}+\epsilon(\alpha,\alpha)_{\Lambda^*}\right\}} \prod_{xy\in B(\Lambda^*)} d\alpha_{xy}, \tag{49}$$

where we have introduced a small regularization term $\epsilon > 0$ to ensure convergence on the space $(\mathbb{Z}^4)^{*1}$, and where $d\alpha$ denotes the Lebesgue measure on $\mathbb{R}$.

Although we are only interested in 1-forms with support in Λ^*, we have shown adjointness of the boundary and co-boundary operators only for forms on $\mathbb{Z}^4$ - see proposition 2.2. Thus we introduce the operator Π_{Λ^*}, the orthogonal projection onto the space $(\Lambda^*)^2$ with respect to the inner product $(\cdot,\cdot)_{\mathbb{Z}^4}$. We then have the adjointness statement

$$(d\alpha, d\alpha)_{\Lambda^*} = (\alpha, \Pi_{\Lambda^*}\delta d\alpha)_{\Lambda^*},$$

from which we see that (49) defines a Gaussian measure of mean 0 and covariance $V_{\Lambda,\epsilon}$, where

$$V_{\Lambda,\epsilon} = \big(\Pi_{\Lambda^*}(\delta d + \epsilon)\big)^{-1}, \tag{50}$$

and inversion is on the space $(\Lambda^*)^1$.

Recall that a general Gaussian measure is characterized by the following Fourier transform relationship:

$$\int e^{i\sum_{xy\in B(\Lambda^*)}\alpha_{xy}\mu_{xy}} d\mu^0_{\Lambda,\epsilon}(\alpha) = e^{-\frac{\beta}{2}(\mu, V_{\Lambda,\epsilon}\mu)_{\Lambda^*}}, \tag{51}$$

for all $\mu \in (\Lambda^*)^1$. We often use the compact notation $\alpha(\mu) \equiv \sum_{xy\in B(\Lambda^*)}\alpha_{xy}\mu_{xy}$.

we now consider the $\epsilon \to 0$ limit of the measure. Suppose μ has support in Λ^*. Then observe that $V_{\Lambda,\epsilon}(\mu) = (1 + \frac{1}{\epsilon}d\delta)(-\Delta+\epsilon)^{-1}\mu$ for Δ the finite-difference Laplacian on Λ with 0 Dirichlet boundary conditions. This is just the result of direct computation:

$$(\delta d + \epsilon)(1 + \frac{1}{\epsilon}d\delta)(-\Delta+\epsilon)^{-1}\mu = (\epsilon + d\delta + \delta d)(-\Delta+\epsilon)^{-1}\mu = \mu.$$

So for μ with $\delta\mu \neq 0$, $V_\epsilon \to \infty$ as $\epsilon \to 0$, implying the right hand side of (51) similarly approaches 0. Another computation using the adjointness of d,δ gives that the orthogonal complement of $\{\mu \in (\Lambda^*)^1 : \delta\mu = 0, \mathrm{supp}(\mu) \in \Lambda^*\}$ is just $\{\mu \in (\Lambda^*)^1 : d\mu = 0, \mathrm{supp}(\mu) \in \Lambda^*\}$. Collecting these results gives the following lemma:

Lemma 3.12. *Define $V_\Lambda = (\Pi_{\Lambda^*}\delta d)^{-1} = (-\Delta)^{-1}$ on the space $\{\mu \in (\Lambda^*)^1 : \delta\mu = 0, \mathrm{supp}(\mu) \in \Lambda^*\}$. Then*

$$\lim_{\epsilon\to 0}\int e^{i\alpha(\mu)} d\mu^0_{\Lambda,\epsilon}(\alpha) = \begin{cases} e^{-\frac{\beta}{2}(\mu, V_\Lambda\mu)_{\Lambda^*}}, if & \delta\mu = 0 \\ 0, & otherwise \end{cases}. \tag{52}$$

This lemma justifies the definition of the measure $d\mu^0_\Lambda(\alpha) = \text{w-lim}_{\epsilon\to 0}\, d\mu^0_{\Lambda,\epsilon}(\alpha)$ on the space of equivalence classes of 1-forms α, where we identify $\alpha \sim \alpha'$ if $d\alpha = d\alpha'$. This measure is determined by its Fourier transform

$$\int e^{i\alpha(\mu)} d\mu^0_\Lambda([\alpha]) = e^{-\frac{\beta}{2}(\mu, V_\Lambda \mu)_{\Lambda^*}} \tag{53}$$

on all 1-forms μ with $\mathrm{supp}(\mu) \subset \Lambda^*$, $\delta\mu = 0$.

Observe that the measure defined by (53) is, apart from the integral constraint on α, exactly the original measure $d\mu^*([\alpha])$ in (48). But we may impose this constraint via a product of delta functions:

$$d\mu^*([\alpha]) = \frac{1}{\Xi} \prod_{xy \in B(\Lambda^*)} \{ \sum_{q'_{xy} \in \mathbb{Z}} \delta(\alpha_{xy} - q'_{xy}) \} d\mu^0_\Lambda([\alpha]). \tag{54}$$

Expanding the delta functions in their Fourier series gives the Sine-Gordon representation of the measure, as described in the following lemma:

Lemma 3.13. *Let $\{z_q\}_{q=1}^\infty \subset \mathbb{R}$ be a sequence with $\sum_{(2\pi)^{-1}q=1}^\infty z_q^{-1} = \frac{1}{2}$. Then*

$$\sum_{q'_{xy} \in \mathbb{Z}} \delta(\alpha_{xy} - q'_{xy}) = \sum_{(2\pi)^{-1} q_{xy} = 1}^{\infty} 2 z_{q_{xy}}^{-1} (1 + z_{q_{xy}} \cos(q_{xy} \alpha_{xy})) \tag{55}$$

Proof. The lemma follows directly from the Poisson summation formula applied to the shifted Dirac distribution $g(n) \equiv \delta(n - \alpha_{xy})$, with Fourier transform

$$\hat{g}(x) = e^{-2\pi i \alpha_{xy} x}.$$

The Poisson summation formula gives that $\sum_{n=-\infty}^\infty g(n)$ can be recovered as the sum of its Fourier transform, sampled at the integers. Thus we get

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{q_{xy}=-\infty}^{\infty} e^{-2\pi i q_{xy} \alpha_{xy}},$$

which through pairing of $q, -q$ and rescaling of q, gives

$$\sum_{n=-\infty}^{\infty} g(n) = 1 + 2 \sum_{(2\pi)^{-1} q_{xy}=1}^{\infty} \cos(q_{xy} \alpha_{xy}).$$

We are free to insert the sequence $z_{q_{xy}}$, due to the constraint $\sum_{(2\pi)^{-1}q=1}^\infty z_q^{-1} = \frac{1}{2}$. □

Before inserting the result of the above lemma into our representation (54), we introduce the following notation:

$$q \equiv \{q_{xy}\}_{xy \in B(\Lambda^*)}, \;\; c_q \equiv \prod_{xy \in B(\Lambda^*)} 2 z_{q_{xy}}^{-1}.$$

