SYMMETRIC HOMOMORPHISMS OF ABELIAN VARIETIES

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1. Overview

This note will give a brief review of the highlights of the theory of abelian varieties, and having set up the prerequisite language, will prove the following result:

Theorem 1.1. *Let X be an abelian variety over a field k, and $f : A \to \widehat{A}$ a symmetric homomorphism. Then there is a finite Galois extension L/k such that ϕ_L has the form $\phi_{\mathscr{L}}$ for some line bundle $\mathscr{L}$ on A_L.*

The two references are David Mumford's book *Abelian Varieties* and Brian Conrad's 2015 course notes, freely available at his webpage.

2. Background on Abelian Varieties

Definition 2.1. A *k-group scheme* is a group object in the category of k-schemes, i.e. a k-scheme G with morphisms $m : G \times G \to G$ (multiplication), $i : G \to G$ (inversion), and an identity point $e : \operatorname{Spec} k \to G$, such that the usual diagrams for the group axioms commute.

Definition 2.2. A smooth, connected, proper k-group scheme $f : A \to \operatorname{Spec} k$ is called an *abelian variety*.

A fundamental fact, which follows from rigidity for integral proper k-schemes, is that every abelian variety is a *commutative* k-group scheme. The study of an abelian variety A relies crucially on the notion of the *dual abelian variety* $\widehat{A}$, defined in terms of the Picard scheme of A, which we now recall. For a k-scheme S, denote by $A_S := A \times_k S$ and $e_S : S \to A_S$ the base change of the identity section, and $f_S : A_S \to S$ the base change of the structure morphism.

Definition 2.3. We define the *Picard functor* $\underline{\operatorname{Pic}}_{A/k} : \operatorname{Sch}_{/k} \to \operatorname{Ab}$ by

$$S \mapsto \{(\mathscr{L}, \iota) \text{ on } A_S\}/ \simeq$$

sending a k-scheme S to the abelian group (under $\otimes$) of isomorphism classes of *rigidified* line bundles $(\mathscr{L}, \iota)$ on A_S, by which we mean line bundles $\mathscr{L}$ on A_S together with a chosen "rigidification" $\iota : e_S^* \mathscr{L} \xrightarrow{\sim} \mathscr{O}_S$, and where isomorphisms $(\mathscr{L}, \iota) \simeq (\mathscr{L}', \iota')$ are those $\mathscr{L} \simeq \mathscr{L}'$ respecting the rigidifications ι and ι'.

This appears to veer from the definition of the usual Picard group due to the rigidification, but the following proposition shows that it only serves to cut out the extra line bundles coming from the base scheme:

Proposition 2.4. *The map $\underline{\operatorname{Pic}}_{A/k}(S) \to \operatorname{Pic}(A_S)/f_S^* \operatorname{Pic}(S)$ given by $(\mathscr{L}, \iota) \mapsto \mathscr{L}$ is an isomorphism.*

Proof. See Conrad, Proposition 2.2.12. □

We now have the following result, due in this case to Grothendieck:

Theorem 2.5. *The functor* $\underline{\mathrm{Pic}}_{A/k}$ *is representable by a locally finite type k-scheme* $\mathrm{Pic}_{A/k}$.

Choosing a natural isomorphism $\mathrm{Hom}_{\mathrm{Sch}_{/k}}(-, \mathrm{Pic}_{A/k}) \xrightarrow{\sim} \underline{\mathrm{Pic}}_{A/k}(-)$ and taking the image of $\mathrm{Id}_{\mathrm{Pic}_{A/k}}$ in $\underline{\mathrm{Pic}}_{A/k}(\mathrm{Pic}_{A/k})$ then gives a distinguished rigidified line bundle $(\mathscr{P}, \theta)$ on $A \times_k \mathrm{Pic}_{A/k}$, with the property that for a rigidified line bundle $\mathscr{L}$ on A_S, there is a unique map $f : S \to \mathrm{Pic}_{A/k}$ such that $(1_A \times f)^* \mathscr{P} \simeq \mathscr{L}$ respecting the rigidifications. By Yoneda's lemma, $\mathrm{Pic}_{A/k}$ is a k-group scheme. Then the connected component of the identity $\widehat{A} := \mathrm{Pic}^0_{A/k}$ is a sub-k-group scheme, which is called the dual abelian variety of A. We check that it deserves the name:

