SURIM Final Report
Zeros of Eisenstein Series and Theta Series from Brandt Matrix

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Abstract

Modular forms are powerful tools used to study problems in number theory, but the spaces of modular forms for $\Gamma_0(N)$ are still not completely understood. In this report we study two problems related to these spaces: Determining the zeros of Eisenstein series, and the formulation of explicit bases for these spaces using Brandt matrices. More specifically, we derive bounds on the imaginary parts of the zeros of level p Eisenstein series, and we examine the subspace of $M_2(\Gamma_0(p))$ generated by the theta series associated to the diagonal of the corresponding Brandt matrices. For the first part, we will derive a formula to find the number of zeros with multiplicity of a modular form of $\Gamma_0(p)$ and use the formula to bound where the zeros of the Eisenstein series of level p prime can be. Then, we study the subspace generated by the elements of Brandt matrices by investigating the properties of the matrix that maps the basis for $M_2(p)$ to the diagonal elements of the Brandt matrices.

1 Introduction

Modular forms are important analytic tools used to study number theory. Modular forms played a key role in the proof of Fermat's Last Theorem and have been used to derive unexpected relationships between arithmetic sequences, elliptic curves and other series. Hence, learning about the spaces of different modular forms and their properties allows us to better understand these powerful functions.

We investigate certain Eisenstein series for $\Gamma_0(N)$. An relevant result about the Eisenstein series of level 1 is they generate all modular forms of level 1. While this result does not generalize to higher level, Eisenstein series are still important modular forms to analyze because they are crucial to the structure of the space of modular forms of level N.

F.K.Rankin and Swinnerton-Dyer prove in [1] that for a weight k level 1 Eisenstein series, all of the zeros lie on the arc $|z| = 1$. Shigezumi applies Rankin and Swinnerton-Dyer's method to $\Gamma_0(p)$ and $\Gamma_0^*(p)$ for small p in [2] and finds a similar result. We try to generalize to higher p and find the zeros of these Eisenstein series.

1.1 Background and Definitions

The congruence subgroup that we will focus on is $\Gamma_0(N)$ which is defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$$

Note that $\Gamma_0(1) = SL_2(\mathbb{Z})$.

Definition 1. *A modular form of weight k and level N for congruence subgroup $\Gamma_0(N)$ is a function $f : \mathbb{H} \to \mathbb{C}$ that is holomorphic on $\mathbb{H}$ and $f(\gamma)$ is holomorphic at ∞ for all $\gamma \in \Gamma_0(N)$. In addition, f*

satisfies the following relation for $z \in \mathbb{H}$, $\gamma \in \Gamma_0(N)$:

$$f(\gamma z) = j(\gamma, z)^k f(z) \tag{1}$$

where $j(\gamma, z) = (cz + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The holomorphicity of f allows us to write the q-expansion of f, which is more useful when computing properties of f. We will denote the q-expansion of f as the following sum, where $q = e^{2\pi i z}$.

$$f = \sum_{n=0}^{\infty} a_n q^n$$

First we will define the level 1 Eisenstein series of weight k as the following function.

Definition 2. *The level 1 Eisenstein series of weight k is $G_k(z)$ where*

$$G_k(z) = \sum_{(c,d) \in \mathbb{Z}^2 - \{(0,0)\}} (cz + d)^{-k}$$

Note that $G_k(z) = 0$ for odd k. For the remainder of the report, we will always assume k is an even positive integer. More often, we will consider the normalized Eisenstein series:

Definition 3. *The normalized level 1 Eisenstein series of weight k is $E_k(z)$ where*

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k}$$

Then the q-expansion of G_k and E_k are as follows:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$E_k(z) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where σ is defined to be:

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$$

It is know that the space of level 1 modular forms is generated by E_4 and E_6.

Now we want to consider the Eisenstein series for $\Gamma_0(N)$:

Definition 4. *A cusp is a $\Gamma_0(N)$-equivalence class set of points in $\mathbb{Q} \cup \{\infty\}$.*

Definition 5. *Let c be a cusp of $\Gamma_0(N)$, then the Eisenstein series centered at the cusp c is defined to be:*

$$E_{k,N}^c(z) = \sum_{\Gamma_c \backslash \Gamma_0(N)} j(\gamma, z)^{-k}$$

Here Γ_c are the matrices of $\Gamma_0(N)$ where that take c to c. A property of these Eisenstein series is that they are zero at all of the cusps they are not centered at.

If we consider $\Gamma_0(p)$ for p prime, then the only cusps are at 0 and ∞. So, we have the following Eisenstein series:

$$E_{k,p}^{\infty} = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (cz + d)^{-k} = \frac{1}{1 - p^k} \left(E_k(z) - p^k E_k(pz) \right)$$

$$E^0_{k,p} = \frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid c}} (cz+d)^{-k} = \frac{1}{1-p^{-k}}\left(E_k\left(z\right) - E_k\left(pz\right)\right)$$

If we consider these series under the Atkin-Lehner involution $W_p = \begin{pmatrix} 0 & -\frac{1}{\sqrt{p}} \\ \sqrt{p} & 0 \end{pmatrix}$ We have the following relation between the two Eisenstein series:

$$(\sqrt{p}z)^{-k} E^{\infty}_{k,p}\left(W_p z\right) = \left(p^{-k/2}\right) E^0_{k,p}\left(z\right)$$

$$(\sqrt{p}z)^{-k} E^{0}_{k,p}\left(W_p z\right) = \left(p^{k/2}\right) E^{\infty}_{k,p}\left(z\right)$$

This means that there is a bijection between the zeros of the two Eisenstein series. In addition, if we define $v_p(f)$ to be the order of f at the point p, then we have that $v_0(E^{\infty}_{k,p}) = v_{\infty}(E^0_{k,p}) = 1$ and $v_{\infty}(E^{\infty}_{k,p}) = v_0(E^0_{k,p}) = 0$.

We want to find and/or bound where the zeros of $E^{\infty}_{k,p}$ will be. Given Rankin and Swinnerton-Dyer and Shigezumi's results both locate the zeros on arcs on the boundary of certain domains, we aim to generalize the domain that Shigezumi uses to find the zeros. We also want to count the number of zeros we expect in a certain area and apply different strategies to locate them.

2 Fundamental Domain

To study where the zeros of $E^{\infty}_{k,p}$ lie, first recall that $E^{\infty}_{k,p}(\gamma z) = j(\gamma, z)^k E^{\infty}_{k,p}(z)$. So, if z is a zero, then so is γz for all $\gamma \in \Gamma_0(p)$. So, we only need to consider $\mathbb{H}/\Gamma_0(p)$:

Definition 6. *A fundamental domain for a congruence subgroup* Γ *is a region* $\mathcal{F} \subset \mathbb{H}$ *such that for every* $z \in \mathbb{H}$*, there is a unique* $\gamma \in \Gamma$ *such that* $\gamma z \in \mathcal{F}$*. Note that there are many fundamental domains for a given* Γ*.*

We want to consider a fundamental domain for $\Gamma_0(N)$. We aim to find the most ideal one for locating zeros. We introduce two fundamental domains, named the *fin* domain and the *bump* domain and discuss the information they contain.

2.1 The Fin Domain

One way to achieve a fundamental domain for $\Gamma_0(N)$ is to look at the coset representatives for $\Gamma_0(N)$ for $SL_2(\mathbb{Z})$. We will define $\mathcal{F}_1$ to denote the fundamental domain of $SL_2(\mathbb{Z})$, pictured here:

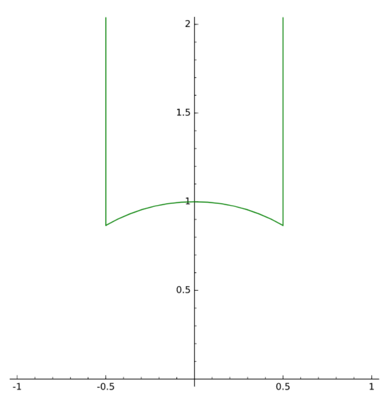

Figure 1: The Fundamental Domain for $SL_2(\mathbb{Z})$

So, to determine what a valid fundamental domain looks like for $\Gamma_0(N)$ for other N, we need to determine a set of representatives S such that

$$SL_2(\mathbb{Z}) = \bigcup_{\alpha \in S} \Gamma_0(N)\alpha$$

To determine these matrices, we first note two formulas from [3]:

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = d_N = N \prod_{d|N} \left(1 + \frac{1}{d}\right)$$

$$\text{Number of cusps } = \varepsilon_\infty = \sum_{d|N} \phi\left(\gcd\left(d, N/d\right)\right)$$

(Note ϕ is the Euler totient function) So, we are looking for d_N matrices, that are all distinct via action of $SL_2(\mathbb{Z})$, and which give ε_∞ different cusps. Since we will consider $N = p$ prime, one such set of representatives is given by

$$S = I \cup \{\alpha_k\}_{k=-(p-1)/2}^{(p-1)/2}$$

Where

$$\alpha_k = \left(\begin{smallmatrix} 0 & 1 \\ -1 & k \end{smallmatrix}\right)$$

This leads to Fin domains that look like these, taken for $p = 11, 13$:

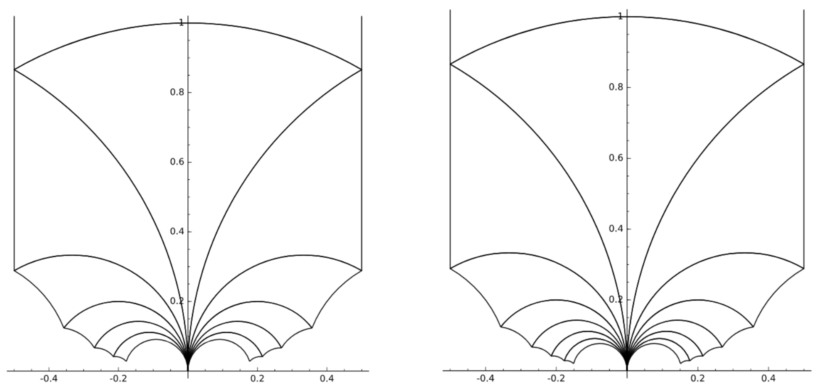

Figure 2: Fin domains for $p = 11, 13$, divided by coset representatives

For composite N, there is a deterministic algorithm to select the desired number of coset representatives, with d_N matrices and ε_∞ different cusps. However, we will hold off on discussing that until the appendix, as the method is unnecessary for our discussions.

However, while these graphs are nice, symmetric, and easy to compute, for the functions we will consider, it is better to define a slightly different fundamental domain:

2.2 The Bump Domain

We consider the fundamental domain that Shigezumi considers in [2]. This fundamental domain we will refer to as the bump domain. We will consider the construction of the bump domain for $\Gamma_0(p)$ for p prime. First, we will parameterize the boundary as the following:

$$\mathcal{A}_{p,f} = \left\{ \frac{e^{i\theta}}{p} - \frac{1}{p} \mid \theta \in [0, \frac{2\pi}{3}] \right\} \cup \left\{ \frac{e^{i\theta}}{p} + \frac{1}{p} \mid \theta \in [\frac{\pi}{3}, \pi] \right\} \cup \left\{ \frac{e^{i\theta}}{p} + \frac{q}{p} \mid \theta \in [\frac{\pi}{3}, \frac{2\pi}{3}], 1 < |q| \leq \frac{p-1}{2}, q \in \mathbb{Z} \right\}$$

$$\mathcal{B}_p = \mathcal{A}_{p,f} \cup \left\{ 1/2 + it \mid t \geq \frac{\sqrt{3}}{2p} \right\} \cup \left\{ -1/2 + it \mid t \geq \frac{\sqrt{3}}{2p} \right\}$$

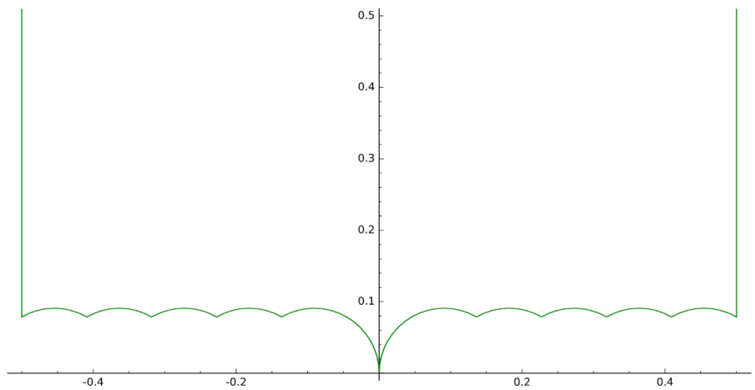

Figure 3: Bump domain for $\Gamma_0(11)$

Lemma 1. *Let* $C_j = \left\{ \frac{e^{i\theta}}{p} + \frac{j}{p} \mid \theta \in [\frac{\pi}{3}, \frac{2\pi}{3}] \right\}$ *be the arc centered at* $\frac{j}{p}$*. Then* C_j *and* C_k *are equivalent via action of* $\Gamma_0(p)$ *if and only if*

$$j \equiv -k^{-1} \ mod\ p \tag{2}$$

Proof. If: Let $j \equiv -k^{-1} \bmod p$, such that $|j| \leq \frac{p-1}{2}$. We need to find an element $\gamma \in \Gamma_0(p)$ such that $C_k = \gamma C_j$. Take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $b = \frac{-jk-1}{p}$. Since $j \equiv -k^{-1} \bmod p$, this is an integer. To show this sends one arc to another, take any $z = \frac{1}{p}(j + e^{i\theta}) \in C_j$. Then

$$\begin{aligned} \gamma z &= \frac{\frac{k}{p}(j + e^{i\theta}) + \frac{-jk-1}{p}}{j + e^{i\theta} - j} \\ &= \frac{1}{p} \frac{ke^{i\theta} - 1}{e^{i\theta}} \\ &= \frac{1}{p}(k - e^{-i\theta}) \in C_k \end{aligned}$$

Specifically, this γ flips the boundaries of the two arcs along the vertical line halfway between them, which will come into play when we talk about equivalences of wedges.