It then follows that

$$\Xi d\mu^*([\alpha]) = \sum_q c_q \prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy} \alpha_{xy})) d\mu^0_\Lambda([\alpha]). \tag{56}$$

By selecting the $\{z\}$ non-negative, we see from (56) a decomposition (convex up to normalization) of the measure into expectations taken against the *continuous* measures

$$\prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy} \alpha_{xy})) d\mu^0_\Lambda([\alpha]), \tag{57}$$

which are in general signed measures. Our goal for the remainder of the section is to derive a geometric representation of the measures (57), by further decomposing the products appearing in (57) into sums over disjoint subsets, with negligible "interactions". It should be noted that the motivation for the remainder of the section comes from the authors' original analysis into the

K-S phase transition in abelian spin systems and Coulomb gases, for which the decomposition has a natural physical interpretation as a splitting into neutral clusters of gas molecules. That the clusters are neutral and sparse allows for strong bounds on their contributions to the partition functions for high β. Our developments for the $U(1)$ gauge theory follow similar arguments, but with different interpretation.

Before proceeding, we formalize the notions of the previous paragraph in the following definition.

Definition 10. *A* ***current distribution****, or* ***density****, is a mapping*

$$\rho : B(\Lambda) \to 2\pi\mathbb{Z}$$

of finite support. Moreover, we define an ***ensemble*** $\mathcal{E}$ *to be a collection* $\{\rho_i\}$ *of current densities, such that the following properties hold:*

1. *For all* i, $\mathrm{supp}(\rho_i) \subset \Lambda^*$.
2. *For* $i \neq j, \mathrm{supp}(\rho_i) \cap \mathrm{supp}(\rho_j) = \varnothing$.

Finally, if $\mathcal{E} = \{\rho_i\}$ *is an ensemble satisfying the additional property that*

$$for i \neq j, \mathrm{dist}(\rho_i, \rho_j) \equiv \mathrm{dist}(\mathrm{supp}(\rho_i), \mathrm{supp}(\rho_j)) \geq 2^{\frac{k}{2}},$$

then $\mathcal{E}$ *is a* ***k-ensemble****.*

Lemma 3.14.

$$\prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) = \sum_{\gamma \in I} c_\gamma \prod_{\rho \in \mathcal{E}^1_\gamma} [1 + K(\rho)\cos(\alpha(\rho))], \tag{58}$$

where

1. *I is a finite set, and each $\mathcal{E}^1_\gamma$ a 1-ensemble.*
2. *for all* $\gamma, c_\gamma > 0$.
3. *The following bound holds on $K(\rho)$, where $N_1(\mathrm{supp}(\rho))$ is the number of links b with* $\mathrm{dist}(\mathrm{supp}(\rho), b) \leq 1$*:*
$$0 < K(\rho) \leq 3^{N_1(\mathrm{supp}(\rho))} \prod_{xy \subset \mathrm{supp}(\rho)} z_{|\rho_{xy}|}.$$

Proof. We begin with the easily verified trigonometric identity

$$\begin{aligned}[1 + K_1\cos(\alpha(\rho_1))][1 + K_2\cos(\alpha(\rho_2))] &= \frac{1}{3}[1 + 3K_1\cos(\alpha(\rho_1))] + \frac{1}{3}[1 + 3K_2\cos(\alpha(\rho_2))] \\ &+ \frac{1}{6}[1 + 3K_1K_2\cos(\alpha(\rho_1 - \rho_2))] + \frac{1}{6}[1 + 3K_1K_2\cos(\alpha(\rho_1 + \rho_2))],\end{aligned} \tag{59}$$

which naturally decomposes the products on the left hand side of (58). We now proceed as follows:

1. First, identify each q_{xy} with a 1-form ρ_{xy} with $\mathrm{supp}(\rho_{xy}) = \{xy\}$, $\rho_{xy}(xy) \equiv q_{xy}$.
2. For a pair of bonds $xy, x'y' \in B(\Lambda^*)$ with $xy \cap x'y' \neq \varnothing$, insert the expansion (59) into the left hand side of (58), and expand the resulting product. This step produces a sum of terms of the same form as the left hand side of (58), but with no summand containing both $xy, x'y'$.
3. Complete step (2) for each term in the resulting sum, and for each pair of overlapping bonds. The resulting expansion is a sum of the form
$$\prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) = \sum_{\lambda \in I} c_\lambda \prod_{\rho \in \mathcal{E}^\lambda} [1 + K'(\rho)\cos(\alpha(\rho))], \tag{60}$$

where the $\mathcal{E}^\lambda$ are ensembles, and each $K'(\rho)$ is a product of factors of 3 (corresponding to the number of iterations $\eta(\rho)$ of the expansion step (2)) and factors $z_{\rho_{xy}}$ for all $xy \in B(\Lambda) \cap \mathrm{supp}(\rho)$. Thus we have the bound

$$|K'(\rho)| \leq 3^{\eta(\rho)} \prod_{xy \in \mathrm{supp}(\rho)} z_{|\rho_{xy}|} \tag{61}$$

4. If any of the $\mathcal{E}^\lambda$ are not 1-ensembles, for any $\rho_1, \rho_2 \in \mathcal{E}^\lambda$ with $\mathrm{dist}(\rho_1, \rho_2) < \sqrt{2}$, applying the expansion (59) produces a larger set $\{\mathcal{E}'\}$, which are 1-ensembles. This procedure terminates, and yields the desired decomposition.

Observe that the c_λ formed through this expansion are products of $\frac{1}{3}, \frac{1}{6}$, implying $c_\lambda > 0$ holds. Similarly, tracing through the above procedure shows $\eta(\rho) \leq N_1(\mathrm{supp}(\rho))$, which combined with (61) gives the bound in the statement of the lemma. □

Corollary 3.14.1.

$$d\mu^*_\Lambda([\alpha]) = \frac{1}{\Xi} \sum_{\gamma \in I} d_\gamma \prod_{\substack{\rho \in \mathcal{N}^1_\gamma \\ \delta\rho = 0}} [1 + K(\rho)\cos(\alpha(\rho))] d\mu^0_\Lambda([\alpha]), \tag{62}$$

where d_γ and $\mathcal{N}^1_\gamma$ have the same properties as in lemma 3.13

Proof. Note it follows immediately from lemma 3.14 and (56) that the following representation of $d\mu^*_\Lambda([\alpha])$ holds:

$$d\mu^*_\Lambda([\alpha]) = \frac{1}{\Xi} \sum_{\gamma \in I} d_\gamma \prod_{\rho \in \mathcal{N}^1_\gamma} [1 + K(\rho)\cos(\alpha(\rho))] d\mu^0_\Lambda([\alpha]). \tag{63}$$

It remains to observe that if $\delta\rho \neq 0$, then since distinct densities $\rho_1, \rho_2 \in \mathcal{N}^1_\gamma$ share no plaquettes in their supports, the product (63) decomposes into a sum over products of the form

$$\prod_{\substack{i \\ \rho_i \in A \subset \mathcal{N}^1_\gamma}} K(\rho_i)\cos(\alpha(\rho_i)) d\mu^0_\Lambda([\alpha]), \tag{64}$$

for subsets $A \subset \mathcal{N}^1_\gamma$. For subsets A containing ρ, we see in light of (52) that

$$\int \cos(\alpha(\rho_i)) d\mu^0_\Lambda([\alpha]) = 0,$$

implying (64) may be omitted (this follows from our definition of $d\mu^0_\Lambda$ via (52)). Therefore, in the sum (63) we only retain current densities satisfying $\delta\rho = 0$.