Proposition 2.6. *The k-group scheme $\widehat{A}$ is an abelian variety.*

Proof. It comes out of the construction of the Picard scheme that $\mathrm{Pic}_{A/k}$ is locally finite type, and then a general result about locally finite type k-group schemes implies that $\widehat{A}$ is irreducible and finite type, so it remains to check that $\widehat{A}$ is smooth and proper. From our concrete description of the functor of points of $\mathrm{Pic}_{A/k}$, it might seem like valuative/infinitesimal criteria are the way to go. For properness, this turns out to be the case, as we verify here, but for smoothness there isn't a good way to make this work, so a more involved proof is necessary, and we refer the interested reader to §5.1 in Conrad's notes. Since we're over a field, separatedness is easy to check by expressing the diagonal morphism as the base change of the composition $\widehat{A} \to \mathrm{Spec}\, k \xrightarrow{e} \widehat{A}$ by the map $\widehat{A} \times \widehat{A} \xrightarrow{(x,y) \mapsto xy^{-1}} \widehat{A}$ (this works for any k-group scheme), so we need to check that for any 1-dimensional regular local k-algebra R with fraction field $K(R)$, any k-morphism $\mathrm{Spec}\, K(R) \to \mathrm{Pic}_{A/k}$ has a factorization $\mathrm{Spec}\, K(R) \hookrightarrow \mathrm{Spec}\, R \to \mathrm{Pic}_{A/k}$ (necessarily unique, by separatedness of $\mathrm{Pic}_{A/k}$). This is equivalent to the statement that any line bundle $\mathscr{L}$ on $A_{K(R)}$ is pulled back from some $\mathscr{L}'$ on A_R (since $\mathrm{Pic}(\mathrm{Spec}\, R) = \mathrm{Pic}(\mathrm{Spec}\, K(R)) = 0$ we needn't worry about rigidifications). By smoothness of A, is enough to show this in the case where $\mathscr{L} = \mathscr{O}(D)$ for some effective Cartier divisor $D \hookrightarrow X_{K(R)}$, but in this case, we can consider the effective Cartier divisor obtained by taking the scheme-theoretic closure D' of D in X_R, and set $\mathscr{L}' = \mathscr{O}(D')$, which by definition satisfies $\mathscr{L}'|_{X_{K(R)}} = \mathscr{O}(D')|_{X_{K(R)}} = \mathscr{O}(D' \cap X_{K(R)}) = \mathscr{O}(D) = \mathscr{L}$, as desired. □

For a line bundle $\mathscr{L}$ on A, there is associated a particular homomorphism $\phi_{\mathscr{L}} : A \to \widehat{A}$, defined functorially for k-schemes S and $s \in A(S)$ by

$$\phi_{\mathscr{L}}(s) = t_s^* \mathscr{L}_S \otimes \mathscr{L}_S^{-1}$$

where $t_s : A_S \to A_S$ is translation by s. Details of this construction are addressed in §4. Such homomorphisms are the primary objects of study for this note, and we will see that they are uniquely characterized by a certain symmetry property, which we now describe. For any homomorphism $f : A \to B$ of abelian varieties, we define the *dual homomorphism* $\widehat{f} : \widehat{B} \to \widehat{A}$ functorially for k-schemes S and $\mathscr{L}_S$ on B_S by $\mathscr{L}_S \mapsto f_S^* \mathscr{L}_S$. Additionally, when we consider the line bundle $\mathscr{P}_A := \mathscr{P}|_{A \times \widehat{A}}$ on $A \times \widehat{A}$, it has a trivialization along $A \times \{0\}$ by the definition of the group structure on $\widehat{A}$, so corresponds to a homomorphism to the dual $\widehat{\widehat{A}}$ of $\widehat{A}$, denoted ι_A, and in fact ι_A is an isomorphism (see Conrad, Theorem 7.3.3 and Example 7.3.4). Now in the particular case of a homomorphism $f : A \to \widehat{A}$, we can compose these

constructions to form a new map $\widehat{f} \circ \iota_A : A \to \widehat{A}$, and we say f is *symmetric* if $f = \widehat{f} \circ \iota_A$. The easy direction of our classification is then the following:

Proposition 2.7. *For any line bundle $\mathscr{L}$ on A, $\phi_{\mathscr{L}} : A \to \widehat{A}$ is symmetric.*

Proof. See Conrad, Proposition 7.4.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The truly remarkable fact is the (near) converse: for any symmetric $f : A \to \widehat{A}$, there is a finite Galois extension L/k over which f_L has the form $\phi_{\mathscr{L}}$ for some line bundle $\mathscr{L}$ on A_L. To prove this, we will want to recast the symmetry condition in more tractable terms, which requires some additional machinery.

3. The Duality Pairing

A map $f : A \to B$ of abelian varieties is called an *isogeny* if it is a finite flat surjection. If we define the *scheme-theoretic kernel* of f to be $\ker f := f^{-1}(e_B) = A \times_{B,e_B} \operatorname{Spec} k$, the finiteness condition then is equivalent to $\ker f \to \operatorname{Spec} k$ being finite. For any finite k-group scheme G, its *Cartier dual* $\widehat{G}$ is defined to be the k-group scheme $\underline{\operatorname{Hom}}_{gp}(G, \mathbf{G}_m)$. A critical result, by an argument in descent theory, is that $\ker \widehat{f}$ is Cartier dual to $\ker f$, i.e. we have a natural duality pairing $\ker f \times \ker \widehat{f} \to \mathbf{G}_m$. The primary case of interest is where $f = [n]_A : A \to A$ is multiplication by n, where we have $\widehat{[n]_A} = [n]_{\widehat{A}}$, so that $\ker f = A[n]$ is the n-torsion subgroup of A, and the duality pairing is a map $A[n] \times \widehat{A}[n] \to \mu_n \subset \mathbf{G}_m$. In this case, we can compile these pairings to obtain something even better: for a prime $\ell \neq \operatorname{char} k$, we define the ℓ-adic Tate module of A to be the inverse limit $T_\ell(A) := \varprojlim A[\ell^n]$ of the of the system of ℓ-power torsion subgroups with the maps $A[\ell^n] \xrightarrow{\times \ell} A[\ell^{n-1}]$, and the duality pairings at finite level respect the maps in the limit, so that we obtain a non-degenerate $\mathbf{Z}_\ell$-bilinear pairing $\langle \cdot, \cdot \rangle_{A,\ell^\infty} : T_\ell(A) \times T_\ell(\widehat{A}) \to \mathbf{Z}_\ell(1)$.