Only If: Assume that there is some $\gamma \in \Gamma_0(p)$ such that $\gamma C_j = C_k$. Now, extend C_j and C_k from just the narrow arc at the top to the whole circles C'_j and C'_k in $\mathbb{C}$. Since mobius inversion sends lines and circles to lines and circles, if $C_k = \gamma C_j$, then $C'_k = \gamma C'_j$. Since γ has all real entries, it must send the rational values on this circle to one another. So, we must have

$$\gamma\left(\frac{j+1}{p}\right) = \frac{k \pm 1}{p}, \quad \gamma\left(\frac{j-1}{p}\right) = \frac{k \mp 1}{p}$$

However, via some simple algebra, we arrive at the conclusion that $\gamma = \left(\begin{smallmatrix} k & b \\ p & -j \end{smallmatrix}\right)$, where $b = \frac{-jk-1}{p}$. For b to be an integer, we must have $j \equiv -k^{-1} \bmod p$. □

Proposition 1. *Let* $\mathcal{F}_p$ *be the region enclosed in* $\mathcal{B}_p$ *including* $\mathcal{B}$*.* $\mathcal{F}_p$ *is a fundamental domain for* $\Gamma_0(p)$*.*

Proof. We note that for $\mathcal{F}_p$ to be a fundamental domain, it suffices to show that the boundary of $\mathcal{F}_p$ maps to itself under $\Gamma_0(p)$. Clearly from Lemma 1, the $\mathcal{A}_p$ maps to itself under $\Gamma_0(p)$. Then under $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$\left\{-1/2 + it \mid t \geq \frac{\sqrt{3}}{2p}\right\}$ maps to $\left\{1/2 + it \mid t \geq \frac{\sqrt{3}}{2p}\right\}$. Thus, $\mathcal{B}_p$ maps to itself under $\Gamma_0(p)$ and hence $\mathcal{F}_p$ will be a fundamental domain. $\square$

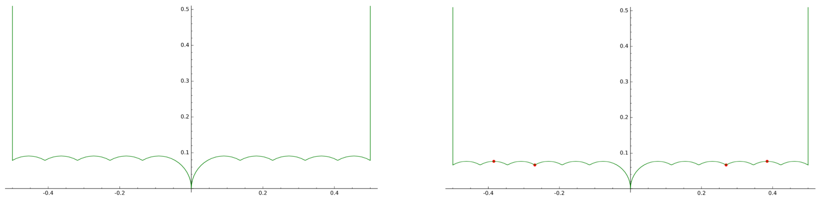

Figure 4: Bump domains for $p = 11, 13$, including elliptic points

Definition 7. *The bump fundamental domain $\mathcal{F}_p$ for $\Gamma_0(p)$ is the area within boundary $\mathcal{B}$ including the $\mathcal{B}_p$.*

Notice that the bump fundamental domain is the fin fundamental domain transformed under the Atkin-Lehner involution. Thus, just like the previous section, there is a formula for the bump domain for arbitrary levels N. However, as before, it is complicated to describe due to the addition of cusps other than at 0 and ∞. This will be discussed briefly in the appendix as well.

2.3 Special Points on $\mathcal{F}_p$

Also important to our understanding of the fundamental domain are the elliptic points of $\Gamma_0(p)$. A point z is an elliptic point for a congruence subgroup Γ (such as $\Gamma_0(p)$) if there is a matrix $I \neq \gamma \in \Gamma$ such that $\gamma z = z$

So, if z is an elliptic point of $\Gamma_0(p)$, then there is some matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), p|c$, such that $\gamma z = z$. Rearranging and applying the relation $ad - bc = 1$, this means that

$$z = \frac{a - d + \sqrt{(a+d)^2 - 4}}{2c}$$

For this to be imaginary, we much have $(a+d)^2 - 4 < 0$, so $a + d \in \{-1, 0, 1\}$. This leads us to two cases:

Definition 8. *A point z is a ρ-elliptic point for $\Gamma_0(p)$ if it is a shift of $\rho = \frac{1+\sqrt{-3}}{2}$, i.e. $z = \gamma\rho$ for some $\gamma \in SL_2(\mathbb{Z})$. These elliptic points are all of the form $z = \frac{q+\sqrt{-3}}{2p}$ for some $q \in \mathbb{Z}$.*

Definition 9. *A point z is an i-elliptic point for $\Gamma_0(p)$ if it is a shift of i, i.e. $z = \gamma i$ for some $\gamma \in SL_2(\mathbb{Z})$. These elliptic points are all of the form $z = \frac{q+i}{p}$ for some $q \in \mathbb{Z}$*

Lemma 2. *The number of ρ-elliptic points of $\Gamma_0(p)$ is $\varepsilon_\rho = 1 + \left(\frac{-3}{p}\right)$, and the number of i-elliptic points of $\Gamma_0(p)$ is $\varepsilon_i = 1 + \left(\frac{-1}{p}\right)$.*

Proof. From the formula above, z is a ρ-elliptic point iff $a + d = \pm 1$. Since we can replace γ with $-\gamma$ without changing how it acts on z, WLOG we have $a + d = -1$, and of course $ad \equiv 1 \bmod p$. This means that there must be some value a such that $a^2 + a + 1 \equiv 0 \bmod p$. Via quadratic reciprocity, one can verify that this is true iff $\left(\frac{-3}{p}\right) = 1$. In this case there are two values of z, otherwise there are none. So, in total there are $\varepsilon_\rho = 1 + \left(\frac{-3}{p}\right)$ total ρ-elliptic points.

Otherwise, if z is a i-elliptic point then we must have $a + d = 0$, and $ad \equiv 1 \bmod p$. This means that there must be some value a such that $a^2 \equiv -1 \bmod p$. Since this is true iff $\left(\frac{-1}{p}\right) = 1$, then in total there are $\varepsilon_i = 1 + \left(\frac{-1}{p}\right)$ total i-elliptic points.

$\square$

3 Counting Zeros of Modular Forms of Level N

When we consider the zeros of a modular form of $\Gamma_0(p)$, it is better to consider the zeros contained in the fundamental domain because the modular invariance condition for modular forms implies that every zero of a modular form corresponds to a zero inside the fundamental domain. Shigezumi in [2] creates valence formulas for $\Gamma_0(2), \Gamma_0(3), \Gamma_0(5)$ and $\Gamma_0(7)$. We generalize this valence formula to general $pandp^2$ for p prime.

Theorem 1. *Let $v_x(f)$ denote the order of the zero of f at a point x. Let f be a modular form of weight k for $\Gamma_0(p)$ where k is even. Let $\rho_{p,1}, \rho_{p,2}$ be the two ρ-elliptic points, and $i_{p,1}, i_{p,2}$ be the two i-elliptic points, if any of them exist. Then the following relation holds:*

$$v_\infty(f) + v_0(f) + \sum_{x \in \mathcal{F}_p} c_x v_x(f) = \frac{p+1}{12}k \tag{3}$$

where

$$c_x = \begin{cases} \frac{1}{3} & \textit{if } x = \frac{q}{2p} + \frac{\sqrt{3}}{2p}i \textit{ for some odd } q \\ \frac{1}{2} & \textit{else if } x \in \mathcal{B}_p \\ 1 & \textit{otherwise} \end{cases}$$

When we consider $\Gamma_0(N)$ and $\Gamma_0(p)$ for composite N and primes, we should discuss the elliptic points and cusps of these congruence subgroups.

We can count the number of elliptic points for $\Gamma_0(N)$ for N composite from [3]. We have denoted the elliptic points of period 2 as i-elliptic points and elliptic points of period 3 as ρ-elliptic points.

Lemma 3. *The number of elliptic points for $\Gamma_0(N)$ of period 2 is*

$$\varepsilon_2(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-1}{p}\right)\right) & \textit{if } 4 \nmid N \\ 0 & \textit{if } 4|N \end{cases}$$

The number of elliptic points for $\Gamma_0(N)$ of period 3 is

$$\varepsilon_3(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right) & \textit{if } 9 \nmid N \\ 0 & \textit{if } 9|N \end{cases}$$

Diamond and Shurman in [3] specify the form of these elliptic points. An elliptic point of period 3 will have the form

$$\Gamma_0(N)\frac{n+\rho}{n^2-n+1}, n^2 - n + 1 \equiv 0 \bmod N$$

We will use S_ρ to denote the set of elliptic points of period 3 of $\Gamma_0(N)$.

The elliptic points of period 2 for $\Gamma_0(N)$ will have the form

$$\Gamma_0(N)\frac{n+i}{n^2+1}, n^2 + 1 \equiv 0 \bmod N$$

We will use S_i to denote the set of elliptic points of period 2 of $\Gamma_0(N)$. We will consider S_∞ to be the set of cusps for $\Gamma_0(N)$. Diamond and Shurman also count the number of cusps for $\Gamma_0(N)$ to be the following:

Lemma 4. *Let ε_∞ be the number of cusps. Then*

$$\varepsilon_\infty\left(\Gamma_0\left(N\right)\right) = \sum_{d|N} \phi\left(\gcd\left(d, N/d\right)\right)$$

Theorem 2. *Let $v_x(f)$ denote the order of the zero of f at a point x. Let f be a modular form of weight k for $\Gamma_0(p^2)$. Then the following relation holds:*

$$\sum_{p_\infty \in S_\infty} v_{p_\infty}(f) + \sum_{x \in \mathcal{F}_p} c_x v_x(f) = \frac{k}{12}(p^2 + \varepsilon_\infty(\Gamma_0(p^2)) - 1) = \frac{k}{12}(p^2 + p) \tag{4}$$

where

$$c_x = \begin{cases} \frac{1}{3} & \text{if } x = \frac{q}{2p^2} + \frac{\sqrt{3}}{2p^2} i \text{ for some odd } q \\ \frac{1}{2} & \text{else if } x \in \mathcal{B}_p \\ 1 & \text{otherwise} \end{cases}$$

3.1 Proof of Theorem 1

Let f be as defined. We know that f will have a finite number of zeros up to equivalence in $\mathcal{F}_p$, so we can create a contour $\mathcal{C}$ that encloses all such zeros except the elliptic points and cusps.

Consider if $\Gamma_0(p)$ has ρ elliptic points. From Lemma 2, we see that $\Gamma_0(p)$ has at most 2 such points, we will denote $\rho_{p,1}$ and $\rho_{p,2}$. Similarly, there are at most 2 i elliptic points, which we will denote as $i_{p,1}, i_{p,2}$. Let us denote $C_{\rho_{p,1}}, C_{\rho_{p,2}}, C_{i_{p,1}}, C_{i_{p,2}}$ as the boundary of the disk of radius $\epsilon_{\rho_{p,1}}, \epsilon_{\rho_{p,2}}, \epsilon_{i_{p,1}}, \epsilon_{i_{p,2}}$.

Now we construct our contour $\mathcal{C}$. We first begin with $\mathcal{A}_{p,f}$ and want to exclude the elliptic points and cusp at 0. To remove these elliptic points, we travel along the arcs $C_{\rho_{p,1}}, C_{\rho_{p,2}}, C_{i_{p,2}}, C_{i_{p,2}}$ inside of $\mathcal{F}_p$ and take the limit where $\epsilon_{\rho_{p,1}}, \epsilon_{\rho_{p,2}}, \epsilon_{i_{p,1}}, \epsilon_{i_{p,2}}$ goes to 0. Similarly, we will travel along C_0, the boundary of the circle centered at 0 with radius ϵ_0. Now, we also add the following lines to $\mathcal{C}$:

$$\ell_{\frac{1}{2}} = \left\{ \frac{1}{2} + it \mid \frac{\sqrt{3}}{p} \le t \le R \right\}$$

$$\ell_{-\frac{1}{2}} = \left\{ -\frac{1}{2} + it \mid \frac{\sqrt{3}}{p} \le t \le R \right\}$$

$$\ell_R = \left\{ t + iR \mid -\frac{1}{2} \le t \le \frac{1}{2} \right\}$$

Now $\mathcal{C}$ is a closed contour and we will apply the residue theorem.

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{df}{f} = \sum_{p \in \mathcal{F}_p - \{\rho_{p,1}, \rho_{p,2}, \psi_{p,1}, \psi_{p,2}\}} v_p(f) \tag{5}$$

Now we will evaluate the right hand side of (5) by splitting into the different arcs of $\mathcal{C}$ to achieve the expression (3).

