

Schubert Calculus on the Grassmanian

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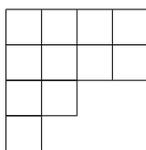
Abstract

Schubert calculus is the study of cells in the Grassmanian. These cells follow neat patterns because they are indexed by Young diagrams in a natural way. Young diagrams also index a generating set for the ring of symmetric polynomials, called the Schur polynomials. Amazingly, the cohomology classes of the Grassmanian cells and the Schur polynomials obey similar multiplication formulas. I explore this connection in a finite- and an infinite-dimensional setting.

1 Introduction

In the late nineteenth century, German mathematician Hermann Schubert introduced a new method of counting linear solutions to geometric problems. His results were compelling but lacked a rigorous foundation, so later mathematicians worked to precisely build a theoretical basis for his enumerative geometry. In fact, making Schubert calculus rigorous was the fifteenth of Hilbert's famous twenty-three problems.

Schubert found a cell structure for the Grassmanian, which is the space of r -dimensional subspaces of a fixed complex vector space for a fixed r . The classes of these cells generate the cohomology of the Grassmanian, so they carry a multiplicative structure. Amazingly, this cohomology ring behaves similarly to the ring of symmetric polynomials. The object that ties these seemingly disparate systems together is the Young diagram. A Young diagram is a collection of boxes arranged in rows such that the row width is nonincreasing as you go down. The following is an example of a diagram $\lambda = (4, 4, 2, 1)$, which we write as a tuple of row widths.



These diagrams index the cells of the Grassmanian. They also index the Schur polynomials, which are an important basis for the symmetric polynomials. For each Young diagram λ , let σ_λ be the cohomology class of the corresponding Schubert cell and let s_λ be the corresponding Schur polynomial. Both of these objects obey the Pieri rule, or the formula for multiplying with a one-row diagram. It says that

$$\sigma_\lambda \cdot \sigma_{(k)} = \sum_{\lambda'} \sigma_{\lambda'} \qquad s_\lambda \cdot s_{(k)} = \sum_{\lambda'} s_{\lambda'}$$

where the sum over λ' is taken over the diagrams obtained from λ by adding k boxes, no two in a column. There is a dual version

$$\sigma_\mu \cdot \underbrace{\sigma_{(1,1,\dots,1)}}_k = \sum_{\mu'} \sigma_{\mu'} \qquad s_\mu \cdots \underbrace{s_{(1,1,\dots,1)}}_k = \sum_{\mu'} s_{\mu'},$$

where the sum over μ' is taken over the diagrams obtained from μ by adding k boxes, no two in a row. These facts are enough to show that the objects follow the same multiplication rule for all pairs of diagrams. Furthermore, this correspondence works if we take Grassmanians in infinite dimensions.

I'll begin by giving the preliminary facts about cohomology and symmetric polynomials needed in this report. After that, I will discuss the finite-dimensional Grassmanian by defining the Schubert cells, proving the Pieri formulas geometrically, and showing how they connect to the ring of symmetric polynomials. Finally, I will discuss the versions of these results when we consider infinite-dimensional Grassmanians.

2 Preliminaries

2.1 Cohomology and the intersection product

The theory of cohomology on manifolds can get quite heavy with machinery, we are only concerned with a simple case and will gloss over most of the technical details. Basically, the only properties of the cohomology ring we need are that the generators are classes of submanifolds, and multiplying is like intersecting those submanifolds. Thankfully, it is easy to work with the Grassmanian because the space has a cell decomposition. I'll explain how to use this cell decomposition to understand the cohomology. For a more rigorous but still elementary introduction to these ideas, look in [2].

Definition 1. *Let X be a compact complex manifold with complex dimension n . A cell decomposition of X is a sequence*

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that each $X_i - X_{i-1}$ is a disjoint union of copies of \mathbb{C}^i , called i -cells. Equivalently, X has a cell decomposition if it can be written as a disjoint union of copies of \mathbb{C}^k for varying k .

Let X be a compact complex manifold with a cellular decomposition. For each $0 \leq k \leq n$, let b_k be the number of k -cells, and let $Z_k^1, \dots, Z_k^{b_k}$ be the closures of these cells. Then, the $2k$ -th homology group $H_{2k}(X)$ is the free abelian group generated by $[Z_k^1], \dots, [Z_k^{b_k}]$. We took $2k$ here instead of k because we are in complex numbers; the odd homology groups are 0.

If Z is a submanifold of X of complex dimension d , then it has a homology class $[Z]$ in $H_{2d}(X)$. If two submanifolds are homotopic, roughly meaning that they can be continuously deformed to each other, then they have the same homology class. There is an *intersection product*

$$H_i(X) \times H_j(X) \rightarrow H_{i+j-n}(X)$$

denoted by “ \cdot ” where if submanifolds Z and W intersect transversally, then

$$[Z] \cdot [W] = [Z \cap W].$$

For our purposes, since we are working with linear spaces, we can always find representatives of homology classes such that intersect transversally or not at all.