This pairing has two symmetry properties which will be key for this argument. First, for a map $f : A \to \widehat{A}$, the maps $T_\ell(f)$ and $T_\ell(\widehat{f})$ are adjoint for the pairings on A and $\widehat{A}$, i.e. for all $(a, \widehat{b}) \in T_\ell(A) \times T_\ell(\widehat{A})$ we have $\langle T_\ell(f)(a), \widehat{b} \rangle_{\widehat{A},\ell^\infty} = \langle a, T_\ell(\widehat{f})(\widehat{b}) \rangle_{A,\ell^\infty}$. Second, for $(a, a') \in T_\ell(A) \times T_\ell(\widehat{A})$ we have $\langle a, a' \rangle_{A,\ell^\infty}^{-1} = \langle a', \iota_A(a) \rangle_{\widehat{A},\ell^\infty}$. Both follow from careful unpacking of the definition of the finite-level pairings. Together, they will let us demonstrate the desired behavior of f, at least as far as its effect on $T_\ell(A)$, and because the collection of $A[\ell^n]$ over all n is dense in A, it turns out that, in general, a map $f : A \to B$ is completely determined by its effect on the ℓ-adic Tate modules for a prime $\ell \neq \operatorname{char} k$. (This may come as a surprise, as $T_\ell(A)$ is built up from a collection of relatively 'small' closed subsets, but the intuition should come from the case over $\mathbb{C}$, where in the dimension 1 case we have $T_\ell(A) = \mathbf{Z}_\ell(1) \times \mathbf{Z}_\ell(1) \subset S^1 \times S^1$, which is visibly dense.) We record this fact here:

Theorem 3.1. *For A, B abelian varieties over k, $\operatorname{Hom}_k(A, B)$ is $\mathbf{Z}$-finite and for $\ell \neq \operatorname{char} k$,*

$$\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \operatorname{Hom}_k(A, B) \to \operatorname{Hom}_{\mathbf{Z}_\ell[G_k]}(T_\ell(A), T_\ell(B)) \subset \operatorname{Hom}_{\mathbf{Z}_\ell}(T_\ell(A), T_\ell(B))$$

is injective.

Proof. See Conrad, §7.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now have everything we need to translate symmetry properties of the pairing to properties of a homomorphism $f : A \to \widehat{A}$. Define the Z_ℓ-bilinear form $e_{f,\ell^\infty}(\cdot, \cdot) : T_\ell(A) \times T_\ell(A) \to \mathbf{Z}_\ell(1)$ to be the composition $T_\ell(A) \times T_\ell(A) \xrightarrow{1 \times T_\ell(f)} T_\ell(A) \times T_\ell(\widehat{A}) \to \mathbf{Z}_\ell(1)$.

Theorem 3.2. *For a homomorphism $\phi : A \to \widehat{A}$, and a prime $\ell \neq \mathrm{char}(k)$, the bilinear form e_{f,ℓ^∞} on $T_\ell(A)$ is skew-symmetric if and only if the homomorphism f is symmetric.*

Proof. Suppose f is symmetric. Then for $a, b \in T_\ell(A)$ we have $e_{f,\ell^\infty}(a,b) = \langle a, T_\ell(f)(b)\rangle_{A,\ell^\infty}$ by definition, and symmetry of f gives

$$\langle a, T_\ell(f)(b)\rangle_{A,\ell^\infty} = \langle a, T_\ell(\widehat{f})(\iota_A(b))\rangle_{A,\ell^\infty}$$

then by adjointness of $T_\ell(f)$ and $T_\ell(\widehat{f})$ we have

$$\langle a, T_\ell(\widehat{h})(\iota_A(b))\rangle_{A,\ell^\infty} = \langle T_\ell(h)(a), \iota_A(b)\rangle_{\widehat{A},\ell^\infty}$$

from which the fact that for any $(a,a') \in T_\ell(A) \times T_\ell(\widehat{A})$ we have $\langle a, a'\rangle_{A,\ell^\infty}^{-1} = \langle a', \iota_A(a)\rangle_{\widehat{A},\ell^\infty}$ implies