□

The representation (62) is motivated by the authors' analysis into the K-S transition in abelian spin systems and Coulomb gases (see [FS81]), for which (62) has a natural physical interpretation. In the setting of Coulomb gases, current densities correspond to roughly neutral clusters of weakly interacting particles. This weak interaction manifests in our case as strong upper bounds on the terms $K(\rho)\cos(\alpha(\rho))$, for large β.

Proof 3: Change of Variables

We next introduce a linear transformation of the coordinate α. Lemma 3.15 illustrates that the combination $D_{\partial\Sigma}(\alpha)] d\mu^0_\Lambda([\alpha])$ takes an especially simple form under this change of variables, leaving us with the task of bounding the multiplicative terms in the measure. We consider this in the following sub-section.

Recall that we have the following relationship between our desired Wilson loop expectation, and the Disorder expectation:

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) = \int D_{\partial\Sigma}(\alpha) d\mu_\Lambda^*([\alpha]),$$

where $\Sigma \subset P(\Lambda)$ is a set of plaquettes with boundary $\partial\Sigma = \mathcal{L}$ the rectangular Wilson loop. Define the following 2-form $\sigma \in (\Lambda^*)^2_{\mathbb{Z}}$, for $p* \in P(\Lambda^*)$ arbitrary:

$$\sigma(p^*) \equiv \begin{cases} 1 \text{ if } (p^*)^* \in \Sigma \\ 0 \text{ otherwise} \end{cases}. \tag{65}$$

Then define the following auxiliary 2-forms τ, ϵ_Λ:

$$\tau \equiv -\delta(\Delta)^{-1}\sigma, \tag{66}$$

and

$$\epsilon_\Lambda \equiv -\Pi_{\Lambda^*}\delta d(\Delta)^{-1}\sigma. \tag{67}$$

We are interested in the measure $d\mu_\Lambda([\alpha])$ under the change of coordinates

$$\alpha \to \alpha + \tau \tag{68}$$

A justification of this transformation comes in the following computational lemma:

Lemma 3.15. *The following properties of the variable transformation $\alpha \to \alpha + \tau$ hold:*

1.

$$\sigma = \epsilon_\Lambda + \Pi_{\Lambda^*} d\tau \tag{69}$$

2. $(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}$ is perimeter-dominated in the infinite volume limit, i.e.

$$(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*} \le C(\beta)(L+T), \tag{70}$$

as $\Lambda^* \nearrow \mathbb{Z}^4$, where $2(L+T)$ is the perimeter of the Wilson loop.

3. Under the change of coordinates (68),

$$D_{\partial\Sigma}(\alpha) d\mu_\Lambda^0([\alpha]) \to e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} d\mu_\Lambda^0([\alpha]), \tag{71}$$

and

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda, \mu_\rho)_{\Lambda^*})], \tag{72}$$

where $\mu_\rho \in (\Lambda^*)^2$ takes values in $2\pi\mathbb{Z}$, and satisfies $\delta\mu_\rho = \rho$.

Proof. Part 1 The proof of (1) is a direct computation:

$$\begin{aligned} \Pi_{\Lambda^*} d\tau = -\Pi_{\Lambda^*} d\delta(\Delta)^{-1}\sigma &= -\Pi_{\Lambda^*}(d\delta + \delta d)(\Delta)^{-1}\sigma + \Pi_{\Lambda^*}\delta d(\Delta)^{-1}\sigma \\ &= -\Pi_{\Lambda^*}\Delta(\Delta)^{-1}\sigma - \epsilon_\Lambda = \sigma - \epsilon_\Lambda, \end{aligned}$$

where we have crucially used that Δ is the finite difference Laplacian with 0 boundary conditions on $\partial\Lambda^*$, and thus that $\Pi_{\Lambda^*}(d\delta + \delta d) = -\Delta$.

Part 2 Next, consider the behavior of $(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}$ in the limit as $\Lambda^* \nearrow \mathbb{Z}^4$. Using the adjointness of δ, d in this limit, we have first that

$$(\Pi_{\Lambda^*} d\tau, \epsilon_\Lambda) \to (dd\tau, d(\Delta)^{-1}\sigma) = 0,$$

implying (using (69))

$$|(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*} - (\sigma, \epsilon_\Lambda)_{\Lambda^*}| \to 0.$$

So it suffices to consider $(\sigma, \epsilon_\Lambda)_{\Lambda^*} \to (d\sigma, d(\Delta)^{-1}\sigma)_{\Lambda^*} = (*d\sigma, *d(\Delta)^{-1}\sigma)_\Lambda$. An elementary computation yields

$$(*d\sigma)_{xy} = \begin{cases} 1 \text{ if } xy \in \mathcal{L} \\ 0 \text{ otherwise.} \end{cases}$$

Thus $*d\sigma$ is supported on the Wilson loop. Next, recall that the gradient of the Green's function of the lattice laplacian, $d(\Delta)^{-1}\sigma$, is evaluated as the convolution of σ with the kernel of $d(\Delta)^{-1}$. This kernel decays asymptotically as $\frac{1}{r^3}$, implying the kernel is everywhere bounded by $\frac{c}{1+r^3}$ for a constant c. So consider $xy \in \mathcal{L}$, and consider the contributions of the plaquettes $P \in \Sigma$ to this convolution. Thus we get the upper bound

$$(*d(\Delta)^{-1}\sigma)_{xy} \leq \sum_{x=1}^{T}\sum_{y=1}^{L} \frac{c}{1+(y^2+x^2)^{\frac{3}{2}}}.$$

Upper bounding this sum by an integral over $\mathbb{R}^2$, we conclude that the contribution is bounded by a constant. Adding the contributions over the perimeter of the Wilson loop, we conclude

$$(*d\sigma, *d(\Delta)^{-1}\sigma)_\Lambda \leq c'(L+T)$$

for some constant c'. This conclude the proof of perimeter decay in the limit.