Since X is compact, and complex manifolds come with a natural orientation, we can apply *Poincaré duality* to turn the homology into cohomology. The duality gives us pairings $H^i(X) \cong H_{n-i}(X)$, so we can think of the cohomology group $H^{2k}(X)$ as the free abelian group on $[Z_{n-k}^1], \dots, [Z_{n-k}^{b_k}]$, or the cell closures in complex *codimension* k . Now, a submanifold Z of complex codimension ℓ has a cohomology class $[Z]$ in $H^{2\ell}(X)$. The intersection product behaves more nicely in cohomology because it sends $H^i(X) \times H^j(X)$ to $H^{i+j}(X)$. This product turns the cohomology ring $H^*(X) = \bigoplus_{k=0}^n H^{2k}(X)$ into a graded ring.

If X is connected, then $H^{2n}(X) \cong \mathbb{Z}$, generated by the class of a point. If the product of two cohomology classes σ_1 and σ_2 lands in $H^{2n}(X)$, then we’ll just write $\sigma_1 \cdot \sigma_2$ as the integer coefficient of $[\{\text{pt}\}]$. If submanifolds Z and W intersect transversally in m distinct points, then $[Z] \cdot [W] = m$. (This property relies on the fact that we are working in a complex manifold).

2.2 Young diagrams and symmetric polynomials

Everything in this following section can be found in the beginning of [1].

First, we’ll need some notation. If λ is a Young diagram, we can write $\lambda = (\lambda_1, \dots, \lambda_r)$ as a tuple of row lengths from top to bottom. Let $|\lambda|$ be the number of boxes in the diagram. A Young *tableau* is a Young diagram with the boxes filled in with integers so that the numbers are strictly increasing as you go down and weakly increasing (nondecreasing) as you go right. The picture on the left is a valid Young tableau, while the picture on the right is not.

1	2	2	2
2	3	5	
4			

1	2	2	2
1	2	3	
2			

Given a tableau T with entries in $\{1, \dots, r\}$, we can produce a monomial x^T in x_1, \dots, x_r where the power of x_i is the number of times i appears in the tableau. Then, we can define *Schur polynomials* for each diagram λ by

$$s_\lambda = \sum_{\text{tableau } T \text{ with shape } \lambda, \text{ entries in } \{1, \dots, r\}} x^T.$$

For example, $s_{(k)} = h_k(x_1, \dots, x_r)$, the complete symmetric polynomial of degree k , and $s_{(1^k)} = e_k(x_1, \dots, x_r)$, the elementary symmetric polynomial of degree k . The Schur polynomials are symmetric. They also obey the Pieri formulas given in the introduction.

The Schur polynomials generate the symmetric polynomials. More specifically, the polynomials s_λ for $|\lambda| = n$ and λ with at most r rows are a basis over \mathbb{Z} for the homogeneous symmetric polynomials of degree n in r variables. If we allow the degree to vary, then we can see that s_λ for λ with at most r rows generate Λ_r , the ring of symmetric polynomials in r variables.

Another generating set for Λ_r is the elementary symmetric functions e_k . Since $s_{(1^k)} = e_k$, the Pieri formulas determine the multiplicative structure for all $s_\lambda \cdot s_\mu$. That rule has a combinatorial formulation which we will not discuss.

In some settings, we want to be agnostic about the number of variables we use because

$$s_\lambda(x_1, \dots, x_r, 0, \dots, 0) = s_\lambda(x_1, \dots, x_r).$$

So, we can define a *symmetric function* to be a sequence of symmetric polynomials $p(x_1, \dots, x_m)$ for every m such that $p(x_1, \dots, x_m) = p(x_1, \dots, x_m, 0, \dots, 0)$. If Λ is the ring of symmetric functions, then it is the inverse limit of the sequence

$$\dots \rightarrow \Lambda_{r+1} \rightarrow \Lambda_r \rightarrow \Lambda_{r-1} \rightarrow \dots,$$

where the maps are $p(x_1, \dots, x_{r+1}) \mapsto p(x_1, \dots, x_r, 0)$.

3 Finite-dimensional Grassmanians

The Grassmanian $G(r, m)$ is the space of r -dimensional subspaces of \mathbb{C}^m . Let $n = m - r$ be the codimension of the subspaces.

We'll need some extra notation for convenience. Fix a basis e_1, \dots, e_m of \mathbb{C}^m . This basis determines a *flag* of subspaces

$$\{0\} = F_0 \subset F_1 \subset \dots \subset F_{m-1} \subset F_m = \mathbb{C}^m,$$

where $F_i = \langle e_1, \dots, e_i \rangle$. We'll also need the *backwards flag* \tilde{F}_\bullet , where $\tilde{F}_i = \langle e_{m-i+1}, \dots, e_m \rangle$. Conversely, given a flag E_\bullet , we say a basis b_j *corresponds* to that flag if each $E_i = \langle b_1, \dots, b_i \rangle$.