$$\langle T_\ell(f)(a), i_A(b)\rangle_{\widehat{A},\ell^\infty} = \langle b, T_\ell(f)(a)\rangle_{A,\ell^\infty}^{-1} = e_{f,\ell^\infty}(b,a)^{-1}$$

and we see that e_{f,ℓ^∞} is skew-symmetric. In the other direction, suppose e_f is skew-symmetric, and let $a, b \in T_\ell(A)$. It suffices to show that $f = \widehat{f} \circ \iota_A$ induce the same maps on $T_\ell(A)$, as then they must agree everywhere by the preceding Theorem. By skew-symmetry we have $e_{f,\ell^\infty}(a,b) = e_{f,\ell^\infty}(b,a)^{-1} = \langle b, T_\ell(f)(a)\rangle_{A,\ell^\infty}^{-1}$, while on the other hand we again use the identities for the ℓ-adic pairing to obtain:

$$e_{f,\ell^\infty}(a,b) = \langle a, T_\ell(f)(b)\rangle_{A,\ell^\infty} = \langle T_\ell(f)(b), \iota(a)\rangle_{\widehat{A},\ell^\infty}^{-1} = \langle b, \widehat{h}(\iota(a))\rangle_{A,\ell^\infty}^{-1}$$

so that $\langle b, h(a)\rangle_{A,\ell^\infty} = \langle b, \widehat{h}(\iota(a))\rangle_{A,\ell^\infty}$ for all $b \in T_\ell(A)$, and since the ℓ-adic pairing is non-degenerate, this implies $\widehat{h}(\iota(a)) = h(a)$. $\qquad\square$

We have now reduced the problem to that of understanding when the $\mathbf{Z}_\ell$-bilinear form associated to a homomorphism is skew-symmetric, and the connection to the maps $\phi_{\mathscr{L}}$ is then made possible by the following theorem:

Theorem 3.3 (c.f. Mumford §20, Theorem 2)**.** *Let A be an abelian variety, and $f : A \to \widehat{A}$ a homomorphism. Then f is symmetric (equivalently, the bilinear form e_{f,ℓ^∞} on $T_\ell(A)$ is skew-symmetric) if and only if there is a line bundle $\mathscr{L}$ on A such that $2f = \phi_{\mathscr{L}}$.*

Proof. If $2f = \phi_{\mathscr{L}}$, then $2e_{f,\ell^\infty} = e_{\phi_{\mathscr{L}},\infty}$ by construction, and we know that $\phi_{\mathscr{L}}$ is a symmetric homomorphism by Proposition 2.7, and so Theorem 3.2 implies that its associated bilinear form is skew-symmetric, hence so is e_{f,ℓ^∞}. In the other direction, suppose e_{f,ℓ^∞} is skew-symmetric. Then by 3.2 f is symmetric. Consider the line bundle $\mathscr{N}_f := (1,f)^* \mathscr{P}_A$ on A. We claim (even without symmetry of f) that $\phi_{\mathscr{N}_f} = f + \widehat{f} \circ \iota_A$ (even without symmetry of f), so when $f = \widehat{f} \circ \iota_A$ is symmetric we get $\phi_{\mathscr{N}_f} = 2f$.

We have that $\phi_{\mathscr{N}_f}(x) \simeq (1,f)^\wedge(\phi_{\mathscr{P}_A}(x, f(x)))$. Using the isomorphism $(A \times \widehat{A})^\wedge \simeq \widehat{A} \times \widehat{\widehat{A}}$ via $\mathscr{L} \mapsto (\mathscr{L}|_{A\times\{0\}}, \mathscr{L}|_{\{0\}\times\widehat{A}})$, we compute the map on each factor:

$$\begin{aligned}
\mathrm{pr}_{\widehat{A}}(\phi_{\mathscr{P}_A}(x, f(x))) &= (1_A \times \{0\})^*(t^*_{(x,f(x))}\mathscr{P}_A \otimes \mathscr{P}_A^{-1}) \\
&\simeq (t_{(0,f(x))} \circ t_{(x,0)} \circ (1_A \times \{0\}))^*\mathscr{P}_A \otimes (1_A \times \{0\})^*\mathscr{P}_A^{-1} \\
&\simeq t_x^*(1_A \times \{f(x)\})^*\mathscr{P}_A \otimes \mathscr{O}_A \\
&\simeq t_x^*(f(x)) \\
&\simeq f(x)
\end{aligned}$$

where we view $f(x)$ as a line bundle on A, and use the fact that $t_x^* f(x) \simeq f(x)$ since $f(x) \in \mathrm{Pic}^0$, and $\phi_{\mathscr{L}}$ is trivial for $\mathscr{L} \in \mathrm{Pic}^0$ (see the discussion following Proposition 4.4). In the other component, we have:

$$
\begin{aligned}
\mathrm{pr}_{\widehat{A}}(\phi_{\mathscr{P}_A}(x, f(x))) &= (\{0\} \times 1_{\widehat{A}})^* (t_{(x,f(x))}^* (\mathscr{P}_A \otimes \mathscr{P}_A^{-1})) \\
&\simeq (t_{(0,f(x))} \circ t_{(x,0)} \circ (\{0\} \times 1_{\widehat{A}}))^* \mathscr{P}_A \otimes (\{0\} \times 1_{\widehat{A}})^* \mathscr{P}_A^{-1}) \\
&\simeq t_{f(x)}^* (\{x\} \times 1_{\widehat{A}})^* \mathscr{P}_A \otimes \mathscr{O}_{\widehat{A}} \\
&\simeq t_{f(x)}^* \iota_A(x) \\
&\simeq \iota_A(x)
\end{aligned}
$$

where we again view $\iota_A(x)$ as a line bundle on $\widehat{A}$, and use the fact that $\iota_A(x) \in \mathrm{Pic}^0$ implies $t_{f(x)}^* \iota_A(x) \simeq \iota_A(x)$. Thus we have $\phi_{\mathscr{P}_A}(x, f(x)) = (f(x), \iota_A(x))$, or specifically as line bundles $\phi_{\mathscr{P}_A}(x, f(x)) \simeq f(x) \boxtimes \iota_A(x)$. Now when we compute the pullback along $(1, f)$:

$$
\begin{aligned}
\phi_{\mathscr{N}_f}(x) &= (1, f)^* \phi_{\mathscr{P}_A}(x, f(x)) \\
&\simeq (1, f)^* (f(x) \boxtimes \iota_A(x)) \\
&\simeq 1_A^* (f(x)) \otimes f^* (\iota_A(x)) \\
&\simeq f(x) \otimes \widehat{f}(\iota_A(x))
\end{aligned}
$$

we obtain the desired result. $\qquad\square$

So we've shown that $2f$ has the desired form $2f = \phi_{\mathscr{N}_f}$, even for a reasonably concrete line bundle on A, but we want to push this further and get a result for f itself. It will suffice to show that, over an algebraically closed field, if $2f = \phi_{\mathscr{L}}$ for some $\mathscr{L}$, then $\mathscr{L} \simeq \mathscr{M}^{\otimes 2}$ for some $\mathscr{M}$, as then we would have $2\phi_{\mathscr{M}} = \phi_{\mathscr{M}^{\otimes 2}} = \phi_{\mathscr{L}} = 2f$, so $f = \phi_{\mathscr{M}}$. This follows from another result in Mumford's book:

Theorem 3.4 (Mumford §23, Theorem 3). *If $\mathscr{L}$ is a line bundle on an abelian variety A over an algebraically closed field and $n \in \mathbf{Z}$, $\mathscr{L} \simeq \mathscr{M}^{\otimes n}$ for some line bundle $\mathscr{M}$ if and only if $\ker \phi_{\mathscr{L}} \supset A[n]$.*

Taking $n = 2$ and $\mathscr{L} = \mathscr{N}_f$, the fact that $\phi_{\mathscr{N}_f} = 2f$ implies $\ker \phi_{\mathscr{N}_f} = \ker f \circ [2]_A \supseteq \ker[2]_A = A[2]$. Thus $f_{\overline{k}} = \phi_{\mathscr{M}}$ for some line bundle $\mathscr{M}$ on $A_{\overline{k}}$. However, we still wish to show that we can find such $\mathscr{M}$ after a finite Galois extension of k. If we set things up in the right generality from the outset, we will get this for free.

4. Cubical Structure and Functorial Properties of $\phi_{\mathscr{L}}$

We first state, for completeness, the necessary properties of $\phi_{\mathscr{L}}$ for a line bundle $\mathscr{L}$ on A. In particular, $\phi_{\mathscr{L}}$ is a homomorphism, but we can get much more out of this fact if we check it in a relative setting. That is, let $\mathscr{L}$ be a line bundle on A_S, and define for S-schemes T and $x \in A_S(T)$ the analogous map $\phi_{\mathscr{L}} : A_S \to \widehat{A}_S$ by $\phi_{\mathscr{L}}(x) = t_x^* \mathscr{L} \otimes \mathscr{L}^{-1}$. In particular, we have:

Theorem 4.1 (The Theorem of the Square). *The map $\phi_{\mathscr{L}}$ is a homomorphism; i.e., for any S-scheme T and $x, y \in A_S(T)$, we have canonical isomorphisms of line bundles on A_T*

$$
t_{x+y}^* \mathscr{L}_T \otimes \mathscr{L}_T^{-1} \simeq t_x^* \mathscr{L}_T \otimes t_y^* \mathscr{L}_T \otimes [(x, y)^* (m^* \mathscr{L} \otimes \mathrm{pr}_1^* \mathscr{L}^{-1} \otimes \mathrm{pr}_2^* \mathscr{L}^{-1} \otimes (e_S^* \mathscr{L})_{A_S}]_{A_T}.
$$