Part 3 Finally, we turn to (3). We approach (71) by transforming the terms independently. First, we consider the transformation of the ϵ-regularized measure, excluding terms with ϵ dependence (which vanish in the limit):

$$d\mu^0_{\Lambda,\epsilon}([\alpha]) \to d\mu^0_{\Lambda,\epsilon}([\alpha])e^{-\frac{1}{\beta}(d\alpha,d\tau)_{\Lambda^*}}e^{-\frac{1}{2\beta}(d\tau,d\tau)_{\Lambda^*}}.$$

Inserting (69) and observing the following (which follow from the definitions of σ, ϵ):

$$(d\alpha, \epsilon_\Lambda)_{\Lambda^*} = 0,$$

$$(d\alpha, d\tau)_{\Lambda^*} = \begin{cases} (d\alpha)_{p^*} \text{ if } p \in \Sigma \\ 0 \text{ otherwise} \end{cases},$$

we finally conclude

$$d\mu^0_{\Lambda,\epsilon}([\alpha]) \to d\mu^0_{\Lambda,\epsilon}([\alpha])e^{\frac{1}{\beta}(\sigma,\epsilon)_{\Lambda^*}}e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)_{\Lambda^*}}\prod_{p\in\Sigma}e^{-\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}}. \tag{73}$$

Now consider $D_{\partial\Sigma}(\alpha)$ under the same change of variables:

$$\begin{aligned} D_{\partial\Sigma}(\alpha) = \prod_{p\in\Sigma} e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}} &\to \prod_{p\in\Sigma} e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}}e^{\frac{1}{\beta}(d\tau)_{p^*}} \\ &= e^{-\frac{1}{\beta}(\sigma,\epsilon_\Lambda)_{\Lambda^*}}\prod_{p\in\Sigma}e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{\frac{1}{2\beta}} \end{aligned} \tag{74}$$

Combining the transformations (73) and (74) give (71) as desired. Thus it remains to show (72). To see this, observe that the variable transformation has the direct effect

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+K(\rho)\cos(\alpha(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+K(\rho)\cos(\alpha(\rho)+\tau(\rho))].$$

Now we use the Poincaré lemma to conclude from $\delta\rho = 0$, that there exists a 2-form μ_ρ taking values in $2\pi\mathbb{Z}$, with $\text{supp}(\mu(\rho)) \subset \Lambda^*$, and $\delta\mu_\rho = \rho$. Thus we may compute

$$\tau(\rho) = (\tau,\rho) = (\tau,\delta\mu_\rho) = (d\tau,\mu_\rho) = (\sigma,\mu_\rho) - (\epsilon_\Lambda,\mu_\rho),$$

implying

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+K(\rho)\cos(\alpha(\rho)+\tau(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+K(\rho)\cos(\alpha(\rho)+(\sigma,\mu_\rho)-(\epsilon_\Lambda,\mu_\rho))].$$

The statement (72) then follows by the 2π periodicity of $\cos(x)$, recognizing $(\sigma,\mu(\rho)) \in 2\pi\mathbb{Z}$.

□

Combining the above lemma with the explicit form of the measure $d\mu^*_\Lambda([\alpha])$ gives the following corollary:

Corollary 3.15.1.

$$\left\langle W(\mathcal{L})\right\rangle_\Lambda(\beta) = \frac{1}{\Xi} e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+K(\rho)\cos(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu^0_\Lambda([\alpha]) \tag{75}$$

The variable transformation focuses attention on the multiplicative factors coming from the activities of current ensembles. With an appropriate bound on the fluctuations of each term from unity (which will arise only for high β), and the perimeter bound on the 2-form ϵ_Λ, we will be close to concluding the proof of Theorem 3.11.

Proof 4: Renormalization and Bounds on Effective Activity

In this section we explore the formal similarities between our expression in (75) and the high temperature expansion of section 3.3. As in the latter expansion, we wish to bound the terms

$$K(\rho)\cos(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}), \tag{76}$$

for β large. Unfortunately, our bounds on $K(\rho)$ from lemma 3.14 are insufficient - thus we first exploit the properties of our ensemble construction to extract an "effective activity" $z(\beta,\bar\rho)$ from (76), with this activity exponentially decreasing with β. This will require the introduction of a "renormalized" ensemble $\rho\to\bar\rho$, the definition of which follows naturally from the following lemma.

Lemma 3.16. *Fix a bond $xy\in B(\Lambda^*)$, and let $G(\alpha)$ be a function with no dependence on α_{xy}. Then one may "integrate out" the link xy in the following sense:*

$$\int e^{i\rho\alpha_{xy}} G(\alpha) d\mu^0_\Lambda([\alpha]) = e^{-\frac{\beta}{2n_{xy}}\rho^2} \int e^{-i\rho\overline{\alpha}_{xy}} G(\alpha) d\mu^0_\Lambda([\alpha]), \tag{77}$$

with $n_{xy} = |\{p^*\in P(\Lambda^*) : xy\in\partial p^*\}|$, and $\overline{\alpha}_{xy} = \frac{1}{n_{xy}}(\delta d\alpha)_{xy} - \alpha_{xy}$.

Before proceeding to the proof, we note that by tracing the definitions of d,δ, it is clear that $\overline{\alpha}_{xy}$ is *independent* of α_{xy}, depending only on values of α on adjacent links. This justifies the interpretation of the lemma as "integrating out" links. Moreover, one may compute $n_{xy}=6$ in 4-D, the case of interest here.

Proof. To ensure convergence of all integrals, we return to the ϵ-regularized measure $d\mu^0_{\Lambda,\epsilon}(\alpha)$. First, recall the definition

$$d\mu^0_{\Lambda,\epsilon}(\alpha) = \frac{1}{N_{\Lambda,\epsilon}} e^{-\frac{1}{2\beta}\left\{(d\alpha,d\alpha)_{\Lambda^*}+\epsilon(\alpha,\alpha)_{\Lambda^*}\right\}} \prod_{xy\in B(\Lambda^*)} d\alpha_{xy}.$$

Observe that the measure naturally factors into a product measure

$$d\mu^0_{\Lambda,\epsilon}(\alpha) = d\rho_{\sim xy}(\alpha) \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy},$$

in which $d\rho_{\sim xy}(\alpha)$ is a measure without any explicit dependence on α_{xy}. Now consider the change of variables

$$\alpha_{xy}\to\alpha_{xy}+i\frac{\beta}{n_{xy}}\rho,$$

under which the left hand side of (77) (with ϵ-regularization) becomes

$$\int e^{i\rho\alpha_{xy}} G(\alpha) d\mu^0_{\Lambda,\epsilon}([\alpha]) = \int \prod_{\substack{p^* \\ xy \in \partial p^* \subset P(\Lambda^*)}} e^{i\rho\alpha_{xy}} G(\alpha) e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha)$$

$$= \int \prod_{\substack{p^* \\ xy \in \partial p^* \subset P(\Lambda^*)}} G(\alpha) e^{-\frac{1}{2\beta}((d\alpha)_{p^*} + i\frac{\beta}{n_{xy}}\rho)^2} e^{i\rho(\alpha_{xy} + i\frac{\beta}{n_{xy}}\rho)} d\alpha_{xy} d\rho_{\sim xy}(\alpha)$$

$$= \int \prod_{\substack{p^* \\ xy \in \partial p^* \subset P(\Lambda^*)}} G(\alpha) e^{-\frac{\beta}{2n^2_{xy}}\rho^2} e^{i\rho(\alpha_{xy} - \frac{1}{n_{xy}}(d\alpha)_{p^*})} e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha)$$

$$= e^{-\frac{\beta}{2n_{xy}}\rho^2} e^{i\rho\alpha_{xy}} e^{-i\rho n^{-1}_{xy}(\delta d\alpha)_{xy}} \int \prod_{\substack{p^* \\ xy \in \partial p^* \subset P(\Lambda^*)}} G(\alpha) e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha)$$

$$= e^{-\frac{\beta}{2n_{xy}}\rho^2} \int e^{-i\rho\bar{\alpha}_{xy}} G(\alpha) d\mu^0_{\Lambda,\epsilon}(\alpha). \quad (78)$$

Taking the $\epsilon \to 0$ limit gives the result.