The Grassmanian has a natural complex manifold structure via the following. Express each $V \in G(r, m)$ as the row space of an $r \times m$ matrix in our basis. Choose r linearly independent columns in this matrix, and use row operations to turn those columns into pivots. Then, fixing those pivots, each choice of entries in the other columns will give us a unique r -dimensional subspace. The following is an example of the matrix representations of elements of $G(4, 8)$ with pivots in columns 2, 3, 6, and 8.

$$\begin{pmatrix} * & 0 & 0 & * & 0 & * & * & 1 \\ * & 0 & 0 & * & 1 & * & * & 0 \\ * & 0 & 1 & * & 0 & * & * & 0 \\ * & 1 & 0 & * & 0 & * & * & 0 \end{pmatrix}$$

We have $r(m - r)$ stars in such a matrix, so we can identify the set of matrices of this form with $\mathbb{C}^{r(m-r)}$. Thus $G(r, m)$ is an $r(m - r)$ -dimensional complex manifold with charts

$$\{V : V \text{ has rank } r \text{ when projected onto } \langle e_{i_1}, e_{i_2}, \dots, e_{i_r} \rangle\} \approx \mathbb{C}^{r(m-r)}.$$

3.1 A cell decomposition

The Grassmanian also has a nice decomposition into cells. For each $V \in G(r, m)$, write V as a matrix and let i_1 be the leftmost nonzero column, i_2 be the leftmost column linearly

independent from i_1 , and so on, so i_r is the leftmost column linearly independent from columns i_1, \dots, i_{r-1} . Then, we can use row operations to uniquely write the matrix such that there are pivots in columns i_j , each column before i_1 is 0, and each column between i_j and i_{j+1} starts with $r - j$ zeroes. Here is an example in $G(3, 8)$ where the leftmost columns are 2, 3, 6, and 8. I have written the pivots in the reverse order from normal in order to make the Young diagram easier to see.

$$\left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 1 & 0 & * & 0 & * & * & 0 \end{array} \right) \rightsquigarrow \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & & \\ \square & & & \\ \square & & & \end{array}$$

The positions of the leftmost columns determine the number of zeros we add per row. If we look at these extra zeroes as boxes, then we get a Young diagram. In general, the Young diagram we get through this process is

$$\lambda = (i_r - r, i_{r-1} - r + 1, \dots, i_2 - 2, i_1 - 1).$$

Note that such a Young diagram must have at most r rows and n columns.

Let $\Omega_\lambda^\circ(F_\bullet)$ be the set of subspaces with leftmost columns i_1, \dots, i_r . This set is homeomorphic to $\mathbb{C}^{r(m-r)-|\lambda|}$, where the coordinates are the values of the stars in the matrix representation. Since each element of $G(r, m)$ yields one choice of leftmost columns, the Grassmanian decomposes into a disjoint union of cells

$$G(r, m) = \coprod_{\lambda \text{ with at most } r \text{ rows, } n \text{ columns}} \Omega_\lambda^\circ(F_\bullet).$$

These cells are called Schubert cells. We can explicitly describe the cell corresponding to a diagram λ as

$$\Omega_\lambda^\circ(F_\bullet) = \{V : \dim(V \cap \tilde{F}_k) = i \text{ for } n + i - \lambda_i \leq k \leq n + i - \lambda_{i+1}, \text{ for } 1 \leq i \leq r\}.$$

Observe that the cell has codimension $|\lambda|$ in the Grassmanian. The closure of the cell, which we call a Schubert variety, is

$$\Omega_\lambda(F_\bullet) = \{V : \dim(V \cap \tilde{F}_{n+i-\lambda_i}) \geq i, 1 \leq i \leq r\}.$$

We can also describe the Schubert variety using the columns of the Young diagram instead of the rows. If λ_j^T is the size of column j in a diagram λ , then

$$\Omega_\lambda(F_\bullet) = \left\{ V : \dim(V \cap \tilde{F}_{n-j+\lambda_j^T}) \geq \lambda_j^T, 1 \leq j \leq n \right\}.$$

Thus the cohomology of $G(r, m)$ is freely generated as an abelian group by $[\Omega_\lambda(F_\bullet)]$ for each diagram λ with at most r rows and n columns. Given two flags F_\bullet and E_\bullet , we can always continuously deform one into the other. So, $\Omega_\lambda(F_\bullet)$ and $\Omega_\lambda(E_\bullet)$ have the same cohomology class, which we can just denote by $\sigma_\lambda = [\Omega_\lambda]$.

Now that we know the additive structure of $H^*(G(r, m))$, we turn to the multiplicative structure.

3.2 Multiplication in $H^*(G(r, m))$: the Pieri formulas

The Pieri formulas tell us how to multiply special elements of $H^*(G(r, m))$. Fortunately, these simple formulas determine the multiplicative structure of the rest of the cohomology ring. I'll restate the Pieri formulas here.