(i) Integrating along ℓ_R, we achieve

$$\frac{1}{2\pi i} \int_{\ell_R} \frac{df}{f} = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f'(u+iR)}{f(u+iR)} du = \frac{1}{2\pi i} \int_{\omega = \{|q| = e^{-2\pi R}\}} \frac{f'(q)}{f(q)} dq = -v_\infty(f)$$

(ii) Integrating along $\ell_{\frac{1}{2}}$ and $\ell_{-\frac{1}{2}}$, we use that $f(Tz) = f(z)$ where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. So, we have that:

$$\int_{\ell_{\frac{1}{2}}} \frac{df}{f} + \int_{\ell_{-\frac{1}{2}}} \frac{df}{f} = \int_{\ell_{\frac{1}{2}}} \frac{df}{f} - \int_{\ell_{\frac{1}{2}}} \frac{df}{f} = 0$$

(iii) Integrating along R_0 as $\epsilon_0 \to 0$, we can define R_0 to be the image of l_R under W_p where $\epsilon_0 \to 0$ as $R \to \infty$. Then if we define $f^0(z) = (\sqrt{p}z)^{-k} f(W_p z)$, we have that

$$\frac{df^0(W_p^{-1}z)}{f^0(W_p^{-1}z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z}$$

So, we have that

$$\frac{1}{2\pi i}\int_{R_0} \frac{df}{f} = \frac{1}{2\pi i}\int_{l_R} \frac{df^0}{f^0} - \frac{1}{2\pi i}\int_{R_0} k\frac{dz}{z}$$

Then as $\epsilon_0 \to 0$, we have that

$$\int_{R_0} k\frac{dz}{z} \to 0$$

Hence we have the following relation:

$$\frac{1}{2\pi i}\int_{R_0} \frac{df}{f} \to \frac{1}{2\pi i}\int_{l_R} \frac{df^0}{f^0} = -v_\infty(f^0) = -v_0(f)$$

(iv) Integrating along $C_{\rho_{p,1}}, C_{\rho_{p,2}}$ as $\epsilon_{\rho_{p,1}}$ and $\epsilon_{\rho_{p,2}}$ go to 0, we have that the angle approaches $\frac{2\pi}{3}$. Thus we have that

$$\frac{1}{2\pi i}\int_{C_{\rho_{p,1}}} \frac{df}{f} \to -\frac{1}{3}v_{\rho_{p,1}}(f)$$

$$\frac{1}{2\pi i}\int_{C_{\rho_{p,2}}} \frac{df}{f} \to -\frac{1}{3}v_{\rho_{p,2}}(f)$$

(v) Integrating along $C_{i_{p,1}}, C_{i_{p,2}}$ as $\epsilon_{i_{p,1}}$ and $\epsilon_{i_{p,2}}$ go to 0, we have that the angle approaches π. Thus we have that

$$\frac{1}{2\pi i}\int_{C_{i_{p,1}}} \frac{df}{f} \to -\frac{1}{2}v_{i_{p,1}}(f)$$

$$\frac{1}{2\pi i}\int_{C_{i_{p,2}}} \frac{df}{f} \to -\frac{1}{2}v_{i_{p,2}}(f)$$

(vi) Integrating along the rest of $\mathcal{A}_{p,f}$ we first consider the arcs $C_1, C_{-1} = \{\frac{e^{i\theta}}{p} \pm \frac{1}{p}\}$. As $\epsilon_{\rho_{p,1}}, \epsilon_{\rho_{p,2}}, \epsilon_{\psi_{p,1}}, \epsilon_{\psi_{p,2}}, \epsilon_0 \to 0$, we have these angles tend to $\frac{2\pi}{3}$. Using the transformation with the matrix $\begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix}$, we have that the sum of integrals for C_1, C_{-1} reduces to

$$\frac{1}{2\pi i}\int_{C=\{e^{i\theta} | \theta \in [0, \frac{2\pi}{3}]\}} k\frac{dz}{z} = \frac{1}{3}k$$

Then we can similarly consider the arcs $C_j, C_{-j} = \{\frac{e^{i\theta}}{p} \pm \frac{j}{p}\}$ who's angles will tend to $\frac{\pi}{3}$. We can similarly find a matrix to transform relate the values of the integrals to each other and find the the integral will tend to $\frac{1}{6}k$.

Now we consider there are $\frac{p-3}{2}$ pairs of arcs C_j, C_{-j}. So in total we have $\frac{p-3}{2}\frac{1}{6}k + \frac{1}{3}k = \frac{p+1}{12}k$. Now note that if there are zeros along the contour, we can create circles around these zeros and take their radii $r \to 0$. Then if the point is of the form $\frac{q}{2p} + \frac{\sqrt{3}}{2p}i$ then the angle with tend towards $\pi/3$. Otherwise the angle tends toward $\pi/2$. Then plugging these values into (5), we achieve (3) and prove Theorem 1. $\square$

3.2 Proof of Theorem 2

Let f be as defined. We know that f will have a finite number of zeros up to equivalence in $\mathcal{F}_{p^2}$, so we can create a contour $\mathcal{C}$ that encloses all such zeros except the elliptic points and cusps.

Now we construct our contour $\mathcal{C}$. We first begin with $\mathcal{A}_{p^2,f}$ and want to exclude the elliptic points and cusps. To remove these elliptic points, we travel along the arcs C_{p_rho}, C_{p_i} inside of $\mathcal{F}_{p^2}$ for $p_\rho \in S_\rho, p_i \in S_i$ and take the limit where the radius $\epsilon_{p_\rho}, \epsilon_{p_i}$ goes to 0. Similarly, we will travel along C_{p_∞} for $p_\infty \in S_\infty$ and take the limit where the radius $\epsilon_{p_\infty} \to 0$. Now, we also add the following lines to $\mathcal{C}$:

$$\ell_{\frac{1}{2}} = \left\{ \frac{1}{2} + it \mid \frac{\sqrt{3}}{p^2} \leq t \leq R \right\}$$

$$\ell_{-\frac{1}{2}} = \left\{ -\frac{1}{2} + it \mid \frac{\sqrt{3}}{p^2} \leq t \leq R \right\}$$

$$\ell_R = \left\{ t + iR \mid -\frac{1}{2} \leq t \leq \frac{1}{2} \right\}$$

Now $\mathcal{C}$ is a closed contour and we will apply the residue theorem.

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{df}{f} = \sum_{p \in \mathcal{F}_{p^2} - S_\rho - S_i - S_\infty} v_p(f) \tag{6}$$

Now we will evaluate the right hand side of (6) by splitting into the different arcs of $\mathcal{C}$ to achieve the expression (4).

(i) Integrating along ℓ_R, we achieve

$$\frac{1}{2\pi i} \int_{\ell_R} \frac{df}{f} = \frac{1}{2\pi i} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f'(u+iR)}{f(u+iR)} du = \frac{1}{2\pi i} \int_{\omega = \{|q| = e^{-2\pi R}\}} \frac{f'(q)}{f(q)} dq = -v_\infty(f)$$

(ii) Integrating along $\ell_{\frac{1}{2}}$ and $\ell_{-\frac{1}{2}}$, we use that $f(Tz) = f(z)$ where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. So, we have that:

$$\int_{\ell_{\frac{1}{2}}} \frac{df}{f} + \int_{\ell_{-\frac{1}{2}}} \frac{df}{f} = \int_{\ell_{\frac{1}{2}}} \frac{df}{f} - \int_{\ell_{\frac{1}{2}}} \frac{df}{f} = 0$$

(iii) Integrating along C_{p_∞} as $\epsilon_{p_\infty} \to 0$, we can define C_{p_∞} to be the image of l_R under an involution W_{p_∞} between cusp at p_∞ and ∞ where $\epsilon_{p_\infty} \to 0$ as $R \to \infty$. Then if we define $f^{p_\infty}(z) = (\sqrt{p}z)^{-k} f(W_p z)$, we have that

$$\frac{df^{p_\infty}(W_{p_\infty}^{-1} z)}{f^{p_\infty}(W_{p_\infty}^{-1} z)} = \frac{df(z)}{f(z)} + k\frac{dz}{z}$$

So, we have that

$$\frac{1}{2\pi i} \int_{C_{p_\infty}} \frac{df}{f} = \frac{1}{2\pi i} \int_{l_R} \frac{df^{p_\infty}}{f^{p_\infty}} - \frac{1}{2\pi i} \int_{C_{p_\infty}} k\frac{dz}{z}$$

Then as $\epsilon_{p_\infty} \to 0$, we have that

$$\int_{C_{p_\infty}} k\frac{dz}{z} \to 0$$

Hence we have the following relation:

$$\frac{1}{2\pi i} \int_{C_{p_\infty}} \frac{df}{f} \to \frac{1}{2\pi i} \int_{l_R} \frac{df^{p_\infty}}{f^{p_\infty}} = -v_\infty(f^{p_\infty}) = -v_{p_\infty}(f)$$

(iv) Integrating along C_{p_ρ} as $\epsilon_{p_\rho} \to 0$, we have that the angle approaches $\frac{2\pi}{3}$. Thus we have that

$$\frac{1}{2\pi i}\int_{C_{p_\rho}} \frac{df}{f} \to -\frac{1}{3}v_{p_\rho}(f)$$

(v) Integrating along C_{p_i} as $\epsilon_{p_i} \to 0$, we have that the angle approaches π. Thus we have that

$$\frac{1}{2\pi i}\int_{C_{p_i}} \frac{df}{f} \to -\frac{1}{2}v_{p_i}(f)$$

(vi) Integrating along the rest of $\mathcal{A}_{p^2,f}$ we first want to consider the arcs surrounding a cusp p_∞. Then as the radii $\epsilon_{p_\rho}, \epsilon_{p_i}, \epsilon_{p_\infty} \to 0$ for all elliptic points, we see that these arcs tend to $\frac{2\pi}{3}$. Then for all of the other arcs, we have the angles tend to $\frac{pi}{3}$. Similar to the $\Gamma_0(p)$ case, there is an equivalence relation between the arcs. Then for each cusp of $\Gamma_0(p^2)$ not at ∞, we will have a factor of $\frac{1}{3}k$. Counting the arcs of $\mathcal{A}_{p^2,f}$, we see that sum of these integrals will be

$$\frac{k}{12}\left(p^2 + \epsilon_\infty\left(\Gamma_0\left(p^2\right)\right) - 1\right)$$

Now note that if there are zeros along the contour, we can create circles around these zeros and take their radii $r \to 0$. Then if the point is of the form $\frac{q}{2p} + \frac{\sqrt{3}}{2p}i$ then the angle with tend towards $\pi/3$. Otherwise the angle tends toward $\pi/2$. Plugging all of these values into the residue theorem, we have achieved (4) and proved Theorem 2. □

4 Finding Zeros of Eisenstein Series

Given observation 2, we aim to locate the zeros of $E_{k,p}^{\infty}$. First we introduce results from [1] and [2] for $SL_2(\mathbb{Z}), \Gamma_0(2), \Gamma_0(3), \Gamma_0(5)$. Then we inspect the areas of $\mathcal{F}_p$ where we would expect the zeros to lie on.

4.1 The Rankin and Swinnerton-Dyer Method

For $SL_2(\mathbb{Z})$, Rankin and Swinnerton-Dyer proved a surprising and important result about the zeros of an Eisenstein series. They defined the function $F_k(\theta)$ and inspected its zeros, which correspond to the zeros of $E_k(z)$ on $|z| = 1$.

$$F_k(\theta) = e^{\frac{ik\theta}{2}} E_k(e^{i\theta}) = \sum_{(c,d)=1} \left(ce^{\frac{i\theta}{2}} + de^{-\frac{i\theta}{2}}\right)^{-k}$$

This function is real on the arc $|z| = 1$ and by considering $\theta \in (\pi/2, 2\pi/3)$, they approximated this function as

$$F_k(\theta) = 2\cos(k\theta/2) + R$$

where $|R| < 2$. Hence, $F_k(\theta)$ is positive or negative when $\theta = \frac{2m\pi}{k}$ for $m \in \mathbb{Z}$ depending on the sign of m. Using the intermediate value theorem for $F_k(\theta)$, we have that there must be a zero in between these values. Hence, we have found $\left\lfloor \frac{k}{3} \right\rfloor - \left\lceil \frac{k}{4} \right\rceil$ zeros on the arc $|z| = 1$ for $\theta \in (\pi/2, 2\pi/3)$.

Then there is an elliptic point at i and ρ that we must consider. We note the multiplicity at i is $c_i = \begin{cases} 0 & \text{if } k \equiv 0 \bmod 4 \\ 1 & \text{if } k \equiv 2 \bmod 4 \end{cases}$. The multiplicity at ρ is $c_\rho = \begin{cases} 0 & \text{if } k \equiv 0 \bmod 6 \\ 2 & \text{if } k \equiv 2 \bmod 6 \\ 1 & \text{if } k \equiv 4 \bmod 6 \end{cases}$.

The valence formula for $SL_2(\mathbb{Z})$ is as follows:

$$v_\infty(E_k) + \frac{1}{2}c_\rho + \frac{1}{3}c_i + \sum_{p \neq i,\rho} v_p(E_k) = \frac{k}{12}$$

Thus we consider the following cases:

- $k \equiv 0 \bmod 12$: We have $\frac{k}{12}$ zeros on the arc, multiplicity 0 for i and multiplicity 0 for ρ. We see that we have all of the expected zeros.
- $k \equiv 2 \bmod 12$: We have $\frac{k-14}{12}$ zeros on the arc, multiplicity 1 for i and multiplicity 2 for ρ. Summing this is $\frac{k}{12}$ zeros, and all of the expected zeros.
- $k \equiv 4 \bmod 12$: We have $\frac{k-4}{12}$ zeros on the arc, multiplicity 0 for i and multiplicity 1 for ρ. Summing this is $\frac{k}{12}$ zeros, and all of the expected zeros.
- $k \equiv 6 \bmod 12$: We have $\frac{k-6}{12}$ zeros on the arc, multiplicity 1 for i and multiplicity 0 for ρ. Summing this is $\frac{k}{12}$ zeros, and all of the expected zeros.
- $k \equiv 8 \bmod 12$: We have $\frac{k-8}{12}$ zeros on the arc, multiplicity 0 for i and multiplicity 2 for ρ. Summing this is $\frac{k}{12}$ zeros, and all of the expected zeros.
- $k \equiv 10 \bmod 12$: We have $\frac{k-10}{12}$ zeros on the arc, multiplicity 1 for i and multiplicity 1 for ρ. Summing this is $\frac{k}{12}$ zeros, and all of the expected zeros.

The following result follows.

Theorem 3. *Let E_k be a Eisenstein series of level 1. All of the zeros of E_k that lie in the fundamental domain are on the arc $|z| = 1$.*

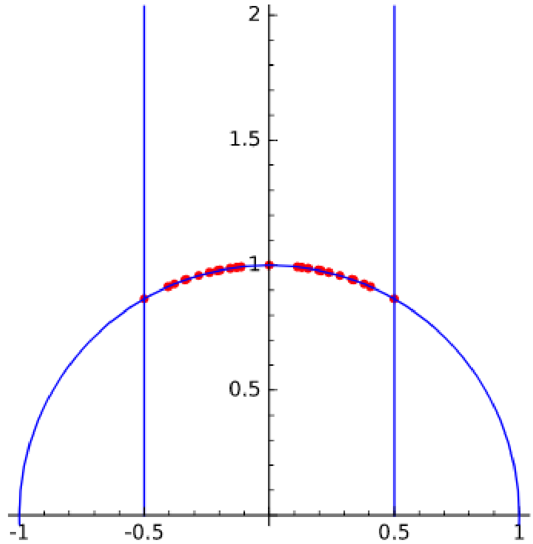

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Figure 5: Zeros of the Eisenstein series of level 1 are on the arc $|z| = 1$

This result can be seen visually from Figure 5.