In particular, $\phi_{\mathscr{L}}$ defines a k-group homomorphism $\mathrm{Pic}_{A/k} \to \underline{\mathrm{Hom}}_{gp}(A, \widehat{A})$.
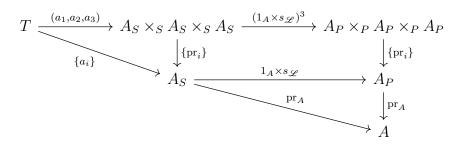
For line bundles on A_S pulled back from A, this is proven in, Corollary 3.2.3 of Conrad's notes, a calculation that essentially boils down to the "Cubical Structure Theorem" for such line bundles. To upgrade Conrad's proof of Theorem 4.1 to work for any line bundles on A_S, we need a suitable version of this result:

Theorem 4.2. *Let $\mathscr{L}$ be a line bundle on A_S. Then for any S-scheme T and $a_1, a_2, a_3 \in A_S(T)$, the line bundle*

$$\theta_{A_S,\mathscr{L}}^{(a_1,a_2,a_s)} := (a_1 + a_2 + a_3)^* \mathscr{L} \otimes (a_1 + a_2)^* \mathscr{L}^{-1} \otimes (a_2 + a_3)^* \mathscr{L}^{-1} \otimes (a_1 + a_3)^* \mathscr{L}^{-1}$$
$$\otimes\, a_1^* \mathscr{L} \otimes a_2^* \mathscr{L} \otimes a_3^* \mathscr{L} \otimes (e^* \mathscr{L}^{-1})_T$$

is canonically trivial.

Proof. For $S = k$ a field, Conrad proves the result using the Theorem of the Cube (3.1.6). Granting this case, we deduce the relative version as follows: Set $P := \mathrm{Pic}_{A/k}$ for brevity. Let S be a k-scheme, $\mathscr{L}$ a line bundle on A_S with $s_{\mathscr{L}}$ the corresponding S-point of P. Given T-points $a_1, a_2, a_3 \in A_S(T)$, consider the following diagram:

with the property that pulling back the Poincare bundle $\mathscr{P}_A$ on A_P recovers the universal case $T = A_S^3$ over A_S for the line bundle $\mathscr{L}$. Pulled back by any triple of T-points (a_1, a_2, a_3), this in turn recovers the general case. Because we're not just looking at P-points of A_P, however, this universal setup is not a special case of the result we're trying to prove (having to do with arbitrary line bundles on A_S), but really a version of the cubical structure theorem for the line bundle $\mathscr{P}_A$ on the abelian scheme A_P and T'-points $b_1, b_2, b_3 \in A_P(T')$ for a P-scheme T', for which we're treating the universal case $T' = A_P^3$. We have the relevant hbox-overfilling line bundle $\theta_{A_P,\mathscr{P}_A}^{(\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3)}$ on $A_P^3 \simeq (A^3)_P$ (we henceforth drop the indices and write θ), and we'll analyze its fibres $\theta|_p$ over $p \in P$ using the case over k to show that $\theta \simeq (\mathrm{pr}_P)^*(\mathrm{pr}_P)_* \theta$ by the Seesaw Theorem, and then show that $(\mathrm{pr}_P)_* \theta$ is trivial by inspection.

Let $p \in P$. To make rigorous the connection to the case over A, we have the following diagram:

which is commutative upon taking the respective maps in each bracketed set, essentially by the definition of the group scheme structure on A_P. As in the familiar case, we have for $i \neq j \neq k$ the identities $(\mathrm{pr}_i + \mathrm{pr}_j + \mathrm{pr}_k) = m$ and $(\mathrm{pr}_i + \mathrm{pr}_j) = m_{ij}$. Now considering the

fiber over p we use the above diagram to compute:

$$\theta|_p = \iota^*\theta = \iota^*\left(m^*\mathscr{P}_A \otimes m_{12}^*\mathscr{P}_A^{-1} \otimes m_{23}^*\mathscr{P}_A^{-1} \otimes m_{13}^*\mathscr{P}_A^{-1} \otimes \mathrm{pr}_1^*\mathscr{P}_A \otimes \mathrm{pr}_2^*\mathscr{P}_A \otimes \mathrm{pr}_3^*\mathscr{P}_A \otimes (e_P^*\mathscr{P}_A^{-1})_{A_P^3}\right)$$

$$\simeq m^*\iota_0^*\mathscr{P}_A \otimes m_{12}^*\iota_0^*\mathscr{P}_A^{-1} \otimes m_{23}^*\iota_0^*\mathscr{P}_A^{-1} \otimes m_{13}^*\iota_0^*\mathscr{P}_A^{-1} \otimes \mathrm{pr}_1^*\iota_0^*\mathscr{P}_A \otimes \mathrm{pr}_2^*\iota_0^*\mathscr{P}_A \otimes \mathrm{pr}_3^*\iota_0^*\mathscr{P}_A$$

where we can drop the $e_P^*\mathscr{P}_A$ term in the second line because $\mathscr{P}_A$ it is canonically trivial along that section. Here we know exactly how to interpret $\iota_0\mathscr{P}_A$: it's just the line bundle $\mathscr{L}_p$ on $A \times_k \operatorname{Spec}\kappa(p)$ corresponding to the point $p \in \operatorname{Pic}_{A/k}$. The expression then becomes

$$\theta|_p = m^*\mathscr{L}_p \otimes m_{12}^*\mathscr{L}_p \otimes m_{23}^*\mathscr{L}_p^{-1} \otimes m_{13}^*\mathscr{L}_p^{-1} \otimes \mathrm{pr}_1^*\mathscr{L}_p \otimes \mathrm{pr}_2^*\mathscr{L}_p \otimes \mathrm{pr}_3\mathscr{L}_p$$

and this is trivial by the (universal case of the) cubical structure theorem for the abelian variety $A \times_k \operatorname{Spec}\kappa(p)$ (Conrad, 3.1.6). $\qquad\square$