Proposition 1 (column version of Pieri). *In $H^*(G(r, m))$, if $k \leq n$, then*

$$\sigma_\lambda \cdot \sigma_{(k)} = \sum_{\lambda'} \sigma_{\lambda'},$$

where the sum is taken over all diagrams obtained from λ by adding k boxes, no two in a column.

Proposition 2 (row version of Pieri). *In $H^*(G(r, m))$, if $k \leq r$, then*

$$\sigma_\lambda \cdot \sigma_{(1^k)} = \sum_{\lambda'} \sigma_{\lambda'},$$

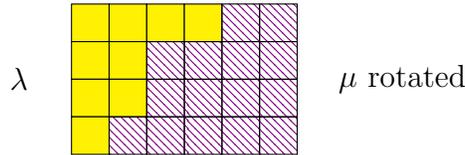
where the sum is taken over all diagrams obtained from λ by adding k boxes, no two in a row.

A proof of the column version can be found in Section 9.4 of [1], so I'll give a proof of the row version. Before that, we'll need a dual basis for the cohomology.

Lemma 1 (Duals of Schubert classes). *Let λ and μ be Young diagrams such that $|\lambda| + |\mu| = rn$*

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \text{if } \lambda_i + \mu_{r+1-i} = n \text{ for all } 1 \leq i \leq r \\ 0 & \text{if } \lambda_i + \mu_{r+1-i} > n \text{ for any } i. \end{cases}$$

Proof sketch. Choose any flag F_\bullet . If the first condition is true, then if we rotate μ by 180 degrees and place it up against λ , then we will get a full $r \times n$ rectangle like so.



Express $V \in G(r, m)$ in matrix form in a basis corresponding to F_\bullet . If $V \in \Omega_\lambda(F_\bullet)$, then we can write the matrix with extra zeros in the shape of λ starting from the top left. If $V \in \Omega_\lambda(\tilde{F}_\bullet)$, then we can write the matrix with extra zeroes in the shape of μ rotated, starting from the bottom right. Because λ and the rotated μ fit together the way they do, the extra zeroes force $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$ to be a single point. If λ and the rotated μ overlap at all (this would happen in the second case), then $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$ is empty. A formal proof of this fact can be found in [1]. \square

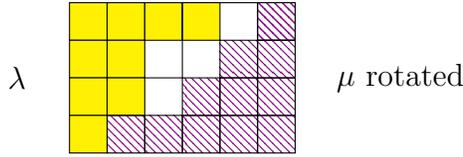
The dual basis gives us a systematic way of computing products in the cohomology ring, which you'll see at the beginning of the proof of Pieri.

Proof of row version of Pieri. We know the product takes the form

$$\sigma_\lambda \cdot \sigma_{(1^k)} = \sum_{|\lambda'|=|\lambda|+k} n_{\lambda'} \sigma_{\lambda'}.$$

For each λ' , if σ_μ is the dual of $\sigma_{\lambda'}$, then $n_{\lambda'} = \sigma_\lambda \cdot \sigma_{(1^k)} \cdot \sigma_\mu$. We'll get this coefficient by intersecting Schubert varieties corresponding to the three diagrams in the subscripts.

First, let's intersect Schubert varieties corresponding to λ and μ . Fix a flag F_\bullet . Place λ at the top left corner of an $r \times n$ rectangle, and rotate μ 180 degrees and place it at the bottom right corner. If the diagrams overlap, then by an argument similar to that of Lemma 1, $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$ is empty, so $n_{\lambda'} = 0$. Let's assume otherwise, i.e. each $\lambda_i + \mu_{r+1-i} \leq n$, which is equivalent to $\lambda'_i \geq \lambda_i$. Now the picture of the diagrams looks something like



where the number of blank squares in each row is $\lambda'_i - \lambda_i$.

Express $V \in \Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$ as a matrix in a basis corresponding to F_\bullet . Then, we can row reduce the matrix so that it has extra zeroes in the shape of λ starting from the top left and extra zeroes in the shape of μ starting from the bottom right. Such a matrix for the above example would look like

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the stars can be anything as long as the matrix is full rank. In general, row i of such a matrix has $\lambda'_i - \lambda_i + 1$ contiguous stars starting at column $r - i + \lambda_i + 1$. This property guarantees that the set of starred columns in one row is never contained in the set of starred columns in another row. So, up to scaling of rows, we can uniquely express each $V \in \Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$ as a matrix with the only nonzero entries in the starred places. Call the rows with only one star *basis rows*. For convenience, let's write all our matrices with the basis rows at the top, and let's replace the stars in the basis rows with 1.

Now that we understand $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$, we need to intersect with another Schubert variety $\Omega_{(1^k)}(E_\bullet)$. Using the column description of the Schubert variety, we can say

$$\Omega_{(1^k)}(E_\bullet) = \{V : \dim(V \cap E_{n+k-1}) \geq k\}.$$

From now on, let's write this variety as $\Omega_{(1^k)}(W)$ for a generic $n+k-1$ dimensional subspace W . If $V \in \Omega_{(1^k)}(W)$, then V contains k linearly independent vectors inside W . To specify these k vectors, we can choose a $k \times r$ matrix C of rank k so that the vectors are the rows of $C \cdot V$.