□

The above lemma illustrates that in the process of integrating out a single link, we extract an exponentially damping factor $e^{-\frac{\beta}{2n_{xy}}\rho^2}$. We will wish to apply this lemma repeatedly to the Disorder loop expectation, but we must be careful to consider only a set of links on *disjoint* plaquettes, i.e. those for which no new dependency relationships may arise over repeated applications of lemma 3.16. Thus, define the geometric constant (for $b \in B(\Lambda)$ a fixed bond)

$$c^{-1} = |\{b' \in B(\Lambda) : b' \neq b, \exists P \in P(\Lambda) s.t. b, b' \in \partial P\}| \quad (79)$$

One may show $c^{-1} = 18$ in 4-D. Next, we construct the desired set of sparse bonds:

Lemma 3.17. *Given a current density ρ, there is a set $\mathcal{B}_\rho \subset \operatorname{supp}(\rho)$ with the following properties:*

- *For all $x_1y_1, x_2y_2 \in \mathcal{B}_\rho$ distinct, there is no plaquette $P \in P(\Lambda^*)$ such that $x_1y_1, x_2y_2 \in \partial P$*
-
$$\sum_{xy \in \mathcal{B}_\rho} |\rho_{xy}|^2 \geq c\|\rho\|_2^2 \quad (80)$$

Moreover, if $\rho_1, \rho_2 \in \mathcal{E}^1$ are distinct current densities in a 1-ensemble $\mathcal{E}^1$, then one may choose $\mathcal{B}_{\rho_1}, \mathcal{B}_{\rho_2}$ independently, such that the above properties hold.

Proof. Given a current density ρ, the construction of $\mathcal{B}_\rho$ follows simply by selecting from each plaquette $P \in P(\Lambda^*)$, $P \cap \operatorname{supp}(\rho) \neq \varnothing$, the bond $b \subset P$ such that $|\rho(b)|$ is maximal among bonds in P. One must be careful to eliminate bonds $b, b' \subset P \in P(\Lambda^*)$ constructed using this method (by taking that bond on which $|\rho|$ takes a larger value). The resulting set $\mathcal{B}_\rho$ is then seen to satisfy the desired properties. Moreover, given distinct densities $\rho_1, \rho_2 \in \mathcal{E}^1$, the property $\operatorname{dist}(\rho_1, \rho_2) \geq \sqrt[2]{\geq} 1$ implies no two bonds $b_1 \in \operatorname{supp}(\rho_1), b_2 \in \operatorname{supp}(\rho_2)$, are subsets of a common plaquette. Thus we may proceed with the construction of $\mathcal{B}_{\rho_1}, \mathcal{B}_{\rho_2}$ independently, such that the desired properties hold for both sets. □

We intend to apply lemma 3.16 to all links $xy \in \mathcal{B}_\rho$ for each current density. This goal motivates the definition of a "renormalized" current density $\bar{\rho}$ via the property

$$\alpha(\bar{\rho}) = \sum_{xy \in \mathcal{B}_\rho} \bar{\alpha}_{xy}\rho_{xy} + \sum_{xy \in \sim\mathcal{B}_\rho} \alpha_{xy}\rho_{xy}, \quad (81)$$

where we recall the definition of $\bar{\alpha}$ from lemma 3.16, and where we define $\sim \mathcal{B}_\rho \equiv supp(\rho) \backslash \mathcal{B}_\rho$. That this definition is appropriate follows from the next lemma, in which we systematically extract a factor exponentially damping for high β from the expression (76).

First, given a current density ρ we define the effective activity $z(\beta, \bar{\rho})$ as follows:

$$z(\beta,\bar\rho) = K(\rho) e^{-\frac{\beta}{2}\sum_{xy\in\mathcal{B}_\rho}\frac{\rho_{xy}^2}{n_{xy}}} \tag{82}$$

From the bounds in (80), and the computation $n_{xy} = 6$ in dimension 4, we get

$$z(\beta,\bar\rho) \le K(\rho) e^{-\frac{\beta}{216}\|\rho\|_2^2}. \tag{83}$$

Lemma 3.18.

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) = \frac{1}{\Xi} e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]), \tag{84}$$

with normalization

$$\Xi = \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho))] d\mu_\Lambda^0([\alpha]) \tag{85}$$

Proof. We need only systematically apply lemma 3.16 to the representation of $d\mu_\Lambda^0([\alpha])$ in 3.15.1. First, we insert the identity

$$\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}) = \frac{1}{2} e^{i(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})} + \frac{1}{2} e^{-i(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}, \tag{86}$$

into (75), yielding

$$\prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]) = \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{i\sigma(\rho)(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}$$

$$= \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{-i\sigma(\rho)(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}} \prod_{xy\in\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}} \prod_{xy\in\sim\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}}. \tag{87}$$

We may now apply lemma 3.16 to the integration over all links $xy \in \mathcal{B}_\rho$, observing that this integration replaces α_{xy} with $\overline{\alpha}_{xy}$, the latter a function of the links adjacent to xy. But recalling the construction of $\mathcal{B}_\rho$, we see the integrations of 3.16 for fixed ρ may each be done independently, yielding

$$\int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha])$$

$$= \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{-i\sigma(\rho)(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}} \prod_{xy\in\mathcal{B}_\rho} e^{-\frac{\beta}{2n_{xy}}\rho_{xy}^2} \int \prod_{xy\in\mathcal{B}_\rho} e^{\mp i\rho_{xy}\overline{\alpha}_{xy}} \prod_{xy\in\sim\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}}$$

$$= \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]). \tag{88}$$

Inserting this equality into the expression (75) proves (84) for the Wilson loop expectation value. The calculation for the partition function (85) is identical. □

In order to complete the goal of showing that the terms (76) are small for large β, we need a bound on the behavior of $K(\rho)$. But this bound follows directly from lemma 3.14, where we showed

$$K(\rho) \le 3^{N_1(\mathrm{supp}(\rho))} \prod_{xy\in\mathrm{supp}(\rho)} z_{\rho_{xy}}.$$

Recall the freedom we had to select suitable $\{z_q\}$, subject only to the constraint $\sum_{(2\pi)^{-1}q=1}^{\infty} z_q^{-1} = \frac{1}{2}$. Thus now we pick

$$z_q = e^{\beta_0 q^2},$$

with β_0 chosen to ensure the constraint. This gives

$$0 < K(\rho) \leq 3^{N_1(\mathrm{supp}(\rho))} \prod_{xy \in \mathrm{supp}(\rho)} e^{\beta_0 |\rho_{xy}|^2} \leq e^{\beta_1 \|\rho\|_2^2}$$

for some β_1. But inserting this bound into (83) gives the desired bound on the effective activity:

$$0 < z(\beta, \bar{\rho}) \leq e^{(\beta_1 - \frac{\beta}{216})|\rho_{xy}|^2}, \tag{89}$$

which has the distinct advantage of being small for β large. In particular, for $\beta > 216\beta_1$ the effective activity satisfies

$$z(\beta, \bar{\rho}) < 1,$$

a bound that will be crucial in the application of Jensen's inequality in the following section.