So for each S-point of $\operatorname{Pic}_{A/k}$, we have an S-group homomorphism $A_S \to \widehat{A}_S$. To make most efficient use of the functorial properties of $\phi_{\mathscr{L}}$, we want to describe it as a map of k-group schemes $\operatorname{Pic}_{A/k} \to \underline{\operatorname{Hom}}_{gp}(A, \widehat{A})$, but this requires making sense of this hom-functor as a group scheme. We set this up now: let X and Y be schemes over a scheme T (we will take $T = \operatorname{Spec} k$. The Hom-functor $\underline{\operatorname{Hom}}(X, Y)$ assigns to any T-scheme S the set $\operatorname{Hom}_S(X_S, Y_S)$. It is a theorem of Grothendieck (using the theory of Hilbert schemes) that if T is locally noetherian, X is proper and T-flat, and X and Y are quasi-projective Zariski-locally over T then this functor is represented by a locally finite type and separated T-scheme H. (That is, there is an H-map $X_H \to Y_H$ that is universal in an evident sense.) We prove an intermediate technical result to get a version of this representing scheme for maps in the category of T-group schemes.

Lemma 1. *Let $f, g : X \rightrightarrows Y$ be a pair of T-morphisms (corresponding to elements of $H(T)$). Then the condition $f_S = g_S$ is represented by a closed subscheme of T.*

Proof. For any T-scheme S, denote by F_S and G_S the maps $S \to H$ corresponding to f and g, so that if $\phi_S : S \to T$ is the structure map, we have $F_S = F_T \circ \phi_S$ and $G_S = G_T \circ \phi_S$. Consider the locus $Z \hookrightarrow T$ where F_T and G_T agree, defined by the Cartesian diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & T \\
\downarrow & & \downarrow{\scriptstyle F_T \times G_T} \\
H & \xrightarrow{\ \Delta\ } & H \times_T H
\end{array}
$$

where the fact that H is separated implies $Z \to T$ is a closed embedding. Clearly $f_S = g_S$ iff $F_S = G_S$, and from the preceding factorizations of F_S and G_S, we have $F_S = G_S$ iff ϕ_S factors through $Z \hookrightarrow T$, hence the condition that $F_S = G_S$ is represented by the closed subscheme Z of S. $\qquad\square$

Proposition 4.3. *Let A and B be abelian varieties over a field k. Then the functor*

$$\underline{\operatorname{Hom}}_{gp}(A, B) : S \mapsto \operatorname{Hom}_{S-gp}(A_S, B_S)$$

is represented by a locally finite type and separated k-group scheme.

Proof. Let $\phi_H : A_H \to B_H$ be the universal morphism of H-schemes for the functor $\underline{\operatorname{Hom}}(A, B)$. Consider the pair of H-maps $f := m_{B_H} \circ (\phi_H \times \phi_H), g := \phi_H \circ m_{A_H} : A_H \times A_H \rightrightarrows B_H$. For an k-scheme S, let $\psi : A_S \to B_S$ be any S-map, and $\Psi : S \to H$ the corresponding k-map. Then ψ is an S-group homomorphism iff $f_S = g_S$, where f_S and g_S are the pullbacks of

$f, g : A_H \times A_H \to B_H$ by Φ. By the lemma, we have $f_S = g_S$ iff Φ factors through a closed subscheme $Z \hookrightarrow H$, so that the subfunctor $\underline{\mathrm{Hom}}_{gp}(A, B) \subset \underline{\mathrm{Hom}}(A, B)$ is represented by $Z \hookrightarrow H$, with the pullback of the universal H-morphism $A_H \to B_H$. Finally, Z is locally finite type and separated because H is so, and is a k-group scheme since it represents a group-valued functor. $\qquad\square$

We next want to show that this group scheme is étale, for which we need the following lemma:

Lemma 2. *Let T be a local noetherian k-scheme, and let A and B be abelian varieties over k. Then if a T-group map $\phi : A_T \to B_T$ vanishes on the special fiber it vanishes everywhere.*