Suppose for some i that λ'_i is at least two bigger than λ_i . Then, there have to be at least $r - k + 1$ rows j where $\lambda'_i = \lambda_i$, so if $V \in \Omega_\lambda(F_\bullet) \cap \Omega_\mu(\tilde{F}_\bullet)$, then its matrix must have at

least $r - k + 1$ basis rows. Using row operations, we can write every matrix C so that the top row is

$$(c_1, \dots, c_{r-k+1}, \underbrace{0, \dots, 0}_{k-1}).$$

Then, if the i -th row of V is e_{j_i} , the top row of $C \cdot V$ is

$$w_1 = c_1 e_{j_1} + \dots + c_{r-k+1} e_{j_{r-k+1}}.$$

We need w_1 to satisfy the $r - k + 1$ equations that cut out W . This condition imposes $r - k + 1$ equations on the $r - k + 1$ variables c_i , so the only solution is zero, which means that C is not full rank. Thus if any $\lambda'_i - \lambda_i > 1$, then $n_{\lambda'} = 0$.

Now suppose all $\lambda'_i - \lambda_i \leq 1$. Then, V must have $r - k$ basis rows $e_{j_1}, \dots, e_{j_{r-k}}$ at the top and k rows v_{r-k+1}, \dots, v_r at the bottom with two stars each. The following is the unique choice of C and V so that the rows of $C \cdot V$ sit in W . Let C have a $k \times k$ identity matrix on the right hand side. Then, the i -th row of $C \cdot V$ is

$$w_i = v_{r-k+i} + \sum_{\ell=1}^{r-k} c_{i\ell} e_{j_\ell}.$$

We need w_i to satisfy the $r - k + 1$ equations for W . We have two variables from each v_{r-k+i} and $r - k$ variables from the $c_{i\ell}$'s. So, we have $r - k + 2$ variables and $r - k + 1$ equations, then we end up with exactly one solution once we mod out by scalar multiplication of v_{r-k+i} .

If there were another solution, then it would be impossible to write C with an identity matrix on the right side. In this case, we could write the top row of C with k zeroes at the end. So, the top row of $C \cdot V$ would have $r - k$ variables, which would not have a solution in $r - k + 1$ equations. Therefore, if each $\lambda'_i - \lambda_i \leq 1$, then $n_{\lambda'} = 1$. \square

The Pieri formulas indicate that $H^*(G(r, m))$ behaves similarly to Λ_r , the ring of symmetric polynomials in r variables. Indeed, we have an abelian group homomorphism $\Lambda_r \rightarrow H^*(G(r, m))$ with the following values on the Schur polynomials:

$$s_\lambda \mapsto \begin{cases} \sigma_\lambda & \text{if } \lambda \text{ has at most } n \text{ columns} \\ 0 & \text{otherwise.} \end{cases}$$

Since Λ_r is generated as a ring by the elementary symmetric polynomials $s_{(1^k)} = e_k$ and both rings follow the row version of the Pieri formula, this map is actually a homomorphism of rings. So, we know that $H^*(G(r, m))$ follows the same multiplication law as the symmetric polynomials in general, which is that

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu,$$

where $c_{\lambda\mu}^\nu$ are the Littlewood–Richardson coefficients.

3.3 Dualization

The row and column versions of the Pieri formula behave in a symmetric fashion: one can be obtained from the other just by interchanging the words “row” and “column.” This fact suggests that the Grassmanian and its cohomology have some sort of duality property. A natural dual map on the Grassmanian is the isomorphism $(\cdot)^\perp : G(r, m) \rightarrow G(m - r, m)$ that takes a subspace V to its complement V^\perp in \mathbb{C}^m . A natural dual map on Young diagrams sends λ to λ reflected across the diagonal, which we call λ^T . These two maps play nicely with each other by the following.

Proposition 3. *If F_\bullet is a flag and \tilde{F}_\bullet the corresponding backwards flag, then $(\Omega_\lambda(F_\bullet))^\perp = \Omega_{\lambda^T}(\tilde{F}_\bullet)$. This means that the map induces a ring isomorphism $H^*(G(r, m)) \rightarrow H^*(G(m - r, m))$ sending σ_λ to σ_{λ^T} .*

Proof. Use the column description of $\Omega_\lambda(F_\bullet)$. Then,

$$\begin{aligned} V \in \Omega_\lambda(F_\bullet) &\iff \dim(V \cap \tilde{F}_{n-j+\lambda_j^T}) \geq \lambda_j^T \\ &\iff \dim(V^\perp \cap \tilde{F}_{n-j+\lambda_j^T}) \leq \dim \tilde{F}_{n-j+\lambda_j^T} - \lambda_j^T = n - j \\ &\iff \dim(V^\perp \cap F_{r+j-\lambda_j^T}) \geq \dim V^\perp - (n - j) = j \\ &\iff V^\perp \in \Omega_{\lambda^T}(\tilde{F}_\bullet). \end{aligned}$$

□

The version of this dual property in symmetric polynomials is ω -involution, which sends each complete symmetric polynomial h_k to the elementary symmetric polynomial e_k . This involution behaves the same way as taking the complement in the Grassmanian, but in the polynomial context it seems much less intuitive.