Proof 5: Perimeter Law Bound

To summarize our progress so far, observe that 3.18 establishes the expectation of the Wilson loop operator as a product of a perimeter-bounded exponential, and a ensemble product of perturbations about unity, the size of these perturbations being *exponentially damped* for large β. In this section we carefully apply Jensen's inequality to establish a lower bound on the expectation value, and apply elementary geometric estimates to establish the desired perimeter bound.

First, note that by (89), for sufficiently large β the following measure is non-negative for all ensembles $\mathcal{N}_\gamma^1$ appearing in the decomposition (84):

$$\prod_{\substack{\rho \in \mathcal{N}_\gamma^1 \\ \delta\rho = 0}} [1 + z(\beta, \bar{\rho}) \cos(\alpha(\bar{\rho}))] d\mu_\Lambda^0([\alpha]).$$

As a positive measure, write $\langle \cdot \rangle_{\mathcal{N}_\gamma^1}$ for expectation against the above measure, with added normalization. As a positive measure, we may apply Jensen's inequality to bound below expectations. To this end, we will use the following lemma:

Lemma 3.19. *For $\alpha, \theta, z \in \mathbb{R}$ with z sufficiently small,*

$$1 + z\cos(\alpha - \theta) \geq (1 + z\cos(\alpha)) e^{E(\alpha,\theta)} e^{O(\alpha,\theta)} e^{F(z,\theta)}, \tag{90}$$

where

$$E(\alpha, \theta) \equiv (1 + z\cos(\alpha))^{-1} z \cos(\alpha)(cos(\theta) - 1),$$
$$O(\alpha, \theta) = (1 + z\cos(\alpha))^{-1} z \sin(\alpha)\sin(\theta),$$

and

$$F(z, \theta) = -2\left(\frac{z}{1-z}\right)^2 \theta^2.$$

Moreover, $E(\alpha, \theta)$ satisfies the following bound:

$$E(\alpha, \theta) \leq \frac{1}{2} \frac{z(\beta, \bar{\rho})}{1 - z(\beta, \bar{\rho})} \theta^2. \tag{91}$$

Proof. To prove (90), we begin with the identity

$$1 + z\cos(\alpha - \theta) = (1 + z\cos(\alpha))\left[1 + \frac{z\cos(\alpha)(\cos(\theta) - 1) + z\sin(\alpha)\sin(\theta)}{1 + z\cos(\alpha)}\right] \equiv (1 + z\cos(\alpha))[1 + g(\alpha, \theta)],$$

for $g(\alpha, \theta)$ with the natural definition. Taylor's theorem with remainder (first in α, then in θ, gives

$$\ln(1 + g(\alpha, \theta)) \geq g(\alpha, \theta) - \frac{1}{2} \frac{1}{(1+c)^2} g(\alpha, \theta)^2 \tag{92}$$

for some $c \in [0, g(\alpha,\theta))$. But a simple computation shows that for z sufficiently small, $|g(\alpha,\theta)| \leq \frac{1}{2}$, so we get

$$g(\alpha,\theta) - \frac{1}{2}\frac{1}{(1+c)^2}g(\alpha,\theta)^2 \geq g(\alpha,\theta) - 2g(\alpha,\theta)^2.$$

Using the elementary identities

$$|\sin(\theta)| \geq |\theta|, |1-\cos(\theta)| \leq \frac{1}{2}\theta^2, 2(a^2+b^2) \geq (a-b)^2, \tag{93}$$

we can bound $g(\alpha,\theta)$ above as

$$\begin{aligned} 2g(\alpha,\theta)^2 &\leq \big(\frac{z}{1-z}\big)^2(\cos(\alpha)^2(\cos(\theta)-1)^2 + \sin(\alpha)^2\sin(\theta)^2) \\ &\leq \left(\frac{z}{1-z}\right)^2 2(1-\cos(\theta)) \leq 2\left(\frac{z}{1-z}\right)^2\theta^2. \end{aligned}$$

Inserting this inequalities into (92) and exponentiating, we recover (90) as desired.

The proof of (91) follows an identical logic, employing the identities in (93). □

The estimates in lemma 3.19 imply we may use Jensen's inequality as follows:

$$\begin{aligned} &\int \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+z(\beta,\bar{\rho})\cos(\alpha(\bar{\rho})-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})]d\mu^0_\Lambda([\alpha]) \\ &\geq \int \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+z(\beta,\bar{\rho})\cos(\alpha(\bar{\rho}))]e^{E(\alpha(\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}e^{O(\alpha(\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}e^{F(z(\beta,\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}d\mu^0_\Lambda([\alpha]) \\ &\geq Z_{\mathcal{N}^1_\gamma}\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} \{e^{-\left\langle E(\alpha(\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}e^{-\left\langle O(\alpha(\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}e^{\left\langle F(z(\beta,\bar{\rho}),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}\}. \end{aligned} \tag{94}$$

Observe $O(\alpha,\theta)$ is odd as a function of α, so its expectation (an even function of its argument) vanishes. Moreover, $F(z,\theta)$ is a constant with respect to the expectation, so its expectation value follows trivially. Now define the following function:

$$\gamma(z) = \frac{1}{2}\frac{z}{1-z} + 2\frac{z^2}{(1-z)^2}.$$

Then by combining the representation of the Wilson loop expectation from lemma 3.18 with the bound on $E(\alpha,\theta)$ from lemma 3.19, we get

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) \geq e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)}\left\{\sum_{\gamma\in I}\lambda_{\mathcal{N}^1_\gamma}\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} e^{-\gamma(z(\beta,\bar{\rho}))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*}}\right\}, \tag{95}$$

where we have defined $\lambda_{\mathcal{N}^1_\gamma} = d_\gamma\frac{Z_{\mathcal{N}^1_\gamma}}{\Xi}$. Recalling the definition of Ξ in 3.18, we see

$$\sum_{\gamma\in I}\lambda_{\mathcal{N}^1_\gamma} = 1. \tag{96}$$

It remains to appropriately bound the terms $\gamma(z(\beta,\bar{\rho}))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*}$. This is accomplished in the following lemma, using geometric estimates:

Lemma 3.20. *There exists $d(\beta)$ such that for β sufficiently large,*

$$\gamma(z(\beta,\bar{\rho}))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*} \leq d(\beta)|\epsilon_\Lambda(p(\rho))|^2, \tag{97}$$

where $p(\rho) \in P(\Lambda^*)$ satisfies the following properties:

1. *There exists* $b \in \operatorname{supp}(\rho)$, $b \subset p(\rho)$

2. *For* $\rho_1, \rho_2 \in \mathcal{N}_\gamma^1$ *distinct current densities,* $p(\rho_1) \neq p(\rho_2)$.