Proof. Let $\mathscr{I}_T$ denote the ideal sheaf corresponding to the closed subscheme $K := \ker \phi$. We wish to show that $\mathscr{I}_T = 0$, i.e. check that the map $\mathscr{O}_{A_T} \to \mathscr{O}_K$ is an isomorphism. It suffices to check this on the stalks at all closed points $p \in X_T$, and since ϕ vanishes on the special fiber we have $p \in K$. By Artin-Rees and the exactness of completions, it suffices to show that the completed map $\mathscr{O}^\wedge_{A_T, p} \to (\mathscr{O}_K)^\wedge_p$ is an isomorphism. This is the limit of the maps $\mathscr{O}_{A_T, p}/\mathfrak{m}^n_p \to \mathscr{O}_{K, p}/\mathfrak{m}^n_p$, each of which is identified with the stalk at p of the base-changed map $\mathscr{O}_{A_{\mathscr{O}_{A, p}/\mathfrak{m}^n_p}} \to \mathscr{O}_{K_{\mathscr{O}_{A, p}/\mathfrak{m}^n_p}}$, so it suffices to show that this base-changed map of structure sheaves is an isomorphism, which is exactly the statement that $\phi_{\mathscr{O}_{A, p}/\mathfrak{m}^n_p}$ vanishes, so we have reduced to the Artin local case. We can then conclude as follows: for finite étale k-schemes X and Y, with T the spectrum of an local Artin k-algebra, a map $X_T \to Y_T$ is formally étale, so behavior over the special fiber extends uniquely in an infinitesimal neighborhood, i.e. extends uniquely to T-points. Now since the map $\phi : A_T \to B_T$ vanishes on the special fiber, the maps $\phi : A_T[\ell^n] \to B_T[\ell^n]$ vanish on the special fiber for all $n \geq 1$, so must vanish. Thus $\ker \phi \supseteq A_T[\ell^n]$ for all $n \geq 1$, and since $\ker \phi$ is closed, the fact that the collection of closed subschemes $A_T[\ell^n]$'s are dense in A_T. implies $\ker \phi = A_T$, so ϕ vanishes. $\qquad\square$

Proposition 4.4. *The k-group scheme Z representing $\underline{\mathrm{Hom}}_{gp}(A, B)$ in Proposition 4.3 is étale.*

Proof. Let Z be the Hom-scheme in 4.3. Then its tangent space at the identity is identified with those maps $\psi : \mathrm{Spec}\, k[\epsilon]/(\epsilon^2) \to Z$ in $Z(\mathrm{Spec}\, k[\epsilon]/(\epsilon^2))$ such that the morphism $\mathrm{Spec}\, k \to Z$ on the closed point is exactly the inclusion of the origin. But if we consider the morphism $\phi : A_{\mathrm{Spec}\, k[\epsilon]/(\epsilon^2)} \to B_{\mathrm{Spec}\, k[\epsilon]/(\epsilon^2)}$ corresponding to ψ, the fact that the closed point of $\mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$ is sent to the origin in Z implies that ϕ vanishes over the special fiber, which by the preceding lemma implies that it vanishes everywhere, so that there is a single map $\mathrm{Spec}\, k[\epsilon]/(\epsilon^2) \to Z$ matching this description, and Z has trivial tangent space at the origin. Base-changing to $\overline{k}$ and translating then shows that it is an étale k-scheme. $\qquad\square$

We henceforth refer to Z by $\underline{\mathrm{Hom}}_{gp}(A, B)$ (consistent with its use earlier in this paper). It is clear from the definition of $\phi_{\mathscr{L}}$ that the map $\mathrm{Pic}_{A/k} \to \underline{\mathrm{Hom}}_{gp}(A, B)$ given on S-points $s_{\mathscr{L}} : S \to \mathrm{Pic}_{A/k}$ by $s_{\mathscr{L}} \mapsto \phi_{\mathscr{L}}$ defines a map of k-group schemes. In particular, it sends the connected component of the identity in $\mathrm{Pic}_{A/k}$ to the zero map, which implies that for any line bundle $\mathscr{L}$ on A_S coming from $\widehat{A}$, $\phi_{\mathscr{L}}$ is trivial, i.e. $t^*_x \mathscr{L}_T \simeq \mathscr{L}_T$ for all $x \in A_S(T)$. More specifically, we define the *Néron-Severi scheme* $\mathrm{NS}(A)$ to be the étale component group of $\mathrm{Pic}_{A/k}$, so by definition this map $\mathrm{Pic}_{A/k} \to \underline{\mathrm{Hom}}_{gp}(A, \widehat{A})$ factors through some k-group map $\Phi : NS(A) \to \underline{\mathrm{Hom}}_{gp}(A, \widehat{A})$. By the result of §2, if $f \in \underline{\mathrm{Hom}}_{gp}(A, \widehat{A})(k)$ is symmetric, then $f_{k_s} \in \underline{\mathrm{Hom}}_{gp}(A, \widehat{A})(k_s)$ is in the image of this map, so is of the form $\Phi(a_{k_s})$ for some

$a_{k_s} \in \mathrm{NS}(A)(k_s)$, but since $\mathrm{NS}(A)$ is étale, this point comes from some $a_L \in \mathrm{NS}(A)(L)$ for L/k finite Galois, and hence $f_L = \Phi(a_L)$, and $f_L = \phi_{\mathscr{L}}$ for some $\mathscr{L}$ on A_L. This completes what we set out to prove.

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