3.4 The Plücker embedding

The Grassmanian can be embedded into projective space by the Plücker embedding

$$\begin{aligned} G(r, m) &\rightarrow \mathbb{P}(\Lambda^r \mathbb{C}^m) \\ \langle v_1, \dots, v_r \rangle &\mapsto v_1 \wedge \dots \wedge v_r. \end{aligned}$$

The image of $G(r, m)$ in $\mathbb{P}(\Lambda^r \mathbb{C}^m)$ is cut out by quadratic equations

$$(v_1 \wedge \dots \wedge v_r) \cdot (w_1 \wedge \dots \wedge w_r) = \sum_{i_1 < \dots < i_k} (v_1 \wedge \dots \wedge w_{i_1} \wedge \dots \wedge w_{i_k} \wedge \dots \wedge v_r) \cdot (v_{i_1} \wedge \dots \wedge v_{i_r} \wedge w_{r+1} \wedge \dots \wedge w_r)$$

for all $k \leq r$, $v_1, \dots, v_r, w_1, \dots, w_r \in \mathbb{C}^m$. The summand is simply $(v_1 \wedge \dots \wedge v_r) \cdot (w_1 \wedge \dots \wedge w_r)$ with v_{i_j} and w_{j+r-k} interchanged for each j . See [1] section 9.1 for a proof of this fact.

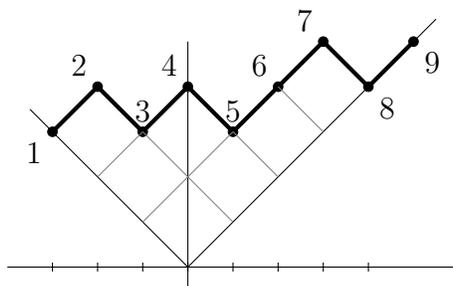
Given $V \in G(r, m)$, the Plücker embedding gives us a fun way to produce the Young diagram corresponding to the Schubert cell containing V . Fix a basis e_i and write V as a matrix. The coefficient of $e_{i_1} \wedge \dots \wedge e_{i_k}$ in the Plücker embedding of V is the determinant of the matrix minor with columns i_1, \dots, i_k . So, if we take the nonzero coordinate with the

smallest possible subscripts, then we will have the indices of the leftmost linearly independent columns of the matrix, which is the information we need to find the cell.

Here is a method to produce the diagram that will extend nicely to infinite dimensions. Label our basis of \mathbb{C}^m as $e_{-r+\frac{1}{2}}, \dots, e_{-3/2}, e_{-1/2}, e_{1/2}, \dots, e_{n-\frac{1}{2}}$. Start with an example: let $V \in G(3, 8)$ and write V with leftmost linearly independent columns 2, 4, and 7. So, we get a matrix like this:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 1 & * & 0 & * & * & 0 & * \end{pmatrix}$$

The term with the smallest subscripts in the Plücker embedding is $e_{-3/2} \wedge e_{1/2} \wedge e_{7/2}$. Draw a graph of $y = |x|$ in the plane. Start at $(-3, 3)$ and execute the following sequence, moving one unit to the right per step. Since $e_{-5/2}$ is missing from our wedge product, let's draw our next point one unit up. Since $e_{-3/2}$ is in the product, let's draw our next point one unit down. Since $e_{-1/2}$ is missing, let's go up. Keep going like this, going up if a basis vector is missing and down if it is there. In the picture, I've labelled the points in order. If we connect the dots, then we can see a rotated Young diagram.



Here is the recipe in general. Choose a basis for \mathbb{C}^m and label it from $-r + \frac{1}{2}$ to $n - \frac{1}{2}$. Given $V \in G(r, m)$, take the Plücker embedding of V into $\mathbb{P}(\Lambda^r \mathbb{C}^m)$ and find the nonzero term with the smallest subscripts. Using this term, draw the picture. Start at $(-r, r)$. If $e_{-r+\frac{1}{2}}$ appears in the wedge product, draw the next point one unit down. Otherwise, go up. Continue like this until you reach (n, n) . Connect the dots to produce the Young diagram. This method produces the correct diagram because a missing coordinate means your next column has as many extra zeroes as the previous column, and a present coordinate means that your next column is a pivot.

4 Infinite-dimensional Grassmanians

We've seen that $H^*(G(r, m))$ is a truncated version of Λ_r , where $s_\lambda \in \Lambda_r$ maps to 0 if λ has too many columns. The fixed codimension in the Grassmanian is the cause of this truncation, so we can try growing this codimension infinitely in order to find a space with cohomology equal to Λ_r . We can also take the dual of this idea, so we can fix the codimension and let the dimension grow and see what happens. Finally, if we limit in both of these directions, we should get some topological space with cohomology equal to Λ , the ring of symmetric functions.