Shigezumi extended this result to $\Gamma_0(2)$, $\Gamma_0(3)$, $\Gamma_0(5)$ by considering the following functions:

$$F_{k,2}(\theta) = e^{ik\theta/2} E_{k,2}^{\infty}\left(\frac{e^{i\theta}}{2} - \frac{1}{2}\right)$$

$$F_{k,3}(\theta) = e^{ik\theta/2} E_{k,3}^{\infty}\left(\frac{e^{i\theta}}{3} - \frac{1}{3}\right)$$

$$F_{k,5}(\theta) = e^{ik\theta/2} E_{k,5}^{\infty}\left(\frac{e^{i\theta}}{5} - \frac{1}{5}\right)$$

By applying the same method that Rankin Swinnerton-Dyer applied, Shigezumi achieved the following result:

Theorem 4. *For $40 \geq k \geq 4$, $E_{k,2}^{\infty}$ has at least $\left\lfloor \frac{k}{4} \right\rfloor - 1$ zeros on the arc $\left\{\frac{1}{2}(e^{i\theta} - 1) \mid \theta \in (0, \pi/2)\right\}$.*

For $40 \geq k \geq 4$, $E_{k,3}^{\infty}$ has at least $\left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on the arc $\left\{\frac{1}{3}(e^{i\theta} - 1) \mid \theta \in (0, 2\pi/3)\right\}$.

For $40 \geq k \geq 4$, $E_{k,5}^{\infty}$ has at least $\left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on the arc $\left\{\frac{1}{5}(e^{i\theta} - 1) \mid \theta \in (0, 2\pi/3)\right\}$.

We find that this result generalizes for larger k and higher p.

Theorem 5. *For $88 \geq k \geq 4$ and $p \geq 3$, $E_{k,p}^{\infty}$ has at least $\left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on the arc $\left\{\frac{1}{p}(e^{i\theta} - 1)\right\}$ for $\theta \in \left(\frac{\pi}{44}, \frac{2\pi}{3}\right)$.*

Proof. We want to apply the RSD method to the arc $\left\{\frac{e^{i\theta}}{p} - \frac{1}{p} \mid \theta \in (0, \frac{2\pi}{3})\right\}$. Let us define the following function:

$$F_{k,p}(\theta) = e^{ik\theta/2} E_{k,p}\left(\frac{e^{i\theta}}{p} - \frac{1}{p}\right) = \sum_{\substack{(c,d)=1 \\ p \nmid d}} \left(c e^{i\theta/2} + (-c+d)\, e^{-i\theta/2}\right)^{-k}$$

Then we have $v_k(c,d,\theta) = |ce^{i\theta/2} + (-c+d)e^{-i\theta/2}|^{-k} = |c^2 + (-c+d)^2 + 2c(-c+d)\cos(\theta)|^{-k/2}$. We note that we can't have $\theta = 0$ because then we will have $v_k(c,d,0) = |c^2 + (-c+d) + 2c(-c+d)\cos(\theta)|^{-k/2}$ which can be 1. Because we want to apply RSD method the way it was applied to the full arcs, this will cause a bound issue. So, instead we will take $\theta \in (\alpha_k, \frac{2\pi}{3})$ where we want $\alpha \leq \frac{2\pi}{k}$ so that we achieve the desired result. We will computationally find that we can only choose such α_k when $k \leq 88$.

We first apply the RSD method and split the sum:

$$F_k(\theta) = 2\cos(k\theta/2) + R$$

Then if we can find $|R| < 2$, we have that there are $\left\lfloor \frac{k}{3} \right\rfloor - \left\lceil \frac{\alpha_k}{2\pi} k \right\rceil$ zeros on the arc.

We can split R by summing over $N = c^2 + (-c+d)^2$. Because this sum will only increase by including all terms and ignoring the $p \not| d$ term, we know that if $|R| < 2$ for some k for all terms, it will for all p. So, for example, we consider the first few terms:

N	Terms
2	$2(2)^{-k/2}$
5	$4(5)^{-k/2} + 4(5 - 4\cos(\alpha_k))^{-k/2}$
10	$4(10)^{-k/2} + 4(10 - 6\cos(\alpha_k))^{-k/2}$
17	$4(17)^{-k/2} + 4(17 - 8\cos(\alpha_k))^{-k/2}$
25	$4(25)^{-k/2} + 4(25 - 24\cos(\alpha_k))^{-k/2}$
31	$4(31)^{-k/2} + 4(31 - 10\cos(\alpha_k))^{-k/2}$
...	...

Then since $\alpha_k \neq 0$, there exists $m \in \mathbb{Z}$ where $m \geq (1 - \cos(\alpha_k))^{-k}$. Then for some $N > M > m$ for M large enough, we can bound all but a finite number of the terms by the sum:

$$S = \sum_{N \geq M} 2\sqrt{N} \left(\frac{N}{m} \right)^{-k/2}$$

We can choose this M so S is much less than 2. We know that such a M exists because by a comparison test, with $k > 3$, which is true for all k that we consider, the series $\sum_{N=1}^{\infty} 2\sqrt{N}(\frac{N}{m})^{-k/2}$ will converges to S. Hence, there exists M such that for all $N \geq M$, $\sum_{N=M} 2\sqrt{N}(\frac{N}{m})^{-k/2} < \epsilon$ for some chosen ϵ.

Then we note that since $\cos(\alpha_k) < 1$, there is some k such that $|R|$ (which is the sum of all of the discrete terms and S) will be less than 2. Formally, if $R(k)$ is the value of the sum for some k,

$$\lim_{k \to \infty} R(k) = 0$$

Hence, there exists some k_{min} such that for all $k \geq k_{min}$, $|R(k)| < 2$.

To acheive the desired result, we want $\left\lceil \frac{\alpha_k}{2\pi} k \right\rceil = 1$. Thus, given different α_k, we want to find when $\frac{1}{k} = \frac{\alpha_k}{2\pi}$. Here are some computed α_k's and k's.

α_k	k	α_k	k	α_k	k	α_k	k
$\pi/6$	12	$\pi/24$	28	$\pi/31$	46	$\pi/38$	66
$\pi/8$	14	$\pi/25$	30	$\pi/32$	48	$\pi/39$	70
$\pi/12$	16	$\pi/26$	32	$\pi/33$	50	$\pi/40$	74
$\pi/15$	18	$\pi/27$	36	$\pi/34$	54	$\pi/41$	78
$\pi/19$	22	$\pi/28$	38	$\pi/35$	56	$\pi/42$	80
$\pi/20$	24	$\pi/29$	40	$\pi/36$	60	$\pi/43$	84
$\pi/22$	26	$\pi/30$	42	$\pi/37$	64	$\pi/44$	88

Unfortunately, for $k > 88$ we have that $\alpha_k > \frac{2\pi}{k}$, so we will not have that $\left\lceil \frac{\alpha_k}{2\pi} k \right\rceil = 1$. Thus for $k \leq 88$, we have that $\left\lfloor \frac{k}{3} \right\rfloor - \left\lceil \frac{k}{88} \right\rceil = \left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on this arc. $\square$

Now we want to show that this is true for $k > 88$. First we will set up some lemmas.

Lemma 5.

$$\lim_{k\to\infty} k^{1/k} = 1$$

Lemma 6.

$$\frac{d}{dx}x^{2/3x} = \frac{2}{3}\left(\frac{1-\ln(x)}{x^2}\right)x^{2/3x}$$

These lemmas can both be shown by direct computation.

Lemma 7. $\frac{k^2}{\pi^2 m^2}\left(\left(Cmk^{1/3}\right)^{2/k} - 1\right)$ *grows with rate* $O\left(\frac{k\ln(k)}{m}\right)$.

Proof. We will show this by taking the limit of the ratio and applying L'Hospital's Rule, Lemma 5 and Lemma 6.

$$\begin{aligned}
\lim_{k\to\infty} \frac{\frac{k^2}{\pi^2 m^2}\left(\left(Cmk^{1/3}\right)^{2/k} - 1\right)}{\frac{k\ln(x)}{m^2}} &= \frac{1}{\pi^2}\lim_{k\to\infty}\frac{\left(Cmk^{1/3}\right)^{2/k} - 1}{\frac{\ln(k)}{k}} \\
&= \frac{1}{\pi^2}\lim_{k\to\infty}\frac{\left(\ln(Cm)\,(Cm)^{2/k}\left(\frac{-2}{k^2}\right) + (Cm)^{2/k}\,\frac{2}{3}\,\frac{1-\ln(k)}{k^2}\right)k^{2/3k}}{\frac{1-\ln(k)}{k^2}} \\
&= \frac{1}{\pi^2}\lim_{k\to\infty}(Cm)^{2/k}\,k^{2/3k}\left(\frac{2}{3} + \frac{2\ln(Cm)}{\ln(k)+1}\right) \\
&= \frac{2}{3\pi^2}
\end{aligned}$$

Since this is a constant, we know that it grows with the rate $O\left(\frac{k\ln(k)}{m}\right)$ with the constant in the limit $\frac{2}{3\pi^2}$. ∎

Theorem 6. *For* $k > 88$*, there are* $\left\lfloor\frac{k}{3}\right\rfloor - 1$ *zeros on the arc centered at* $\frac{-1}{p}$.

Proof. We again define and analyze the following function:

$$F_{k,p}^{-1}(\theta) = e^{ik\theta/2}E_{k,p}\left(\frac{e^{i\theta}}{p} - \frac{1}{p}\right) = \sum_{\substack{(c,d)=1\\ p\nmid d}}\left(ce^{i\theta/2} + (-c+d)\,e^{-i\theta/2}\right)^{-k}$$

We choose $\alpha_k < \frac{2\pi}{k}$ and consider $\theta \in (\alpha_k, \frac{\pi}{44})$. We recall that using the RSD method before we have that $F_{k,p}^{-1}(\theta) = 2\cos(k\theta/2) + R$. However, for $k > 88$, $\alpha_k < \frac{2\pi}{k}$, $|R|$ was not necessarily less than 2 because of the terms of the form $c = \pm n$, $-c+d = \pm(n+1)$ or $c = \pm(n+1)$, $-c+d = \pm n$. However, if we choose M such that M is trivially small (for instance let $M = 3/2$, then $M^{-k/2} < 10^{-8}$), there exists N such that for all $n \geq N$, $n^2 + (n+1)^2 - 2n(n+1)\cos(\alpha_k) > M$. Note that when $M = 3/2$, we have that $N < \frac{k}{8}$. Now we will split the sum as follows:

$$F_{k,p}^{-1}(\theta) = 2\cos(k\theta/2) + D + S$$

where

$$\begin{aligned}
D = \frac{1}{2}\sum_{n=1}^{N}\left(ne^{i\theta/2} + (n+1)\,e^{-i\theta/2}\right)^{-k} + \left(ne^{i\theta/2} - (n+1)\,e^{-i\theta/2}\right)^{-k} \\
+ \left((n+1)\,e^{i\theta/2} + ne^{i\theta/2}\right)^{-k} + \left((n+1)\,e^{i\theta/2} - ne^{i\theta/2}\right)^{-k}
\end{aligned}$$

and S is the sum of all of the remaining terms. Using the methods from Theorem 5, we can show that $|S| < 2$. Now we want to show that at $\theta = \frac{2m\pi}{k}$ for $m \in \mathbb{Z}$,

$$\operatorname{sgn}(\cos(m\pi)) = \operatorname{sgn}(D)$$

Note that

$$\left|\left(ne^{i\theta/2} + (n+1)\,e^{-i\theta/2}\right)^{-k} + \left((n+1)\,e^{i\theta/2} + ne^{-i\theta/2}\right)^{-k}\right| \leq 2\left|(2n+1)^2\cos^2(\theta/2) + \sin^2(\theta/2)\right|^{-k/2}$$

Recall $\theta \in (\alpha_k, \pi/44)$, so that this is less than $2|2n|^{-k/2}$.

However, $(ne^{i\theta/2} - (n+1)e^{-i\theta/2})^{-k} + ((n+1)e^{i\theta/2} - ne^[-i\theta/2])^{-k}$ can not be bounded in the same way. Instead let us express it in different terms:

$$r = \left(\cos^2(\theta/2) + (2n+1)^2\sin^2(\theta/2)\right)^{1/2}$$

$$\varphi = \sin^{-1}\left(\frac{2n+1}{r}sin(\theta/2)\right)$$

Then

$$\left((n+1)\,e^{i\theta/2} - ne^{i\theta/2}\right)^{-k} + \left(ne^{i\theta/2} - (n+1)e^{-i\theta/2}\right)^{-k} = r^{-2k}(r^k e^{ik\varphi} + r^k e^{-ik\varphi}) = 2r^{-k}\cos(k\varphi)$$

We will apply the small angle approximation here where if $\theta = \frac{2\pi m}{k}$ where $m \in \mathbb{Z}$ and $\theta < \frac{\pi}{4}$, and approximate $\sin(\theta/2) \approx \frac{\pi m}{k}$ with an error less than $\frac{1}{6}(\frac{\pi}{44})^3 < 0.00007$. Then we consider $\frac{2n+1}{r}\frac{m\pi}{k} < \frac{m(2n+1)\pi}{k}$.