Proof. Recall that by the Poincaré lemma, we have $\operatorname{supp}(\mu_\rho) \subset \Omega_\rho$, where for given ρ, Ω_ρ is the smallest hypercube containing $\operatorname{supp}(\rho)$. Thus we have

$$|(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}| \leq \max_{P \in \Omega_\rho} |\epsilon_\Lambda(P)| \max_{P \in P(\Lambda^*)} |\mu_\rho(P)||\Omega_\rho|, \tag{98}$$

with $|\Omega_\rho|$ denoting the cardinality of Ω_ρ. Given the ensemble $\mathcal{N}_\gamma^1 \ni \rho$, select a plaquette $P(\rho)$ such that properties (1) and (2) of the lemma hold. This is possible due to the definition of a 1-ensemble, guaranteeing non-overlapping supports of constituent densities. Thus it remains to bound the individual terms on the right hand side of (98). It immediately follows by the Poincaré lemma that

$$\max_{P \in P(\Lambda^*)} |\mu_\rho(P)| \leq \|\rho\|_1 \leq \|\rho\|_2^2,$$

with the last inequality following immediately from ρ taking values in $2\pi\mathbb{Z}$. Next, observe isoperimetric inequalities give a bound on the cardinality of Ω_ρ, i.e.

$$|\Omega_\rho| \leq dL(\rho)^4,$$

for some constant d, where $L(\rho)$ is the number of links in the support of ρ. A similar argument shows

$$\max_{p \in \Omega_\rho} |\epsilon_\Lambda(p)| \leq bL(\rho)^4 |\epsilon_\Lambda(p(\rho))|.$$

Collecting the above results gives

$$|(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}| \leq CL(\rho)^8 |\epsilon_\Lambda(p(\rho))| \|\rho\|_2^2, \tag{99}$$

for some constant C. Now observe that there are finitely many current densities $\rho \in \mathcal{N}_\gamma^1$ (and finitely many γ), so for β sufficiently large we have $z(\beta, \overline{\rho}) \leq 1 - \delta$ for all ρ, and for a fixed $\delta > 0$. From this we conclude

$$\gamma(z(\beta, \overline{\rho}))(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}^2 \leq d(\beta) |\epsilon_\Lambda(p(\rho))|^2, \tag{100}$$

as desired. □

Combining the bound in (97) with the lower bound (95) on the Wilson loop expectation, we get that for β sufficiently large,

$$\langle W(\mathcal{L}) \rangle_\Lambda (\beta) \geq e^{-\{\frac{1}{2\beta} + d(\beta)\}(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}}. \tag{101}$$

Recalling the perimeter behavior (70) of the inner product appearing in (101), we see that the perimeter behavior of the Wilson loop expectation follows. This completes the proof of theorem 3.11, as all estimates hold in the $\Lambda \to \mathbb{Z}^4$ limit.

3.5 Results on Non-abelian Gauge Theory

Discussion of Results

Thus far, we have only considered non-abelian gauge theories in the perturbative β small regime. Proofs of the phase structure of non-abelian theories in higher dimensions turn out to be quite difficult, and it remains unknown whether the 4-D SU(3) theory of the strong nuclear force is confining, as is experimentally observed. However, a result due to Fröhlich [Frö79] establishes a general relationship between confinement in G-gauge theories, and $Z(G)$-theories, where $Z(G)$ is the center of G. This relationship proves insufficient to show confinement in most interesting cases, but the techniques offer a unified way of viewing confinement in general Lie group theories as a product of confinement in subgroups of the center. In this section we convey the proof in [Frö79], which is simplified by excluding the coupling of Higgs fields. However, the general proof is a simple extension of what is presented below.

Our major goal is the following theorem:

Theorem 3.21. *Given a compact Lie group G with center $Z(G)$, let χ^q be an irreducible character determining the action of the Wilson loop, and χ the irreducible character used in the definition of the Yang–Mills action. Suppose χ^q is non-trivial on $Z(G)$. Then if the area law holds for the theory with gauge group $Z_\chi \equiv \chi(Z(G))$, it additionally holds for the full G-lattice theory.*

Recall that one describes condition χ^q being non-trivial on $Z(G)$ as having "fractionally charged" quarks. The above theorem is of intrinsic interest, but the following result will allow us to approach confinement in $Z(G)$-theories by related questions in classical spin systems of lower dimensions (such results are called "dimensional reduction" results). These relationships, reflected in the following theorem, will be useful in applications:

Theorem 3.22. *Under the same assumptions as in theorem 3.21, if exponential clustering of spins obtains in a $\nu - 1$ dimensional generalized Ising model with spins in Z_χ, then the area law holds in the ν dimensional Z_χ-gauge theory.*

Using results on the generalized Ising models and theorem 3.22 as input, the ultimate interest will lie in the following two corollaries, establishing general confinement for low dimensional non-abelian lattice gauge theory:

Corollary 3.22.1. *Given a compact Lie group G, the two dimensional lattice theory corresponding to G confines fractionally charged quarks for all β.*

Corollary 3.22.2. *In three dimensions, the $\mathrm{U}(n)$ gauge theory with n arbitrary confines fractionally charged quarks for all β.*

Proof

Proof of Theorem 3.21. We first recall definitions of the lattice gauge measure under consideration. Let χ^q be the irreducible character associated with the quarks, such that $\chi^q(\tau) \not\equiv 1$ for $\tau \in Z(G)$. Additionally, let χ be another irreducible character on G with which we define the Wilson action:

$$S_W(\{g_{xy}\}) = -\beta \sum_{P \in P(\Lambda)} \mathrm{Re}(W_g(P)), \tag{102}$$

and associated probability measure

$$d\mu_\Lambda(\{g_{xy}\}, \beta) = \frac{1}{Z_\Lambda} e^{-S_W(\{g_{xy}\})} \prod_{xy \in B(\Lambda)} d\sigma_{xy}, \tag{103}$$

where we recall that W is the Wilson loop, i.e. the ordered product of link variables comprising the plaquette P. To simplify notation, in this section we will write $g_C \equiv \prod_{xy \in C} g_{xy}$, given a configuration $\{g_{xy}\}$ on the lattice and rectangular loop $C \subset \Lambda$. Moreover, recall the notation $d\theta_C$ for the sum of angular coordinates along a curve C, in the complex representation $g = e^{i\theta}$.