4.1 Fixing dimension and growing codimension

Let e_1, e_2, \dots be a basis of $\mathbb{C}^{\oplus\infty}$, henceforth written as \mathbb{C}^∞ . Observe that our \mathbb{C}^∞ is a direct sum, so every vector is a finite linear combination of basis elements. Let $G(r, m)$ be the space of r -dimensional subspaces of $\langle e_1, \dots, e_m \rangle$. Then, the inclusion of \mathbb{C}^m into \mathbb{C}^{m+1} induces an inclusion of $G(r, m)$ into $G(r, m+1)$. The direct limit of the sequence of inclusions

$$G(r, r) \hookrightarrow \dots \hookrightarrow G(r, m) \hookrightarrow G(r, m+1) \hookrightarrow \dots$$

is the space of r -dimensional subspaces of \mathbb{C}^∞ , which we will write as $G(r, \infty)$. Consider the cohomology rings of the Grassmanians in the sequence. There are pullback maps $H^*(G(r, m+1)) \rightarrow H^*(G(r, m))$ that send σ_λ to σ_λ if λ has at most n columns and 0 otherwise. These maps give us the sequence in the following proposition.

Proposition 4. *The ring Λ_r is the inverse limit of the sequence*

$$\dots \rightarrow H^*(G(r, m+1)) \rightarrow H^*(G(r, m)) \dots \rightarrow H^*(G(r, r)).$$

Proof. If λ has n columns, then σ_λ appears in $H^*(G(r, m))$ for all $m \geq r+n$. If we write the inverse limit R as a subgroup of the direct product of all $H^*(G(r, m))$, then the following gives an isomorphism $\Lambda_r \cong R$:

$$s_\lambda \mapsto (\dots, \sigma_\lambda, \sigma_\lambda, \sigma_\lambda, \underbrace{0, \dots, 0}_{n-1})$$

□

Assuming that the cohomology of the limit is the limit of the cohomology, this means that $\Lambda_r \cong H^*(G(r, \infty))$. Indeed, all of our previous logic follows through. Since our basis is numbered starting from 1, we still have a notion of leftmost linearly independent columns, so we can still see the Young diagrams in the matrix representations of subspaces. Since we are allowed to have arbitrarily many columns, the cells are indexed by Young diagrams with at most r rows and any number of columns. Pieri's formula holds because given arbitrary classes σ_λ and σ_μ , we can always find a finite Grassmanian for both of them to live in, as long as we set the codimension to be big enough.

4.2 Fixing codimension and growing dimension

Fix a codimension n , and let $\dots, e_{-3}, e_{-2}, e_{-1}, e_1, e_2, \dots, e_n$ be a basis for $\mathbb{C}^{\oplus\infty}$. For $a \leq b$, let $E_{a,b} = \langle e_a, e_{a+1}, \dots, e_b \rangle$. Let $G(r, r+n)$ be the space of codimension n subspaces of $E_{-r,n}$, which we identify with \mathbb{C}^{r+n} . Then, we have an inclusion of Grassmanians $G(r, r+n) \hookrightarrow G(r+1, r+1+n)$ sending $\langle v_1, \dots, v_r \rangle$ to $\langle e_{-r-1}, v_1, \dots, v_r \rangle$. We wish to analyze the direct limit of the sequence

$$G(0, n) \hookrightarrow \dots \hookrightarrow G(r, r+n) \hookrightarrow G(r+1, r+1+n) \hookrightarrow \dots,$$

which we will call $G(\infty, \infty+n)$. This task is best done using the Plücker embedding $G(r, m) \rightarrow \mathbb{P}(\Lambda^r \mathbb{C}^m)$. Consider the inclusion $\Lambda^r \mathbb{C}^m \hookrightarrow \Lambda^{r+1} \mathbb{C}^{m+1}$ sending $v_1 \wedge \dots \wedge v_r$ to $e_{-r-1} \wedge v_1 \wedge \dots \wedge v_r$.

Lemma 2. *The direct limit of the sequence $\Lambda^r \mathbb{C}^{r+n} \hookrightarrow \Lambda^{r+1} \mathbb{C}^{r+1+n} \hookrightarrow \dots$ is an infinite-dimensional vector space with basis*

$$\{\cdots \wedge e_{-r-2} \wedge e_{-r-1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_\ell} : r \geq 0, -r \leq i_1 < \cdots < i_\ell \leq n, \\ \text{and } \#(i_j > 0) = k - 1 - \#(i_j < 0)\}.$$

Denote this direct limit by $\Lambda^\infty \mathbb{C}^{\infty+n}$.

In English, this basis consists of infinite wedge products where the number of positive indexed e_i 's equals the number of missing negative indexed e_j 's. For instance, $\cdots e_{-5} \wedge e_{-1} \wedge e_2 \wedge e_3$ satisfies this property.