In order for $\operatorname{sgn}(\cos(k\varphi)) = \operatorname{sgn}(\cos(m\pi))$, we want that

$$\left|\sin\left(\frac{m(2n+1)\pi}{k}\right) - \frac{m(2n+1)\pi}{k}\right| < \frac{1}{k} < \frac{\pi}{k}$$

We will take $\frac{1}{k}$ to make the inequality easier, although $\frac{\pi}{k}$ suffices to show negativity. This will allow for $\cos(k\varphi)$ to have a larger magnitude.

Then, we evaluate and find that

$$\left|\sin\left(\frac{m(2n+1)\pi}{k}\right) - \frac{m(2n+1)\pi}{k}\right| < \frac{1}{6}(\frac{m(2n+1)\pi}{k})^3 < \frac{1}{k}$$

This gives us that for $n < \frac{k^{2/3}}{2(6)^{1/3}m} - \frac{1}{2}$, $\operatorname{sgn}(\cos(k\varphi)) = \operatorname{sgn}(\cos(m\pi))$. To simplify the expression for further manipulation, we take $c_n = \frac{1}{2(6)^{1/3}}$ and say $n \leq \left\lfloor \frac{ck^{2/3}}{m} \right\rfloor$.

Then for

$$\sum_{n=1}^{\left\lfloor \frac{ck^{2/3}}{m} \right\rfloor} \left(ne^{i\theta/2} - (n+1)\,e^{-i\theta/2}\right)^{-k} + \left((n+1)\,e^{i\theta/2} - ne^{-i\theta/2}\right)^{-k} = \operatorname{sgn}(\cos(m\pi))\, c_D' c_n \frac{k^{2/3}}{m}$$

where c_D' is a constant that comes from the fact that $|\cos(k\varphi)| < 1$. Since we choose $\frac{1}{k}$ instead of $\frac{\pi}{k}$, we expect that $c_D' > \frac{1}{4}$. Now, let $c_D = c_D' c_n$.

Now we want that the magnitude of the sum of the other terms in D to be less than $c_D \frac{k^{2/3}}{m}$. The number of other terms would be $\frac{k}{8} - c_n \frac{k^{2/3}}{m}$. So let

$$N' = \left|\frac{\frac{k}{8} - c_n \frac{k^{2/3}}{m}}{c_D \frac{k^{2/3}}{m}}\right| > C' m k^{1/3}$$

where $C' = \frac{1}{8c_D}$. Then we want to find for which n,

$$2\left(\cos^2(\theta/2) + (2n+1)^2\sin^2(\theta/2)\right)^{-k/2} < \frac{1}{N'}$$

Rearranging we have

$$\left(1 + (4n+4n^2)\sin^2(\theta/2)\right)^{k/2} > N$$

where $N = 2N' = Cmk^{1/3}$ where $C = 2C'$. Thus we have that

$$n > \frac{-1}{4} + \frac{\sqrt{1 + \frac{k^2}{\pi^2 m^2}(N^{2/k} - 1)}}{4}$$

Now from Lemma 7 to show that the minimum n such that this is true, denoted n_{min}, grows at the rate of $\frac{\sqrt{k\log(k)}}{m}$ and $n_{min} < \frac{\sqrt{k\ln(k)}}{2m}$ since the constant in the front from the limit is $\frac{\sqrt{\frac{2}{3\pi^2}}}{4} < 1$.

Then for $k > 88$ we know that

$$\frac{k\ln(k)}{2m} < \frac{k^{2/3}}{2(6)^{1/3}m}$$

. Thus the magnitude of the sum of the terms that do not satisfy the property $\operatorname{sgn}(\cos(k\varphi)) = \operatorname{sgn}(\cos(m\pi))$ is less than the magnitude of the sum created by the terms that satisfy the property. Thus, we have that $\operatorname{sgn}(D) = \operatorname{sgn}(\cos(m\pi))$.

Finally, we have that

$$F_{k,p}^{-1}(\theta) = 2\cos(k\theta/2) + D + S$$

will be positive or negative depending on the parity of m, indicating a zero in between each of these values. Thus looking at the arc $\theta \in (\alpha_k, \pi/44)$, we have found the rest of the zeros not on $\theta \in (\pi/44, 2\pi/3)$. So, there are $\left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on the arc centered at $\frac{-1}{p}$ for all $k \geq 4$. $\square$

Now that we know there are $\left\lfloor \frac{k}{3} \right\rfloor - 1$ zeros on the arc centered at $\frac{-1}{p}$, then we have found approximately $\frac{4}{p+1}$ of the expected zeros from the valence formula.

4.2 Zeros at Elliptic Points

We can show that there are zeros at elliptic points for particular k.

Lemma 8. *For $k \not\equiv 0 \bmod 4$, there are zeros of $E_{k,p}$ at the i-elliptic points.*

Proof. Let $p_i = \frac{q}{p} + \frac{i}{p}$ be an elliptic point of $\Gamma_0(p)$. Then $\gamma p_i = p_i$ when $\gamma = \begin{pmatrix} q & p^{-1}(-1-q^2) \\ p & -q \end{pmatrix} \in \Gamma_0(p)$ $\square$

So, we know that $E_{k,p}(p_i) = E_{k,p}(\gamma p_i) = (i)^k E_{k,p}(p_i)$. Thus, if $k \not\equiv 0 \bmod 4$, this forces $E_{k,p}(p_i) = 0$.

Lemma 9. *For $k \not\equiv 0 \bmod 6$, there are zeros of $E_{k,p}$ at the ρ-elliptic points.*

Proof. Let $p_\rho = \frac{q}{2p} + \frac{\sqrt{3}i}{2p}$ be an elliptic point of $\Gamma_0(p)$. Then $\gamma p_\rho = p_\rho$ when $\gamma = \begin{pmatrix} \frac{q+1}{2} & p^{-1}(-1-\frac{q^2-1}{4}) \\ p & -\frac{q-1}{2} \end{pmatrix} \in \Gamma_0(p)$. Then we know that $E_{k,p}(p_\rho) = E_{k,p}(\gamma p_\rho) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^k E_{k,p}(p_\rho)$. Thus, if $k \not\equiv 0 \bmod 6$, this forces a zero at p_ρ. $\square$

4.3 Inspecting the Boundary of $\mathcal{F}_p$

Given our observations of the zeros, we graph the value of $E_{k,p}(z)$ along the arcs $\left\{\frac{e^{i\theta}}{p} + \frac{q}{p} \mid \theta \in [\pi/3, 2\pi/3]\right\}$ where red corresponds to the point $\theta = \pi/3$ and $\theta = 2\pi/3$. Here we have plotted $F^q_{k,p}(\theta) = e^{ik\theta/2}E_{k,p}(\frac{e^{i\theta}}{p} + \frac{q}{p})$. Note that there is bijection between the zeros of $F_{k,p}$ and $E_{k,p}$.

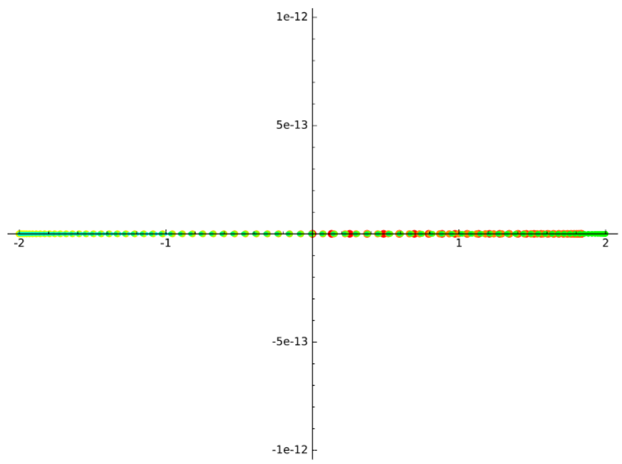


Figure 6: $F^1_{22,13}(\theta)$

Note that in Figure 6 we have that all of the points are real as expected. This allows us to run RSD on the arc and achieve the Theorem 5.

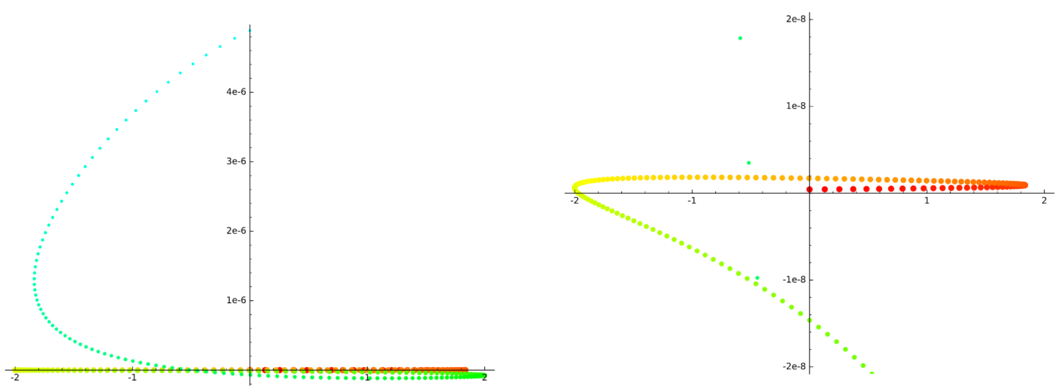


Figure 7: $F^2_{22,13}(\theta)$, at various zooms

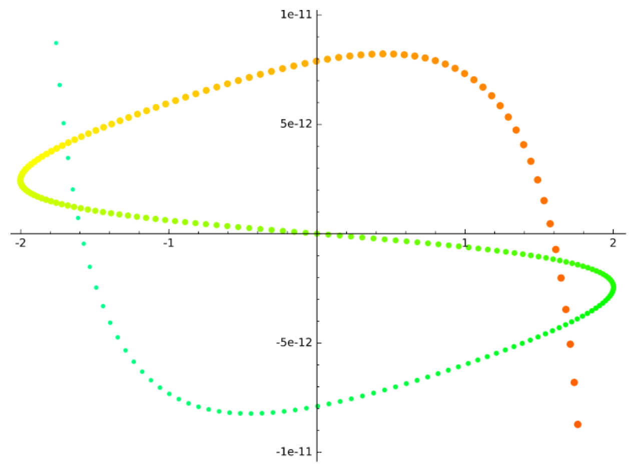


Figure 8: $F^5_{22,13}(\theta)$

We want to inspect Figure 7 more carefully to see if there are zeros on the second arc. However looking at the zoomed in version, we see that there are no zeros on this arc. Similarly, when looking closer at the other arcs (ex: 8), we see that the only zeros that occur on the arcs are at the elliptic points or on the first arc. More images are in the appendix. After trying this for many different values of k, p, this leads us to the observation:

Observation 1. *The only zeros on the arcs of $\mathcal{F}_p$ are the elliptic points and the arcs centered at $\pm\frac{1}{p}$.*

4.4 Bounds on Zeros

We have shown that there are zeros of $E^{\infty}_{k,p}(z)$ on the arcs centered at $\pm\frac{1}{p}$, as well as at the various elliptic points, if they exist. But aside from this, we can still bound the imaginary component of any zero. To do this, consider $E(z) = (p^k - 1)E^{\infty}_{k,p}(z)$.

Then, we have

$$E(z) = p^k - 1 - \lambda_k \sum_{n=1}^{\infty} a_n q^n$$

where

$$a_n = \begin{cases} \sigma_{k-1}(n) & p \nmid n \\ \sigma_{k-1}(n) - p^k \sigma_{k-1}(n/p) & p \mid n \end{cases}$$

and $\lambda_k = \frac{(2\pi i)^k}{(k-1)!\zeta(k)}$. So, note that we can bound

$$|a_n| < p\sigma_{k-1}(n) < pn^{k-1}\zeta(k-1)$$

Now, assume that $z = x + iy$ is a zero of $E_{k,p}^{\infty}$. Then we have

$$\begin{aligned}
0 = |E_{k,p}^{\infty}(z)| &= |E(z)| \\
&= |p^k - 1 - \lambda_k \sum_{n=1}^{\infty} a_n q^n| \\
&\geq p^k - 1 - |\lambda_k \sum_{n=1}^{\infty} a_n q^n| \\
&\geq p^k - 1 - |\lambda_k| \sum_{n=1}^{\infty} |a_n q^n| \\
&\geq p^k - 1 - \frac{(2\pi)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} p\zeta(k-1) n^{k-1} e^{-2\pi yn} \\
p^k - 1 &\leq \frac{(2\pi)^k p}{(k-1)!} \frac{\zeta(k-1)}{\zeta(k)} \sum_{n=1}^{\infty} n^{k-1} e^{-2\pi yn}
\end{aligned}$$

Here, the first few lines are just applying the lower and upper bounds of the triangle inequality. Now, recall that

$$\sum_{n=1}^{\infty} n^{k-1} e^{-2\pi yn} \approx \int_0^{\infty} t^{k-1} e^{-2\pi yt} dt = (2\pi y)^{-k} \Gamma(k) = (k-1)!$$

So, for some $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} n^{k-1} e^{-2\pi yn} < (1+\epsilon)(2\pi y)^{-k}(k-1)!$$

Then

$$\begin{aligned}
p^k - 1 &\leq \frac{(2\pi)^k p}{(k-1)!} \frac{\zeta(k-1)}{\zeta(k)} (1+\epsilon)(2\pi y)^{-k}(k-1)! \\
&= \frac{(1+\epsilon)p}{y^k} \frac{\zeta(k-1)}{\zeta(k)}
\end{aligned}$$

So, this implies that

$$y^k \leq \frac{(1+\epsilon)p}{p^k - 1} \frac{\zeta(k-1)}{\zeta(k)}$$

Now, since $1 < \zeta(k), \zeta(k-1) < 2$, the ratio $\zeta(k-1)/\zeta(k) < 2$. So, for some small $1 < c < 2$, we have

$$y \leq \frac{1}{p}(cp)^{1/k}$$

So, for $x + iy$ to be a zero of E, and therefore of $E_{k,p}^{\infty}$, we must have y less than the above bound. This means that all of the zeros are concentrated close to, if not below, the line $y = \frac{1}{p}$. Combined with the bump fundamental domain we use for this function, the zeros must all lie either on the arcs of the domain, or between any adjacent pair.