Now we observe some algebraic aspects of χ and χ^q. As characters on $Z(G)$, both χ, χ^q are irreducible characters on an abelian group. Thus the representations corresponding to the characters are 1-dimensional when restricted to $Z(G)$. But this implies the representation U^χ, restricted to $Z(G)$, satisfies

$$U^{\chi}(g) = \chi(g) \in \mathbb{C},$$

i.e. the representation acts via complex multiplication by the character value (similarly for χ^q). Moreover, G compact and $Z(G)$ a closed subgroup together give that the image $\chi(Z(G))$ is a compact subgroup of $\mathbb{C}$, and thus we have $\chi(Z(G)) \subset S^1$. This allows us to parameterize $\chi(Z(G))$ in exponential form $e^{i\theta}$, $\theta \in [-\pi, \pi)$, and to write the Haar measure on $\chi(Z(G))$ as $d\lambda(\theta)$. As irreducible representations of subgroups of S^1, Without loss of generality there exists integer q such that we can write

$$\chi^q(\tau) = e^{iq\theta}, \chi(\tau) = e^{i\theta}, \tag{104}$$

for some θ depending on τ. The following identites will be useful, following from τ a central element of the group:

$$\chi^q((g\tau)_C) = \chi^q(g_C\tau_C) = \chi^q(g_C)e^{iqd\theta_C} \tag{105}$$

$$\chi((g\tau)_C) = \chi(g_C\tau_C) = \chi(g_C)e^{id\theta_C} \tag{106}$$

Now we turn to the computation of the Wilson loop expectation value, where we represent the latter using the character χ^q.

$$\begin{aligned}\langle \chi^q(g_C) \rangle(\beta) &= \frac{1}{Z_\Lambda} \int \chi^q(g_C) e^{-S_W(\{g_{xy}\})} \prod_{xy \in B(\Lambda)} d\sigma_{xy} \\ &= \frac{1}{Z_\Lambda} \int \prod_{xy \in B(\Lambda)} d\sigma_{xy} \int \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta) \chi^q((\tau g)_C) e^{-S_W(\{\tau g_{xy}\})} \\ &= \frac{1}{Z_\Lambda} \int \prod_{xy \in B(\Lambda)} d\sigma_{xy} \chi^q(g_C) \int e^{iqd\theta_C} e^{-S_W(\{\tau g_{xy}\})} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta).\end{aligned} \tag{107}$$

Writing out $S_W(\{\tau g_{xy}\})$, we see

$$\begin{aligned}S_W(\{\tau g_{xy}\}) &= -\beta \operatorname{Re} \sum_{P \in P(\Lambda)} \chi(g_P \tau_P) \\ &= -\beta \operatorname{Re} \sum_{P \in P(\Lambda)} e^{id\theta_P} \chi(g_P).\end{aligned}$$

But we may expand the complex exponential, giving

$$\begin{aligned}\operatorname{Re}\left(e^{id\theta_P}\chi(g_P)\right) &= \operatorname{Re}\left((\cos(d\theta_P) + i\sin(d\theta_P))(\operatorname{Re}\chi(g_P) + i\operatorname{Im}\chi(g_P))\right) \\ &= \cos(d\theta_P)\operatorname{Re}\chi(g_P) - \sin(d\theta_P)\operatorname{Im}\chi(g_P) \\ &\equiv \cos(d\theta_P)J_P - \cos\Big(d\theta_P + \frac{\pi}{2}\Big)K_P.\end{aligned} \tag{108}$$

If one defines the probability measure

$$d\mu_{J,K} \equiv \frac{1}{Z'(g)} e^{\beta \sum_{P \in P(\Lambda)} \cos(d\theta_P)J_P - \cos\left(d\theta_P + \frac{\pi}{2}\right)K_P} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta) \equiv \frac{e^{\beta \sum_{P \in P(\Lambda)} \cos(d\theta_P)J_P - \cos\left(d\theta_P + \frac{\pi}{2}\right)K_P} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta)}{\int e^{\beta \sum_{P \in P(\Lambda)} \cos(d\theta_P)J_P + \cos\left(d\theta_P + \frac{\pi}{2}\right)K_P} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta)} \tag{109}$$

with expectation denoted $\langle \cdot \rangle_{J,K}$, we may rewrite (107) as

$$\langle \chi^q(g_C) \rangle(\beta) = \frac{1}{Z_\Lambda} \int \prod_{xy \in B(\Lambda)} d\sigma_{xy} \chi^q(g_C) Z'(g) \langle e^{iqd\theta_C} \rangle_{J,K}. \tag{110}$$

Our goal is to apply a correlation inequality to appropriately bound the expectation $\langle \cdot \rangle_{J,K}$, which is a measure on Z_χ^Λ with couplings K_P, J_P a function of an ambient configuration $\{g_{xy}\}$.

Correlation inequalities are useful for these types of uniform bounds, in which we aim to dominate a theory with fluctuating coupling constants by one with specified constants.

Before stating the desired inequality, originally proven in [MMSP78], we first define the gauge theory on Z_χ. For a configuration $g : B(\Lambda) \to Z_\chi$, define an action

$$A(g) \equiv -\beta \sum_{P \in P(\Lambda)} \cos(d\theta_P),$$

and associated gauge theory measure

$$d\mu'_\Lambda(\beta') = \frac{1}{Z'} e^{-A} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta) \tag{111}$$

with expectation denoted $\langle \cdot \rangle_{Z_\chi}$.

Lemma 3.23. *For arbitrary $\alpha \in \mathbb{R}$, $q \in \mathbb{Z}$, assuming*

$$\beta[|J_P| + |K_P|] \leq \beta' \tag{112}$$

for all $P \in P(\Lambda)$, we have

$$\pm \left\langle \cos(qd\theta_C + \alpha) \right\rangle_{J,K} \leq \left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}. \tag{113}$$

Proof. See [MMSP78]. □

If d is the dimension of the representation corresponding under which quarks transform, then since $\chi^q(1) = d$, and $|\chi^q(g)| \leq d$ for all g, we conclude that (112) is satisfied if $2d\beta \leq \beta'$. We therefore conclude

$$|e^{iqd\theta_C}(\beta)| \leq 2\left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}(2d\beta). \tag{114}$$

Inserting (114) into (110), and using the bound on $|\chi^q(g)|$, we conclude

$$|\left\langle \chi^q(g_C) \right\rangle(\beta)| \leq 2d\left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}. \tag{115}$$

For confining Z_χ-theory, the right hand side of (115) is upper bounded by an area law. Thus the desired bound follows for the full gauge theory, proving the theorem.

□

We do not discuss here the proof of 3.22, but interested readers are referred to [Frö79] and the references therein. Assuming the two theorems, corollary 3.22.1 follows from the observation that the Z_χ generalized Ising model in 1 dimension has exponential clustering, a fact that may be obtained through explicit analysis of the latter spin model.

Corollary 3.22.2 follows from the identification of Z(U(n)) with a U(1) subgroup, and the application of the well known result that U(1) is confining in 3-D for all β (see [GM81]).

Note it does not follow from 3.21that confinement obtains in SU(n) gauge theories in 3-D and 4-D, as $Z(\mathrm{SU}(n)) = \mathbb{Z}_n$ is known to have phases with non-confining behavior in those dimensions [FS82].

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