Proof. Choosing r distinct indices from of $\{-r, -1, 1, n\}$ automatically ensures that the number of positive indices equals the number of missing negatives, just by virtue of how we indexed the set. So, every basis element $e_{i_1} \wedge \cdots \wedge e_{i_\ell}$ of $\Lambda^r \mathbb{C}^{r+n}$ has this property. The maps from $\Lambda^r \mathbb{C}^{r+n}$ to the direct limit are simply

$$v_1 \wedge \cdots \wedge v_r \longmapsto \cdots \wedge e_{-r-2} \wedge e_{-r-1} \wedge v_1 \wedge \cdots \wedge v_r.$$

Notice that attaching $\cdots \wedge e_{-r-2} \wedge e_{-r-1}$ to the wedge product contributes no positive indices or missing negative indices, so the property is preserved. Thus the basis for $\Lambda^\infty \mathbb{C}^{\infty+n}$ is taken from the bases of each $\Lambda^r \mathbb{C}^{r+n}$ and attaching an infinite wedge product of basis elements with indices up to $-r - 1$. \square

Using the embeddings $G(r, m) \rightarrow \mathbb{P}(\Lambda^r \mathbb{C}^m)$, the universal property of the direct limit gives us an embedding $G(\infty, \infty + n) \rightarrow \mathbb{P}(\Lambda^\infty \mathbb{C}^{\infty+n})$. The image of $G(\infty, \infty + n)$ will be the direct limit of the Plücker images of $G(r, r + n)$. So, in coordinates, the equations that cut out this image will be

$$(\cdots \wedge e_{-r-1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_\ell}) \cdot (\cdots \wedge e_{-p-1} \wedge e_{j_1} \wedge \cdots \wedge e_{j_\ell}) \\ = \sum (\cdots \wedge e_{-r'-1} \wedge e_{i'_1} \wedge \cdots \wedge e_{i'_\ell}) \cdot (\cdots \wedge e_{-p'-1} \wedge e_{j'_1} \wedge \cdots \wedge e_{j'_\ell}),$$

where the sum is over exchanging a fixed set of k indices from j_1, \dots, j_ℓ with some k indices from i_1, \dots, i_ℓ , preserving order. Note that this exchange will preserve the property that the number of positives is the number of missing negatives.

Now that we can describe $G(\infty, \infty + n)$ as a topological space, let's find its cell structure. Let $V \in G(\infty, \infty + n)$. Then, we can find some r where $V \in G(r, r + n)$. Take the finite Plücker coordinates of V , find the nonzero coordinate with the smallest indices, and draw a picture as shown in Section 3.4. This picture will give us the Young diagram corresponding to the cell containing V . Since r can be arbitrary large, the diagrams that index the cells have at most n columns but any number of rows.

4.3 Limiting in both directions

We have examined sequences of inclusions in two directions:

$$\begin{array}{ccccc}
 G(r, m) & \hookrightarrow & G(r+1, m+1) & \hookrightarrow & \dots \\
 \downarrow & & \downarrow & & \\
 G(r, m+1) & \hookrightarrow & G(r+1, m+2) & \hookrightarrow & \dots \\
 \downarrow & & \downarrow & & \\
 \vdots & & \vdots & &
 \end{array}$$

Let's call the direct limit of this system $G(\infty, \infty + \infty)$. Combining the ideas from the previous two sections, the Plücker embedding of $G(\infty, \infty + \infty)$ lives in $\mathbb{P}(\Lambda^\infty \mathbb{C}^{\infty + \infty})$, where we index the basis of $\mathbb{C}^{\infty + \infty}$ by $\dots, e_{-2}, e_{-1}, e_1, e_2, \dots$. A basis for $\Lambda^\infty \mathbb{C}^{\infty + \infty}$ is the set of infinite wedge products of basis vectors where the number of positive indices is the number of missing negative indices. Now, when we draw the diagram associated with a wedge product of e_i 's, we can get Young diagrams of any size, so the cells in $G(\infty, \infty + \infty)$ are indexed by all Young diagrams.

Consider the cohomology of the Grassmanians:

$$\begin{array}{ccccc}
 H^*(G(r, m)) & \longleftarrow & H^*(G(r+1, m+1)) & \longleftarrow & \dots \\
 \uparrow & & \uparrow & & \\
 H^*(G(r, m+1)) & \longleftarrow & H^*(G(r+1, m+2)) & \longleftarrow & \dots \\
 \uparrow & & \uparrow & & \\
 \vdots & & \vdots & &
 \end{array}$$

Taking the limits of the columns gives us

$$\Lambda_r \leftarrow \Lambda_{r+1} \leftarrow \dots,$$

where the map takes $p(x_1, \dots, x_{r+1})$ to $p(x_1, \dots, x_r, 0)$. The inverse limit of the above sequence is Λ , the ring of symmetric functions. So, we have an isomorphism $\Lambda \cong G(\infty, \infty + \infty)$ given by $s_\lambda \mapsto \sigma_\lambda$.

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