4.5 Converging to Ellipses

We want to find zeros of $E_{k,p}$ near the arcs of the bump fundamental domain since the bound in section 4.4 shows that the zeros must be near the arcs. We consider the following function where $r \geq 1$:

$$F_{k,p,r}^{q}(\theta) = r^{k/2} e^{ik\theta/2} E_{k,p}\left(\frac{re^{i\theta}}{p} + \frac{q}{p}\right)$$

We note that if $E_{k,p}$ has a zero on the arc $\left\{\frac{re^{i\theta}}{p}+\frac{q}{p}\right\}$ then so will $F^q_{k,p,r}$. Now we expand the function to see that

$$F^q_{k,p,r}(\theta)=\frac{1}{2}\sum_{\substack{(c,d)=1\\ p\nmid d}}\left(cr^{1/2}e^{i\theta/2}+(qc+d)r^{-1/2}e^{-i\theta/2}\right)$$

Now consider $(c,d)=(0,\pm1),(1,-q),(-1,q)$. Then we have the terms

$$2r^{k/2}e^{ik\theta/2}+2r^{-k/2}e^{-ik\theta/2}=2\left(r^{k/2}+r^{-k/2}\right)\cos\left(k\theta/2\right)+2i\left(r^{k/2}-r^{-k/2}\right)\sin\left(k\theta/2\right)$$

So, we will write our expression in the form:

$$F_{k,p,r}(\theta)=(r^{k/2}+r^{-k/2})\cos(k\theta/2)+R_{real}+i(r^{k/2}-r^{k/2})\sin(k\theta/2)+R_{imag}$$

Note that $(r^{k/2}+r^{-k/2})\cos(k\theta/2)+i(r^{k/2}-r^{-k/2})\sin(k\theta/2)$ describes an ellipse that never attains the value 0 if we take $r>1$. This will be the ellipse that $F_{k,p,r}(\theta)$ converges to.

Then we note that for both R_{real} and R_{imag} we have

$$|R|<(2+2\cos(\theta))^{-k/2}+(2-2\cos(\theta))^{-k/2}+\sum_{N=5}^{\infty}2\sqrt{N}\left(\frac{N}{2}\right)^{-k/2}=V$$

As a result, we want to compare the magnitude of V to $r^{k/2}-r^{-k/2}$ to determine when $F^q_{k,p,r}(\theta)$ values converge to this ellipse.

Thus we have want

$$(2+2\cos\left(\theta\right))^{-k/2}+(2-2\cos\left(\theta\right))^{-k/2}+2^{k/2+1}\left(\zeta\left(\frac{k-1}{2}\right)-4^{1/2-k/2}-3^{1/2-k/2}-2^{1/2-k/2}-1\right)>\epsilon\left(r^{k/2}-r^{-k/2}\right)$$

If we let $x=\frac{1}{\epsilon}\left((2+2\cos\left(\theta\right))^{k/2}+(2-2\cos\left(\theta\right))^{-k/2}+2^{k/2+1}\left(\zeta\left(\frac{k-1}{2}\right)-4^{1/2-k/2}-3^{1/2-k/2}-2^{1/2-k/2}-1\right)\right)$, then we have that

$$r^{k/2}>\frac{x+\sqrt{x^2-4}}{2}$$

and if $x^2-4<0$, this inequality holds true for all $r\geq 1$.

So now we want to find when $x^2-4<0$ for which α and ϵ pairs where we take $\theta\in(\pi/2-\alpha,\pi/2+\alpha)$ because then $F^q_{k,p,r}(\theta)$ will have no zeros for $\theta\in(\pi/2-\alpha,\pi/2+\alpha)$ if we take ϵ small enough (ex: $\epsilon=0.05$).

For $x^2-4<0$, we can bound x to be $2^{-k/2}+(2-2\sin\left(\alpha\right))^{-k/2}+2^{k/2+1}\left(\zeta\left(\frac{k-1}{2}\right)-4^{1/2-k/2}-3^{1/2-k/2}-2^{1/2-k/2}-1\right)$. Then we can solve and get

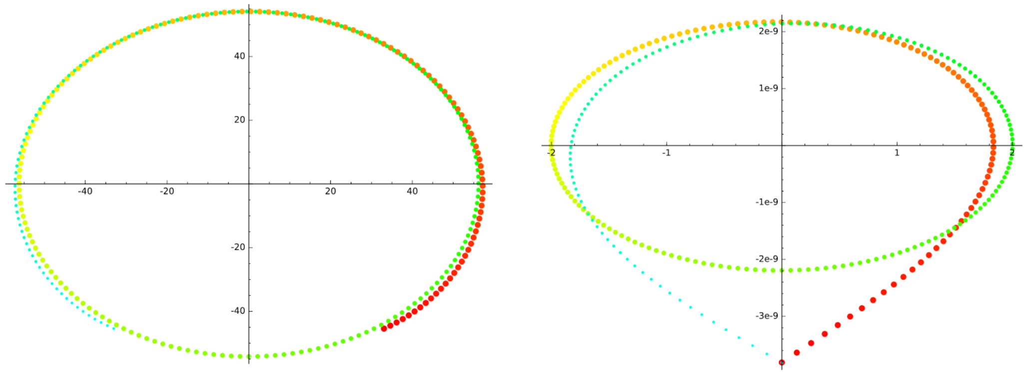
$$\alpha<\sin^{-1}\left(1-\frac{\left(2\epsilon-2^{-k/2}-2^{k/2+1}\left(\zeta\left(\frac{k-1}{2}\right)-4^{1/2-k/2}-3^{1/2-k/2}-2^{1/2-k/2}-1\right)\right)^{-2/k}}{2}\right)$$

This proves the following result:

Theorem 7. *For $\theta\in(\pi/2-\alpha,\pi/2+\alpha)$, $r>1$, $F^q_{k,p,r}(\theta)$ will be ϵ away from an elliptic shape and have no zeros, where*

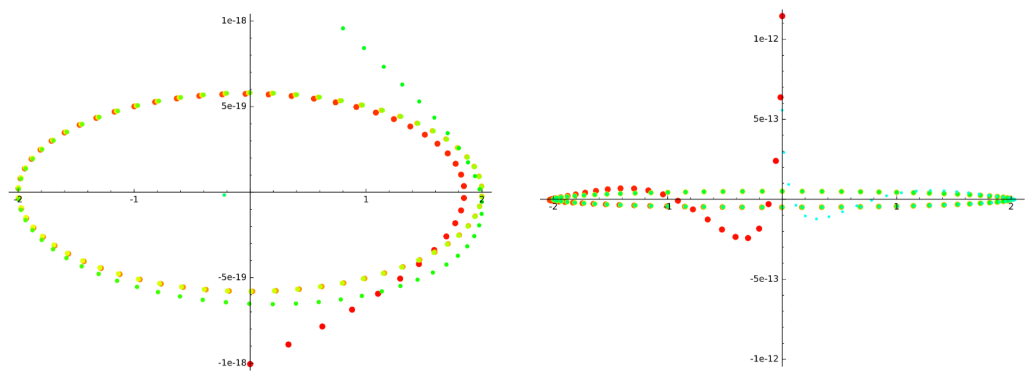
$$\alpha=\sin^{-1}\left(1-\frac{\left(2\epsilon-2^{-k/2}-2^{k/2+1}\left(\zeta\left(\frac{k-1}{2}\right)-4^{1/2-k/2}-3^{1/2-k/2}-2^{1/2-k/2}-1\right)\right)^{-2/k}}{2}\right)$$

Note that the limit as $k\to\infty$, we have $\alpha\to\sin^{-1}(1-\frac{1}{2})=\pi/6$. Thus this bound pushes the places where the zeros of $E_{k,p}$ can be to a small area near the wedges.

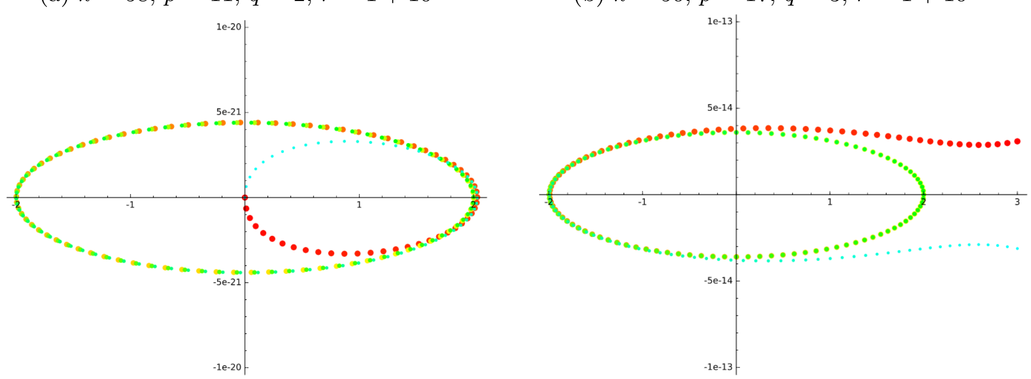


(a) $r = 1.2$ (b) $r = 1 + 10^{-10}$

Figure 9: Graphs of weight 22 level 7 $F^q_{k,p,r}(\theta)$ on arc 3 for different radii for $\theta \in (\pi/3, 2\pi/3)$



(a) $k = 58$, $p = 11$, $q = 2$, $r = 1 + 10^{-20}$ (b) $k = 50$, $p = 17$, $q = 3$, $r = 1 + 10^{-15}$



(c) $k = 44$, $p = 23$, $q = 4$, $r = 1 + 10^{-22}$ (d) $k = 36$, $p = 13$, $q = 4$, $r = 1 + 10^{-15}$

Figure 10: Graphs of $F^q_{k,p,r}(\theta)$ for $\theta \in (\pi/3, 2\pi/3)$

4.6 Further Research

Given this bound on the imaginary component of possible zeros of $E^{\infty}_{k,p}(z)$, the next step would be to further narrow this bound towards the arcs of $\mathcal{F}_p$ and create a way of counting zeros associated to an arc. In addition, we observe that other than elliptic points, there seem to be no zeros on the arcs of $\mathcal{F}_p$ except those centered at $\pm\frac{1}{p}$.

5 Brandt Series Matrix and Modular Forms

In the following sections, we will discuss a basis for $M_2(\Gamma(p))$, which is a finite dimensional vector space (for more detailed information on the dimension of the space of modular forms and construction of its basis, refer to the appendix 8.3). We first introduce some general theories on quaternion algebras over a field to define a Brandt series matrix for a quaternion algebra $\mathcal{B}_p$ ramified precisely at p and ∞. Then, we will examine the properties of the subspace generated by the theta series on the diagonal of the Brandt series matrix, which are weight 2 modular forms, and determine whether the diagonal entries form a new basis for the space $M_k(\Gamma_0(p))$.

5.1 Introduction

It was well known that there is a close relation between the study of modular forms of weight 2 on $\Gamma_0(N)$ and the arithmetical theory of quaternion algebras. After Hecke conjectured in 1940 that all weight 2 cusp forms of weight 2 on $\Gamma_0(p)$ are linear combinations of theta series attached to the norm of some quaternion algebra, Eichler, Pizer and many others studied the connection between modular forms and quaternion algebras [4]. In this section, we explain the connection between quaternion algebras and modular forms in more detail by introducing Brandt matrices, and demonstrate some computational results on for which prime numbers p Hecke's conjecture holds.

5.2 Quaternion Algebras and Brandt Matrix

We first define the n-th Brandt Matrix using the language of quaternion algebras. We will follow the notations used in [5] throughout this section. Also, throughout the section, let p be a prime number.

Definition 10. *Let $\mathbb{F}$ be either $\mathbb{Q}$, $\mathbb{Q}_p$, the field of p-adic numbers, or $\mathbb{R}$. A quaternion algebra A over $\mathbb{F}$ is a central simple algebra of dimension 4 over $\mathbb{F}$.*

For any quaternion algebra A over $\mathbb{F}$, there is a basis $1, i, j, k$ over $\mathbb{F}$ such that $i^2 = a, j^2 = b, ij = k = -ji$, for some $a, b \in \mathbb{F}^\times$. If $\mathbb{F} = \mathbb{Q}$, we will denote $A = (a, b)$. Similarly, when $\mathbb{F} = \mathbb{Q}_p$, denote $(a, b)_p$, and when $\mathbb{F} = \mathbb{R}$, $(a, b)_\infty$. For $\alpha = w + xi + yj + zk \in A$, we define the norm of α to be

$$\mathrm{Nor}(\alpha) = w^2 - ax^2 - by^2 + abz^2$$

For example, if $\mathbb{F} = \mathbb{R}$, and $a = b = -1$, we get the Hamiltonian quaternions, $H = (-1, -1)_\infty$.

$$H = \{w + x \cdot i + y \cdot j + z \cdot k : x, y, z, w \in \mathbb{R}\}, \quad \mathrm{Nor}(\alpha) = w^2 + x^2 + y^2 + z^2$$

Now if A is a quaternion algebra over $\mathbb{Q}$, we can consider $A_p = A \otimes_{\mathbb{Q}} \mathbb{Q}_p$, which is a quaternion algebra over $\mathbb{Q}_p$. Similarly, we let $A_\infty = A \otimes_{\mathbb{Q}} \mathbb{R}$. It is a well-known fact that over $\mathbb{Q}_p$ or $\mathbb{R}$, there are only two quaternion algebras up to isomorphism: the 2×2 matrix algebra and a unique quaternion division algebra. For example, the only quaternion algebras over $\mathbb{R}$ are $M(2, \mathbb{R})$ or the Hamiltonian quaternions. A prime p of $\mathbb{Q}$ can either ramify or split in a quaternion algebra A over $\mathbb{Q}$.

Definition 11. *A prime p is said to ramify in A if $A_p = A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra, denoted H_p, and to split in A if A_p is the 2 by 2 matrix algebra over, denoted $M(2, \mathbb{Q}_p)$.*

If we count the infinity as a prime, the set of primes ramifying in A is finite and even in number. Also, any set S consisting of an even number of primes uniquely determines the quaternion algebra A over $\mathbb{Q}$, ramified precisely at the primes in S.

Similar to a ring of integers in a number field, we can define an order $\mathcal{O}$ of a quaternion algebra A and a left-ideal class of an order $\mathcal{O}$.

Definition 12. *An order $\mathcal{O} \subset A$ is a subring of A that is a free Z-submodule of rank 4 (a lattice on A) satisfying the relation $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. An order $\mathcal{O}$ of A is called maximal, if it is not properly contained in any other order of A. A left-ideal I is a lattice on A satisfying $I = \mathcal{O}a$ for some $a \in A^\times$. Two left-ideals I, J are in the same class if $I = J\tilde{a}$ for some $\tilde{a} \in A^\times$.*

For a quaternion algebra A over $\mathbb{Q}$, a maximal order in A is not unique. Instead, there are finitely many maximal orders in A up to conjugation. For a left $\mathcal{O}$-ideal I, we can also define the right order of I to be the set $\{a \in A : Ia \subseteq I\}$. For a left $\mathcal{O}$-ideal I, we define the following operations:

Definition 13. *The norm of I, Nor(I) is the positive rational number which generates the fractional ideal of $\mathbb{Q}$, $\{N(\alpha) : \alpha \in I\}$. The inverse of I, denoted I^{-1}, is given by $I^{-1} = \{\alpha \in A : I\alpha I \subseteq I\}$.*

With these definitions, we can now state an important proposition that leads to the definition of the Brandt matrices.

Proposition 2. *Let $\mathcal{O}$ be a maximal order of a quaternion algebra A over $\mathbb{Q}$. Then, the number of distinct left-ideal classes of $\mathcal{O}$ (called a class number and denoted h) is finite, and for a quaternion algebra A ramified precisely at p and ∞, the class number $h = g+1$ where g is the genus of the modular curve $X_0(p)$. Further, if $I_1, \ldots, I_h$ are the distinct left-$\mathcal{O}$ ideal classes and $\mathcal{O}_j$ are the right order of I_j for $j = 1, \ldots, h$, then $I_j^{-1}I_1, \ldots, I_j^{-1}I_h$ is a complete set of representatives of the left $\mathcal{O}_j$-ideal classes. Further, the $\mathcal{O}_j$'s represent all the conjugacy classes of maximal orders in A, with possible duplication.*

Definition 14. *Let $\mathcal{B}_p$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞, and let $\mathcal{O}$ be a maximal order of $\mathcal{B}_p$. Also, let $I_1, I_2, \ldots, I_h$ be representatives of the left ideal classes of $\mathcal{O}$ and e_j be the number of units of the right order of I_j. Then the n-th Brandt matrix B(n) has i,j-th entry*

$$b_{ij}^{(n)} = e_j^{-1} \cdot |\{\alpha : \alpha \in I_j^{-1}I_i \text{ and } \mathrm{Nor}(\alpha)\mathrm{Nor}(I_j)/\mathrm{Nor}(I_i) = n\}|.$$

When $n = 0$, $b_{ij} = 1/e_j$.

Brandt matrices give a representation of the Hecke algebra (algebra of Hecke operators). Now for each entry $b_{ij}^{(n)}$ of $B(n)$, we can define a Brandt series to be $\sum_{n=0}^{\infty} b_{ij}^{(n)} \exp(n\tau)$, which can be rewritten as

$$\theta_{ij}(\tau) = \sum_{n=0}^{\infty} b_{i,j}^{(n)} \exp(n\tau) = e_j^{-1} \sum_{\alpha \in I_j^{-1}I_i} \exp(\tau \mathrm{Nor}(\alpha)\mathrm{Nor}(I_j)/\mathrm{Nor}(I_i))$$

Since $\mathrm{Nor}(\alpha)\mathrm{Nor}(I_j)/\mathrm{Nor}(I_i)$ is a quadratic form on $\alpha \in I_j^{-1}I_i$, $\theta_{ij}(\tau)$ is a generalized theta series. Therefore, we established a connection between the Brandt matrices and modular forms, which is stated in the following theorem in [4].

Theorem 8. *The entries of the Brandt series matrix (a $h \times h$ matrix where h is the class number)*

$$\theta_{ij}(\tau) = \sum_{n=0}^{\infty} b_{ij}^{(n)} \exp(n\tau)$$

are modular forms of weight 2 on $\Gamma_0(p)$.

Remark 1. *Note that $\dim(M_2(\Gamma_0(p))) = g+1$ where g is the genus of $X_0(p)$, and by proposition 2, the class number for a maximal ideal $\mathcal{O}$ of a quaternion algebra $\mathcal{B}_p$ ramified at p and ∞, $h = g+1$ as well.*

Hence, through the Brandt series matrix, we can obtain numerous new modular forms, each one a generalized theta series associated with some quadratic form. Let $\dim(M_2(\Gamma_0(p))) = h$. Then, the Brandt series matrix is a h by h matrix with some modular forms as its h^2 entries. Using the above theorem along with some theory on the arithmetic of quaternion algebras, it can be shown that the entries in the Brandt series matrix are non-cusp forms, but the difference of any two series in the same column is a cusp form on $\Gamma_0(p)$.

5.3 Subspace of $M_2(\Gamma_0(p))$ Spanned by Elements of Brandt Series Matrix

Again, let h be the dimension of the space of weight 2 modular forms on $\Gamma_0(p)$. It is an interesting question to check whether h elements in the Brandt series matrix form a basis for $M_2(\Gamma_0(p))$ or not. In this project, with h diagonal elements in the Brandt series matrix, we checked whether those elements form a new basis for $M_2(\Gamma_0(p))$ or not, for various prime numbers.

Using a modular forms package in SAGE, it is possible to obtain a canonical basis for $M_2(\Gamma_0(p))$ in its q-expansion form (which will be denoted q-basis in this section). For each of h selected entry from Brandt series matrix, we calculated the coefficients in the linear combination of the modular forms in the q-basis that produces the theta series in the Brandt matrix. Then, we obtained a $h \times h$ matrix from those coefficients, and computed the determinant of that matrix to determine whether the entries from the Brandt series matrix form a new basis for $M_2(\Gamma_0(p))$ or not. We introduce some computation results along with an interpretation.

5.4 Results

p	Determinant	Rank-Corank*	Class Number
3	1	1	1
7	1	1	1
11	2	1*	1
19	-2	1*	1
23	12	3	3
31	-4	3	3
43	0	1*	1
47	-208	5	5
59	192	3*	3
67	0	1*	1
71	704	7	7
79	0	5	5
83	0	3*	3
103	0	5	5
107	0	3*	3
127	0	5	5
131	0	5*	5
139	0	3*	3
151	0	7	7
163	0	1*	1
167	0	11	11
179	0	5*	5
191	0	13	13
199	0	9	9
211	0	3*	3
223	0	7	7
227	0	5*	5
239	0	15	15
251	0	7*	7
263	0	13	13
271	0	11	11
283	0	3*	3

Table: Results for $p \equiv -1 \pmod 4$ up to 300.
Rank − Corank was divided with 2 when it was an even number (numbers with *).

On the above table, the class number is the class number of the number field $\mathbb{Q}(\sqrt{-p})$. As can be noticed in this table, the entries in the Brandt series matrix stopped forming a basis for the space $M_2(\Gamma_0(p)$ for primes greater than 71. There was no pattern observed in the rank, co-rank, or the ratio of the rank and the dimension of the transformation matrix. However, an interesting correspondence between Rank - Corank and the class number of $\mathbb{Q}(\sqrt{-p})$ could be observed. The implication of this correspondence is still being discussed.

Pizer provides an explanation for the observation that the theta series on the diagonal of the Brandt series matrix stop forming a basis for $M@2(\Gamma_0(p))$ in [4]. Class number of a maximal ideal $\mathcal{O}$ is strictly greater than the number of conjugacy classes of right orders (which is referred to as the type number in [4]) for primes greater than 71, and this results in a linear dependence among the theta series on the diagonal of the Brandt matrix. This explains why the diagonal entries fail to form a basis for the space $M_2(\Gamma_0(p))$.

6 Conclusion

There is a lot more to investigate about the location of the zeros of the Eisenstein series. We want to learn more about their placement near the arcs of $\mathcal{A}_{p,f}$ and bound their location. Using the (3) we can find the number of zeros expected and using the bounds on their location can hopefully determine where all of the zeros lie.

Diagonal entries of the Brandt series matrix formed a new basis for the space $M_k(\Gamma_0(p))$ for primes up to 31, but there were some primes with transformation matrices of determinant 0, and primes greater than 71 all failed to form a complete basis for $M_k(\Gamma_0(p))$. A correspondence between Rank -Corank and the class number of $\mathbb{Q}(\sqrt{-p})$ was found, but its implication is not well understood. We hope to understand and discover interesting properties of the elements of the Brandt series matrix through further research.

7 Acknowledgment

We would like to thank George Schaeffer and Jesse Silliman for mentoring our project.

8 Appendix

8.1 Cusps and fundamental domains for prime power/composite levels

One thing that was important in our construction of $\mathcal{F}_p$ was the fact that for prime N, $\varepsilon_\infty = 2$, so we only needed to consider the two cusps at 0 and ∞. However, for prime powers and composites, this is not true.

Before we delve into the algorithm, it's worth noting that while this method gives a list of cusps and cosets, as far as we are aware it doesn't have a closed form. That is, on a computer one can compute–relatively quickly–the set of representatives, but that computation is specific to that N.

First, we need to determine a set of representatives of cusps. Note that certain cusps are equivalent via $\Gamma_0(N)$, such as for $N = 9$,

$$-\frac{1}{3} = \left(\begin{smallmatrix} 5 & -1 \\ -9 & 2 \end{smallmatrix}\right)\left(\frac{1}{6}\right)$$

Now, two cusps x, y are equivalent if there is some $\gamma \in \Gamma_0(N)$ such that $y = \gamma x$. Setting $\gamma = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)$, with $ad - Nbc = 1$, we see that

$$\begin{aligned} y &= \frac{ax+b}{Ncx+d} \\ Ncxy + dy - ax - \frac{ad}{Nc} &= b - \frac{ab}{Nc} \\ (d + Ncx)(a - Ncy) &= 1 \end{aligned}$$

So, if there exist integer a, b, c, d such that this is true, then x and y are equivalent as cusps. Now, from here we choose cusps of the form $x = \frac{1}{x'}, x' \in \mathbb{Z}$. Then we can bash out a set of cusps using a computer using the following code:

```
def bruteForceInverseCusps(N):
    # Calculates the number of cusps
    totalCusps = -1
    for d in range(N):
        if N%(d+1) == 0:
            totalCusps += euler_phi(gcd(d+1, N/(d+1)))

    # This is going to be a set of inverses of cusps,
    # i.e. if x is a cusp, then 1/x will be in this list
    invCusps = [0];  x = 2
    while x < N and len(invCusps) < totalCusps:
        if gcd(x, N) > 1:
            newCusp = true
            for y in invCusps:
                if gcd(y, N) == gcd(x, N) and newCusp:
                    newCusp = false
                    xP = x/gcd(x, N); yP = y/gcd(y, N); NP = N/gcd(x, N)
                    arg = xP*inverse_mod(int(yP), int(NP))
                    try:
                        crt([1, int(arg)], [int(y), int(NP)])
                    except:             # This is actually good, it means that x and y are
                        newCusp = true  #NOT equivalent of level N
            if newCusp:
                invCusps.append(x)
        x = toggleCount(x)
    return ret
```

Here, `toggleCount` simply counts numbers in the order {0, 1, -1, 2, -2, ...}, so we get cusps spread evenly over the positive and negative x-axis.

From here, we need to construct d_N matrices that aren't equivalent via $\Gamma_0(N)$. Since we already know that we can't get between different cusps that we've chosen, we need to find a number of matrices for any particular cusp. So, for the cusp $x = \frac{1}{c}$, define

$$\beta_{c,k} = \left(\begin{smallmatrix} 1 & k \\ c & 1+ck \end{smallmatrix}\right)$$

Then, if $\beta_{c,k} \equiv \beta_{c,l}$, this means that

$$\begin{aligned} \beta_{c,k} &= \gamma\beta_{c,l} \\ \beta_{c,k}\beta_{c,l}^{-1} &= \gamma \in \Gamma_0(N) \\ \left(\begin{smallmatrix} * & * \\ p* & * \end{smallmatrix}\right) &= \left(\begin{smallmatrix} 1+cl & -l \\ -c & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+c(l-k) & k-l \\ c^2(l-k) & 1+c(k-l) \end{smallmatrix}\right) && \in \Gamma_0(N) \\ \left(\begin{smallmatrix} 1 & k \\ c & 1+ck \end{smallmatrix}\right) \end{aligned}$$

So, this is true if $N|c^2(k-l)$. So, we take the several values of k, following the `toggleCount` method of alternating + and -, such that $c^2(k_1 - k_2)$ is never divisible by N for different k_1, k_2. This also includes the identity matrix, since 0 is an element of our list of inverse cusps.

Now, all of these cosets form a set of representatives for $SL_2(\mathbb{Z})$ of $\Gamma_0(N)$, so we can write

$$SL_2(\mathbb{Z}) = \bigcup_{\alpha \in S} \Gamma_0(N)\alpha$$

for $S = \{\alpha_k\} \cup \{\beta_{c,k}\}$

This construction of cosets gives an idea of what the fin fundamental domain looks like for prime powers and composites. Here is an example for $N = 9$:

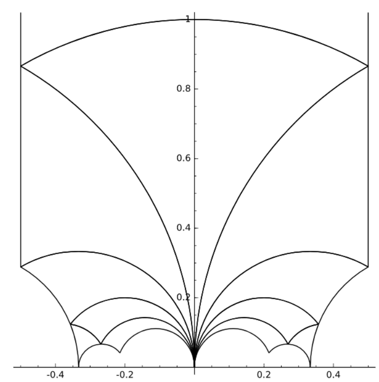


Figure 11: Fin domains for $p = 9$, divided by coset representatives

Now we consider the bump domain for $N = p^2$, as we can explicitly construct a set of cusps for these N. We know that $\Gamma_0(p^2)$ will have $\varepsilon_\infty = p + 1$ different cusps, which can be verified to be at $\frac{1}{kp}$ for $0 < |k| \leq (p-1)/2$

Then, we define the fundamental domain as the area above all the arcs

$$\mathcal{A}_{p^2} = \bigcup_{(q,p)=1} \{\frac{e^{i\theta} + q}{p^2} \mid \theta \in [0, \pi]\}$$

$$\mathcal{B}_{p^2} = \mathcal{A}_{p,f} \cup \{1/2 + it \mid t \geq \frac{\sqrt{3}}{2p}\} \cup \{-1/2 + it \mid t \geq \frac{\sqrt{3}}{2p}\}$$

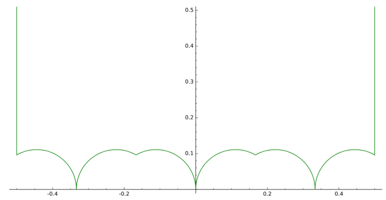
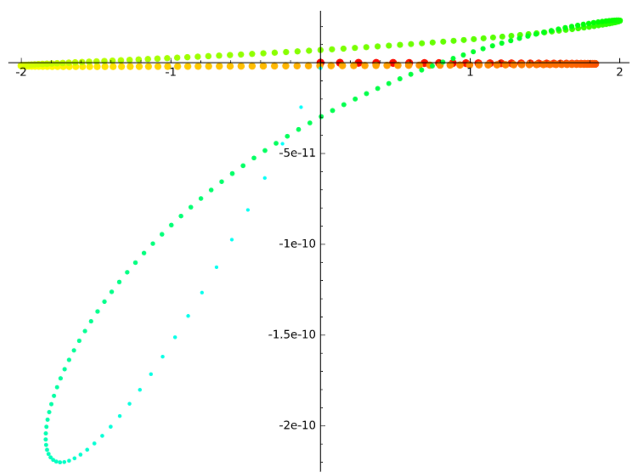


Figure 12: Bump domains for $N = 9$

Then the fundamental domain $\mathcal{F}_{p^2}$ is the area within $\mathcal{B}_{p^2}$ including $\mathcal{B}_{p^2}$.

8.2 Graphs of $F^q_{k,p}(\theta)$



[H]

Figure 13: $F^3_{22,13}(\theta)$

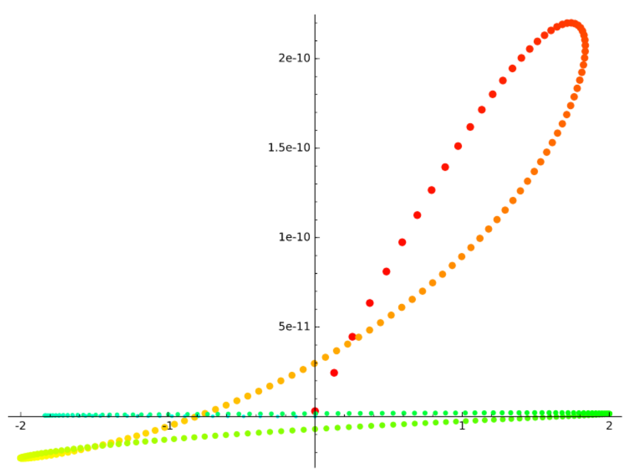


Figure 14: $F^4_{22,13}(\theta)$

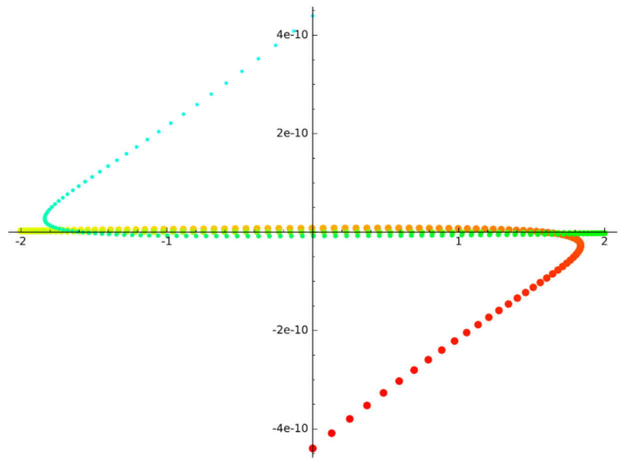


Figure 15: $F^{5}_{22,13}(\theta)$

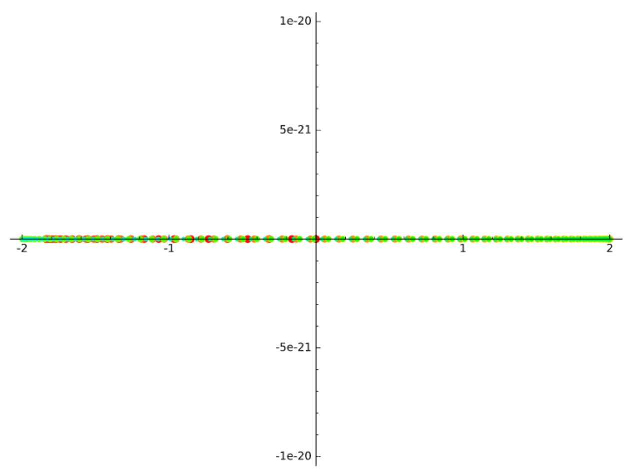


Figure 16: $F^{1}_{28,11}(\theta)$

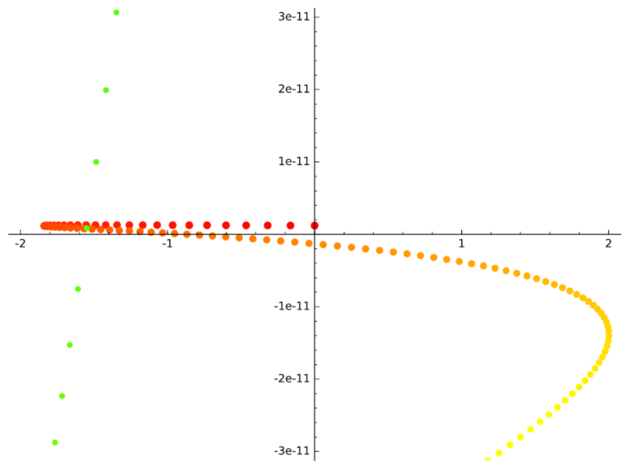


Figure 17: $F^2_{28,11}(\theta)$

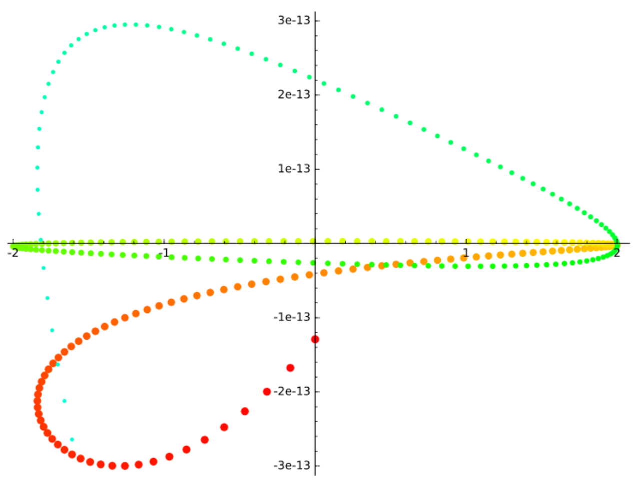


Figure 18: $F^3_{28,11}(\theta)$

8.3 Dimension Formula for $M_k(\Gamma)$ and Eigenforms

One interesting fact about the space of weight k, level N modular forms is that it forms a finite dimensional vector space over $\mathbb{C}$. Holomorphicity of modular forms on the extended upper half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ makes the space $M_k(\Gamma)$ finite dimensional. An explicit formula for the dimension of $M_k(\Gamma)$ also exists. The formula is derived from the study of modular curves ($X(\Gamma)$) as a Riemann Surface.

Hecke operators are commuting endomorphisms of the vector space $M_k(\Gamma)$ and its subspace $S_k(\Gamma)$, the space of cusp forms. By studying their properties, it is possible to find a canonical basis for the vector space which are simultaneous eigenvectors for the Hecke operators.

8.3.1 Dimension Formula for $M_k(\Gamma)$

In this section, we state some important results about modular forms without proof to state the dimension formula for the space of weight k modular forms for a congruence subgroup Γ, $M_k(\Gamma)$. We will follow notations in [3] in this section.

Definition 15. *For any congruence subgroup Γ of $SL_2(\mathbb{Z})$ acting on the upper half plane from the left, the modular curve $Y(\Gamma)$ is defined as the quotient space of orbits under Γ*

$$Y(\Gamma) = \Gamma \setminus \mathbb{H} = \{\Gamma\tau : \tau \in \mathbb{H}\}$$

The modular curve for $\Gamma_0(N)$ is denoted $Y_0(N)$.

By composing the natural projection map from the upper half plane to $Y(\Gamma)$ defined as $\pi : \tau \mapsto \Gamma\tau$ with appropriate coordinate maps, $Y(\Gamma)$ can be given a local coordinate chart to be treated as a Riemann surface. However, $Y(\Gamma)$ is not a compact surface due to the cusps, which are the Γ-equivalence class of $\mathbb{Q} \cup \{\infty\}$. Thus, $Y(\Gamma)$ can be compactified by adjoining the cusps, and the resulting compact Riemann surface is denoted $X(\Gamma)$. Topologically, $X(\Gamma)$ is a torus with genus g. Using the Riemann-Hurwitz formula, it can be shown that the genus of $X(\Gamma)$ is given by the following formula:

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2} \tag{7}$$

where $\varepsilon_2, \varepsilon_3$ and ε_∞ are the number of elliptic points and cusps for $X(\Gamma)$, and d is the index $[SL_2(\mathbb{Z}) : \Gamma]$, as in the previous section.

We established that $X(\Gamma)$ is a compact Riemann surface, and obtained a formula for the genus g of $X(\Gamma)$. Now we can state the dimension formula for the space $M_k(\Gamma)$ and $S_k(\Gamma)$ (Derivation of these formulas involve the Riemann-Roch Theorem and some other theories on meromorphic functions and meromorphic differentials. For more detailed proof for these formulas, one can refer to [3]).

Theorem 9. *Let k be an even integer, and Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Then,*

$$\dim(M_k(\Gamma)) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty \quad k \geq 2 \tag{8}$$

and

$$\dim(S_k(\Gamma)) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + (\frac{k}{2} - 1)\varepsilon_\infty \quad k \geq 4. \tag{9}$$

$\dim(M_k(\Gamma)) = 1$ *if $k = 0$, and* $\dim(S_k(\Gamma)) = g$ *if $k = 2$, and the dimensions are 0 otherwise.*

The dimension formulas for odd k also exists, but we will not state them in this paper.

8.3.2 Hecke Operators and Eigenforms

Hecke operators are special type of operators acting on modular forms that take $M_k(\Gamma)$ to itself and $S_k(\Gamma)$ to itself. They form a group of endomorphisms of $M_k(\Gamma)$ and $S_k(\Gamma)$. The T_n operators, which are to be defined below, are a family of Hecke operators with some useful properties. Along with the definition we introduce some useful properties of the T_n Hecke operators.

Definition 16. *Let n be a positive integer. Then, the n-th Hecke operator T_n acts on a modular form $f(\tau)$ of weight k by the following formula:*

$$T_n f(\tau) = n^{k-1} \sum_{a\geq 1, ad=n, 0\leq b<d} d^{-k} f\left(\frac{a\tau + b}{d}\right) \tag{10}$$

The T_n operator maps a weight k modular form (resp. cusp form) to another weight k modular form (resp. cusp form). Also, the family of T_n operators commute with each other, i.e. $T_nT_m = T_mT_n$ for positive integers m and n. Furthermore, for relatively prime m and n, we have that $T_mT_n = T_{mn}$.

Other than being commuting endomorphisms of vector spaces M_k and S_k (modular forms and cusps forms of weight k), Hecke operators have a very nice property that they are normal operators under the Petersson inner product. For weight k modular forms f, g, the Petersson inner product is given by

$$\langle f, g\rangle = \frac{1}{V_\Gamma} \int_{X(\Gamma)} f(\tau)\overline{g(\tau)}(\mathrm{Im}(\tau))^k d\mu(\tau)$$

where $V_\Gamma = \int_{X(\Gamma)} d\mu(\tau)$, and μ is the hyperbolic measure. (Formally, the Petersson inner product can only be defined for $\langle,\rangle : M_k(\Gamma) \times S_k(\Gamma) \mapsto \mathbb{C}$.)

The Spectral Theorem of linear algebra states that a normal operators on a finite dimensional inner-product space has an orthogonal basis of eigenvectors for the operator. Also from linear algebra, commuting family of normal operators can have simultaneous eigenvectors (since the space has modular forms as its elements, they will be called eigenforms). Thus, we have the following conclusion.

Theorem 10. *The space $M_k(\Gamma)$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators.*

In [3], a construction of a canonical basis for the space $M_k(\Gamma)$ is explained in detail. This is achieved by further developing the relationship between the Fourier coefficients of eigenforms and the T_n operators. However, for the purpose of this report, it is enough to note that there is a canonical basis of the space of modular forms that is an orthogonal list of simultaneous eigenforms for the Hecke operators.

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